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A best proximity point theorem for α-proximal Geraghty non-self mappings

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Abstract

In this paper, we search some best proximity point results for a new class of non-self mappings $T : A \longrightarrow B$ called α-proximal Geraghty mappings. Our results extend many recent results appearing in the literature. We suggest an example to support our result. Several consequences are derived. As applications, we investigate the existence of best proximity points for a metric space endowed with symmetric binary relation.

MSC: 47H10; 54H25

Keywords: Best proximity points; α-Proximal Geraghty non-self mappings on metric spaces

1 Introduction

One of the famous generalizations of the Banach contraction principle for the existence of fixed points for self mappings on metric spaces [5] is the theorem by Geraghty [8]. Consider A and B to be two nonempty subsets of a metric space (X,d). Let $T : A \longrightarrow B$ be a non-self mapping. Then the best proximity points of T are the points $x \in A$ satisfying $d(x,Tx) = d(A,B)$. Recently, several works on best proximity point theory were studied by giving sufficient conditions assuring the existence. Thus, several known results were derived. For additional information, see Refs. [2, 3, 7, 10, 12, 19–22], and [25].

Recently, Jleli, Karapinar, and Samet in [11] have introduced a new class of contractive mappings called α–ψ-contractive type mappings. They have provided some results on the existence and uniqueness of best proximity points of such non-self mappings. There are many papers in the literature about α-contractions, see, for example, [1, 4, 13, 17], and [15, 16, 24]. Recently, Ayari in [9] proposed an extension for the case of α–β-proximal quasi-contractive mappings. We are interested in extending these works for the Geraghty functions by introducing the notion of α-proximal Geraghty non-self mappings. The purpose of all of this is to provide a theorem on the existence and uniqueness of best proximity point for such mappings.

Kumam and Mongkolekeha in [14] proved new common best proximity point theorems for proximity commuting mappings using the concept of Geraghty theorem in complete metric spaces. Also Biligili, Karapinar, and Sadarangani in [6] also suggested a best proximity point theorem for a pair (A,B) of subsets on a metric space X satisfying the P-property. This was accomplished by introducing the notion of generalized Geraghty-contraction.

In this work, we have established a new result on the existence and uniqueness of best proximity point for α-proximal Geraghty non-self mappings defined on a closed subset of a complete metric space. Our result generalized results existing in the literature. Moreover, we have shown that from our main theorems we are able to deduce some other theorems of best proximity points for the case of metric spaces endowed with symmetric binary relations. We also have deduced the main fixed point theorem of Geraghty [8].

The paper is divided into five different sections as follows. Section 2 is dedicated to the notation adopted and to providing definitions. Moreover, best proximity point theorem is stated in Sect. 3 with its proof illustrated by an example. Then, several consequences are obtained in Sect. 4. Finally, the existence of best proximity points on metric spaces endowed with symmetric binary relations and a fixed point result are given in Sect. 5.

2 Preliminaries and definitions

Let (A, B) be a pair of nonempty subsets of a metric space (X, d). We adopt the following notations:

$d(A,B) := \inf\{d(a,b) : a \in A, b \in B\}$;

$A_0 := \{a \in A : \text{ there exists } b \in B \text{ such that } d(a,b) = d(A,B)\}$;

$B_0 := \{b \in B : \text{ there exists } a \in A \text{ such that } d(a,b) = d(A,B)\}$.

Definition 2.1 ([22]) Let $T : A \to B$ be a mapping. An element x^* is said to be a best proximity point of T if $d(x^*, Tx^*) = d(A,B)$.

Definition 2.2 ([18]) Let (A, B) be a pair of nonempty subsets of a metric space (X, d) such that A_0 is nonempty. Then the pair (A, B) is said to have the P-property iff $d(x_1, y_1) = d(x_2, y_2) = d(A,B) \Longrightarrow d(x_1, x_2) = d(y_1, y_2)$, where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Definition 2.3 ([11]) Let $T : A \to B$ and $\alpha : A \times A \longrightarrow [0, +\infty)$. We say that T is said to be α-proximal admissible if $\alpha(x_1, x_2) \geq 1$ and $d(u_1, Tx_1) = d(u_2, Tx_2) = d(A,B) \Longrightarrow \alpha(u_1, u_2) \geq 1$ for all $x_1, x_2, u_1, u_2 \in A$.

Let us introduce the set F that is the class of all functions $\beta : [0, \infty) \longrightarrow [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \longrightarrow 1$ implies $t_n \longrightarrow 0$.

Definition 2.4 ([8]) Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a β-Geraghty contractive mapping if there exists $\beta \in F$ such that

$$d(Tx, Ty) \leq \beta\big(d(x,y)\big)d(x,y)$$

for all $x, y \in X$.

3 Main results and theorems

First, we introduce the following concept which is a natural generalization of the definition of Geraghty.

Definition 3.1 Let (X, d) be a metric space and (A, B) be a pair of nonempty subsets of X. A non-self mapping $T : A \to B$ is called α-proximal Geraghty mapping, where $\alpha : A \times A \longrightarrow$

$[0, +\infty)$, if there exists $\beta \in F$ such that

$$\alpha(x,y)d(Tx,Ty) \leq \beta\big(d(x,y)\big)d(x,y)$$

for all $x, y \in A$.

We propose the following best proximity point theorem.

Theorem 3.2 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $\alpha : A \times A \longrightarrow [0,+,\infty)$ and $\beta \in F$. Consider an α-proximal Geraghty non-self mapping $T : A \longrightarrow B$ satisfying the following assertions*:

(1) *$T(A_0) \subset B_0$ and the pair (A,B) satisfies the P-property*;
(2) *T is α-proximal admissible*;
(3) *There exist elements $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq 1$*;
(4) *If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\lim_{n\longrightarrow+\infty} x_n = x_* \in A$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x_*) \geq 1$ for all k.*

Then T has a unique best proximity point $x_* \in A$ such that $d(x_*, Tx_*) = d(A,B)$.

Proof Thanks to condition (3), there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq 1$. As $T(A_0) \subset B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A,B)$.

As T is α-proximal admissible and using $\alpha(x_0,x_1) \geq 1$, $d(x_1, Tx_0) = d(x_2, Tx_1) = d(A,B)$, this implies that $\alpha(x_1,x_2) \geq 1$.

In a similar fashion, by induction, we can construct a sequence $\{x_n\} \subset A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A,B) \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.1}$$

Our next step is to prove that the sequence $\{x_n\}$ is a Cauchy sequence. Let us first prove that $\lim_{n\longrightarrow+\infty} d(x_n, x_{n-1}) = 0$. Since $d(x_{n+1}, Tx_n) = d(A,B)$ and $d(x_n, Tx_{n-1}) = d(A,B)$, using the P-property, we get $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$. As T is α-proximal Geraghty mapping and $\alpha(x_{n-1}, x_n) \geq 1$, then

$$d(x_{n+1},x_n) \leq \beta\big(d(x_{n-1},x_n)\big)d(x_{n-1},x_n) \leq d(x_{n-1},x_n) \quad \text{for all } n \geq 1. \tag{3.2}$$

We can suppose that $d(x_{n-1},x_n) > 0$. From (3.2) we conclude that

$$\frac{d(x_{n+1},x_n)}{d(x_{n-1},x_n)} \leq \beta\big(d(x_{n-1},x_n)\big) \leq 1 \quad \text{for all } n \geq 1. \tag{3.3}$$

Let $r = \lim_{n\longrightarrow+\infty} d(x_{n-1},x_n)$. Using equation (3.2) and letting $n \longrightarrow +\infty$, we obtain that

$$\frac{r}{r} = 1 \leq \lim_{n\longrightarrow+\infty} \beta\big(d(x_{n-1},x_n)\big) \leq 1.$$

Thus $\lim_{n\longrightarrow+\infty} \beta(d(x_{n-1},x_n)) = 1$. Using the definition of the function β, we conclude that

$$\lim_{n\longrightarrow+\infty} d(x_{n-1},x_n) = 0. \tag{3.4}$$

Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that, for all positive integers k such that $m(k) > n(k) > k$ $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ and $d(x_{m(k)}, x_{n(k)-1}) < \epsilon$. Using the triangular inequality, we get

$$\begin{aligned} \epsilon &\leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \\ &\leq \epsilon + d(x_{n(k)-1}, x_{n(k)}), \quad \forall k. \end{aligned} \tag{3.5}$$

Taking limit as $k \longrightarrow +\infty$ in the above inequality (3.5) and using (3.4), we conclude that

$$\lim_{n \longrightarrow +\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{3.6}$$

Using again the triangular inequality,

$$d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)}). \tag{3.7}$$

On the other hand, using triangular inequality and inequality (3.7), we have

$$\epsilon \leq d(x_{m(k)+1}, x_{n(k)+1}) \leq d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}). \tag{3.8}$$

Letting $k \longrightarrow +\infty$ and using (3.4) and (3.6), we get

$$\lim_{n \longrightarrow +\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \epsilon. \tag{3.9}$$

Since T is an α-proximal Geraghty mapping, we obtain

$$\begin{aligned} d(x_{m(k)+1}, x_{n(k)+1}) &\leq \beta\bigl(d(x_{m(k)}, x_{n(k)})\bigr) d(x_{n(k)}, x_{n(k)}) \\ &\leq d(x_{m(k)}, x_{n(k)}), \quad \forall k. \end{aligned} \tag{3.10}$$

Therefore

$$\frac{d(x_{m(k)+1}, x_{n(k)+1})}{d(x_{m(k)}, x_{n(k)})} \leq \beta\bigl(d(x_{m(k)}, x_{n(k)})\bigr) \leq 1. \tag{3.11}$$

Letting $k \longrightarrow +\infty$ in the above inequality (3.11), we conclude that

$$\lim_{k \longrightarrow +\infty} \beta\bigl(d(x_{m(k)}, x_{n(k)})\bigr) = 1.$$

Hence

$$\lim_{k \longrightarrow +\infty} d(x_{m(k)}, x_{n(k)}) = 0 < \epsilon,$$

which is a contradiction.

Thus, the sequence $\{x_n\}$ is a Cauchy sequence in the closed subset A of the space (X, d).

The fact that (X,d) is complete and A is closed assures that the sequence $\{x_n\}$ converges to some element $x_* \in A$.

Using hypothesis (4) of the theorem, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x_*) \geq 1$ for all k. Since T is a generalized α-proximal Geraghty mapping, then we have

$$\begin{aligned} d(Tx_{n(k)}, Tx_*) &\leq \alpha(x_{n(k)},x_*)d(Tx_{n(k)}, Tx_*) \\ &\leq \beta\bigl(d(x_{n(k)},x_*)\bigr)d(x_{n(k)},x_*), \quad \forall k. \end{aligned} \tag{3.12}$$

By the triangular inequality and (3.1), we have

$$\begin{aligned} d(x_*, Tx_*) &\leq d(x_*,x_{n(k)+1}) + d(x_{n(k)+1}, Tx_{n(k)}) + d(Tx_{n(k)}, Tx_*) \\ &= d(x_*,x_{n(k)+1}) + d(A,B) + d(Tx_{n(k)}, Tx_*). \end{aligned} \tag{3.13}$$

We obtain that

$$d(Tx_{n(k)}, Tx_*) \geq d(x_*, Tx_*) - d(x_*,x_{n(k)+1}) - d(A,B), \quad \forall k. \tag{3.14}$$

Using (3.12) and (3.14), we get

$$\begin{aligned} d(x_{n(k)},x_*) &\geq \beta\bigl(d(x_{n(k)},x_*)\bigr)d(x_{n(k)},x_*) \\ &\geq d(x_*, Tx_*) - d(x_*,x_{n(k)+1}) - d(A,B), \quad \forall k. \end{aligned}$$

As $k \longrightarrow +\infty$, we get

$$d(x_*, Tx_*) - d(A,B) \leq 0.$$

So we deduce that $d(x_*, Tx_*) = d(A,B)$. Therefore x_* is a best proximity point for the non-self mapping T.

For the uniqueness, suppose that there are two distinct best proximity points for T such that $x_* \neq y_*$. Thus $r = d(x_*,y_*) > 0$. Since $d(Tx_*,x_*) = d(Ty_*,y_*) = d(A,B)$, using the P-property, we conclude that $r = d(Tx_*, Ty_*)$.

Since T is an α-proximal Geraghty non-self mapping, we obtain $r \leq \beta(r)r$. Thus $\beta(r) \geq 1$. Since $\beta(r) \leq 1$, we conclude that $\beta(r) = 1$; and therefore $r = 0$, which is a contradiction. □

Example Consider the complete Euclidian space $X = \mathbb{R}^2$ with the metrics $d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Let $A = \{(a,0), a \in [2,3]\}$ and $B = \{(b,0), b \in [\frac{1}{3},\frac{1}{2}]\}$.

Consider the non-self mapping $T : A \longrightarrow B$ such that $T(a,0) = (\frac{1}{a},0)$. $T(A) = B$ and A and B are closed subsets on the complete space $(\mathbb{R}^2,d)$. It is easy to see that the couple (A,B) satisfies the P-property. The function $\alpha(x,y) = 1$ for all $x,y \in A$. We have $d(T(2,0),(2,0)) = |\frac{1}{2} - 2| = \frac{3}{2} = d(A,B)$. So hypotheses (2), (3), and (4) of the theorem are satisfied. What is remaining is to prove hypothesis (5) of Theorem 3.2 that T is a proximal Geraghty mapping.

In fact, for all $a, a' \in A = [2,3]$, we can prove that

$$d\big(T(a,0), T\big(a',0\big)\big) = \left|\frac{1}{a} - \frac{1}{a'}\right| \leq \frac{|a-a'|}{1+|a-a'|} = \frac{d((a,0),(a',0))}{1+d((a,0),(a',0))}.$$

This inequality is true for $a = a'$.

If $a \neq a'$, we have for all $a, a' \in A = [2,3]$, $|aa'| \geq 4$; meanwhile $1 + |a - a'| \leq 2$. Thus hypothesis (5) of our Theorem 3.2 is satisfied for the function $\beta : [0,+\infty) \longrightarrow [0,1]$ such that $\beta(u) = \frac{1}{1+u}$. So the conclusion is the existence and uniqueness of best proximity point of the mapping T which is $(2,0)$.

4 Consequences

For the case $\alpha = 1$, the definition of Geraghty is the following.

Definition 4.1 Let (X,d) be a metric space and (A,B) be a pair of nonempty subsets of X. A non-self mapping $T : A \to B$ is called a proximal Geraghty mapping if there exists $\beta \in F$ such that

$$d(Tx,Ty) \leq \beta\big(d(x,y)\big)d(x,y)$$

for all $x, y \in A$.

Several consequences of our main theorem are suggested in this section.

Corollary 4.2 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $\beta \in F$. Consider a non-self mapping $T : A \longrightarrow B$ satisfying the following assertions*:

(1) *$T(A_0) \subset B_0$ and the pair (A,B) satisfies the P-property*;
(2) *There exist elements $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A,B)$*;
(3) *T is a proximal Geraghty mapping.*

Then T has a unique best proximity point $x_* \in A$ such that $d(x_*, Tx_*) = d(A,B)$.

Proof This is an immediate consequence of our main Theorem 3.2 by taking $\alpha(x,y) = 1$ for all $x, y \in A$. □

We can also suggest some corollary for the cases $\beta(u) = e^{-ku}$, where $k > 0$, $\beta(u) = \frac{1}{u+1}$, $\beta(u) = \begin{cases} 1; & u = 0, \\ \frac{\ln(1+u)+1}{u}; & u > 0 \end{cases}$ and $\beta(u) = \frac{1}{2-\frac{2}{\pi}\arctan(\frac{1}{u^{\alpha}})}$ where $0 < \alpha < 1$, $u > 0$.

Corollary 4.3 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Consider a non-self mapping $T : A \longrightarrow B$ satisfying the following assertions*:

(1) *$T(A_0) \subset B_0$ and the pair (A,B) satisfies the P-property*;
(2) *There exist elements $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A,B)$*;
(3) *The non-self mapping T satisfies*

$$d(Tx,Ty) \leq \exp\big(-kd(x,y)\big)d(x,y)$$

for all $x, y \in A$, for some constant $k > 0$.

Then T has a unique best proximity point $x_* \in A$ such that $d(x_*, Tx_*) = d(A,B)$.

Corollary 4.4 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Consider a non-self mapping $T:A\longrightarrow B$ satisfying the following assertions*:

(1) *$T(A_0)\subset B_0$ and the pair (A,B) satisfies the P-property*;
(2) *There exist elements $x_0,x_1\in A$ such that $d(x_1,Tx_0)=d(A,B)$*;
(3) *The non-self mapping*

$$d(Tx,Ty)\leq \frac{d(x,y)}{1+d(x,y)}$$

for all $x,y\in A$.

Then T has a unique best proximity point $x_*\in A$ such that $d(x_*,Tx_*)=d(A,B)$.

Corollary 4.5 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Consider a non-self mapping $T:A\longrightarrow B$ satisfying the following assertions*:

(1) *$T(A_0)\subset B_0$ and the pair (A,B) satisfies the P-property*;
(2) *There exist elements $x_0,x_1\in A$ such that $d(x_1,Tx_0)=d(A,B)$*;
(3) *The non-self mapping T satisfies*

$$d(Tx,Ty)\leq \ln\bigl(1+d(x,y)\bigr)$$

for all $x,y\in A$.

Then T has a unique best proximity point $x_*\in A$ such that $d(x_*,Tx_*)=d(A,B)$.

Corollary 4.6 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Consider a non-self mapping $T:A\longrightarrow B$ satisfying the following assertions*:

(1) *$T(A_0)\subset B_0$ and the pair (A,B) satisfies the P-property*;
(2) *There exist elements $x_0,x_1\in A$ such that $d(x_1,Tx_0)=d(A,B)$*;
(3) *The non-self mapping T satisfies*

$$d(Tx,Ty)\leq \frac{d(x,y)}{2-\frac{2}{\pi}\arctan(\frac{1}{d(x,y)^{\alpha}})},\quad 0<\alpha<1$$

for all $x,y\in A$.

Then T has a unique best proximity point $x_*\in A$ such that $d(x_*,Tx_*)=d(A,B)$.

5 Applications

Our first consequence is the theorem of Geraghty for the existence of fixed point.

Corollary 5.1 ([8]) *Let (X,d) be a metric space and T be an operator. Suppose that there exist $\beta:[0,\infty)\longrightarrow[0,1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n)\longrightarrow 1$ implies $t_n\longrightarrow 0$.*

If T satisfies the following inequality

$$d(Tx,Ty)\leq \beta\bigl(d(x,y)\bigr)d(x,y)$$

for all $x,y\in X$, then T has a unique fixed point.

Proof By considering $A = B = X$ and the function $\alpha(x,y) = 1$ in Theorem 3.2, we guarantee the existence and uniqueness of a fixed point of such a self mapping T. □

In order to apply our results on best proximity points on a metric space endowed with symmetric binary relation, we need some preliminaries.

Let (X,d) be a metric space and $\mathcal{R}$ be a symmetric binary relation over X.

Definition 5.2 ([11]) A non-self mapping $T : A \longrightarrow B$ is a proximal comparative mapping if $x\mathcal{R}y$ and $d(u_1, Tx) = d(u_2, Ty) = d(A,B)$ for all $x, y, u_1, u_2 \in A$, then $u_1\mathcal{R}u_2$.

Definition 5.3 ([23]) We say that $(X,d,\mathcal{R})$ is regular if, for a sequence $\{x_n\}$ in X, we have $x_n\mathcal{R}x_{n+1}$ for all $n \in \mathbb{N}_0$ and $\lim_{n\to\infty} d(x_n,x) = 0$ for some $x \in X$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)}\mathcal{R}x$ for all $k \in \mathbb{N}_0$.

Definition 5.4 Let X be a nonempty set. A non-self mapping $T : A \to B$ is called Geraghty contractive if there exists $\beta \in F$ such that

$$x, y \in A : x\mathcal{R}y \quad \Longrightarrow \quad d(Tx, Ty) \leq \beta\bigl(d(x,y)\bigr)d(x,y).$$

We have the following best proximity point result.

Corollary 5.5 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $\mathcal{R}$ be a symmetric binary relation over X. Consider a non-self mapping $T : A \longrightarrow B$ satisfying the following assertions*:

(1) *$T(A_0) \subset B_0$ and the pair (A,B) satisfies the P-property*;

(2) *T is proximal comparative mapping*;

(3) *There exist elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A,B)$ and $x_0\mathcal{R}x_1$*;

(4) *If $(A,d,\mathcal{R})$ is regular*;

(5) *There exists $\beta \in F$ such that $T : A \to B$ is Geraghty contractive.*

Then T has a unique best proximity point $x_* \in A$ such that $d(x_*, Tx_*) = d(A,B)$.

Proof Let us introduce the function

$$\alpha : A \times A \longrightarrow [0,+\infty) \quad \text{by: } \alpha(x,y) = \begin{cases} 1 & \text{if } x\mathcal{R}y, \\ 0 & \text{otherwise.} \end{cases}$$

In order to apply our Theorem 3.2, we have to prove that T is α-admissible.

Assume that $\alpha(x,y) \geq 1$, and $d(u_1, Tx) = d(u_2, Tx) = d(A,B)$, for some $x, y, u_1, u_2 \in A$. By the definition of α, we get $x\mathcal{R}y$ and $d(u_1, Tx) = d(u_2, Tx) = d(A,B)$. Condition (2) of Corollary implies $u_1\mathcal{R}u_2$, which gives us $\alpha(u_1,u_2) \geq 1$.

Condition (3) means that $d(x_1, Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq 1$.

The condition $T : A \to B$ is Geraghty contractive means that T is an α-proximal Geraghty mapping. Also the condition $(A,d,\mathcal{R})$ is regular implies that if $\{x_n\}$ is a sequence in A such that $\alpha(x_n,x_{n+1}) \geq 1$ and $\lim_{n\longrightarrow+\infty} x_n = x_* \in A$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x_*) \geq 1$ for all k.

Now all the hypotheses of Theorem 3.2 are satisfied, which implies the existence and uniqueness of a proximity point for the non-self mapping T. □

6 Conclusion

We recall that we managed in this paper to propose a new best proximity point for α-proximal Geraghty non-self mappings. This was achieved by introducing the notion of α-proximal Geraghty non-self mappings which is an extension of the definition of Geraghty for the case of self mappings. As applications, we have established not only the existence but also the uniqueness of best proximity point results for the case of non-self-mappings on metric spaces endowed with symmetric binary relations.

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