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Random fixed point theorems in Banach spaces applied to a random nonlinear integral equation of the Hammerstein type

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Abstract

The purpose of this paper is to define a new random operator called the generalized ϕ-weakly contraction of the rational type. This new random operator includes those studied by Khan *et al.* (Filomat 31(12):3611–3626, 2017) and Zhang *et al.* (Appl. Math. Mech. 32(6):805–810, 2011) as special cases. We prove some convergence, existence, and stability results in separable Banach spaces. Moreover, we produce some numerical examples to demonstrate the applicability of our analytical results. Furthermore, we apply our results in proving the existence of a solution of a nonlinear integral equation of the Hammerstein type.

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Keywords: Random fixed point; Bochner integrable; Generalized ϕ-weakly contraction of the rational type; Random iterative process; Almost sure T-stability; Nonlinear integral equation of the Hammerstein type

1 Introduction

The process of solving some real life problems is characterized with uncertainties, ambiguities, and difficulties. To develop an approach for solving probabilistic models, probabilistic functional analysis has emerged as one of the indispensable mathematical disciplines and tools. Consequently, it has attracted the attention of well-known mathematicians over the years in view of its applications in diverse areas from pure mathematics to engineering and applied sciences. Random nonlinear analysis, which is an important branch of probabilistic functional analysis, deals with the solution of several classes of random operator equations and related problems. Interestingly, the development of random methods has revolutionized the financial markets and related sectors in many world economies (see, e.g., [27]). Whenever the mathematical models or equations used to describe certain phenomena in the physical, engineering, and biological systems that contain some parameters or coefficients that have specific interpretations, but whose values are not known, then it is more realistic to study such equations as random operator equations (see, e.g., Graef *et al.* [16]). Random fixed point theorems are stochastic generalizations of classical or deterministic fixed point theorems and are required for the theory of stochastic dynamic programming, random equations, random matrices, random partial differential equations, and various classes of random operators arising in physical systems (see, e.g., [20, 27]). Random fixed point theory was initiated in 1950s by Prague school of probabilists. Spacek [35] and Hans [17] established a stochastic analogue of the Banach fixed point theorem in a separable complete metric space. Itoh [19] in 1979 generalized and extended Spacek and Han's theorem to a multivalued contraction random operator. The survey article by Bharucha-Reid [12] in 1976, in which he established the sufficient conditions for a stochastic analogue of Schauder's fixed point theorem for random operators, gave wings to random fixed point theory. This area of research over the years has received the attention of several well-known mathematicians leading to the development of several interesting techniques to obtaining the solution of nonlinear random systems (see, e.g., [7–10, 15–17, 19, 20, 22, 23, 27–30, 33–35, 37]).

In 2011, Zhang *et al.* [37] proved some stability and convergence results for some random iterative sequence generated by a ϕ-weakly contractive random operator. Okeke and Abbas [27] introduced the concept of generalized ϕ-weakly contraction random operators and proved some interesting random fixed point theorems for this kind of random operators in separable Banach spaces. There results unify and generalize the results of Zhang *et al.* [37]. Recently, Khan *et al.* [22] introduced a generalized random operator and proved some convergence and stability results for those kinds of random operators in separable Banach spaces. Their results generalize and improve several known random fixed point results in the literature, including the results of Okeke and Abbas [27].

In 2012, Chugh *et al.* [13] proposed a new three step iterative scheme, called the CR iterative scheme. They proved that the CR iterative scheme is equivalent to and converges faster than all of Picard, Mann [25], Ishikawa [18], Agarwal et al. [2], Noor [26], and SP [32] iterative schemes for certain contractive operators in the sense of Berinde [11]. In 2013, Karahan and Ozdemir [21] proposed a new three step iterative scheme. They proved that this iteration process is faster than all of Picard, Mann [25], and S [2] iteration processes in the sense of Berinde [11].

Motivated by the results above, we introduce a random operator, called the generalized ϕ-weakly contraction of the rational type. This new random operator includes those studied by Khan *et al.* [22] and Zhang *et al.* [37] as special cases. We introduce the random versions of the CR iterative scheme and the Karahan–Ozdemir iterative scheme. We prove some existence, convergence, and stability results for the generalized ϕ-weakly contraction of the rational type via these random iteration schemes. Some numerical examples are given to demonstrate the applicability of our analytical results. Furthermore, we apply our results in proving the existence of a solution of a nonlinear integral equation of the Hammerstein type. Our results unify, extend, and generalize several deterministic fixed point theorems in stochastic version, including the results of Akewe *et al.* [4], Akewe and Okeke [3], Chugh *et al.* [13], and Karahan and Ozdemir [21] among others.

2 Preliminaries

Let (Ω, Σ, μ) be a complete probability measure space and $(E, B(E))$ be a measurable space, where E is a separable Banach space, $B(E)$ is Borel sigma algebra of E, (Ω, Σ) is a measurable space (Σ—sigma algebra), and μ is a probability measure on Σ, that is, a measure with total measure one. A mapping $\xi : \Omega \to E$ is called (a) E-valued random variable if ξ is $(\Sigma, B(E))$-measurable; (b) strongly μ-measurable if there exists a sequence $\{\xi_n\}$ of μ-simple functions converging to ξ μ-almost everywhere. Due to the separability of a Banach space E, the sum of two E-valued random variables is an E-valued random variable.

A mapping $T : \Omega \times E \to E$ is called a random operator if, for each fixed e in E, the mapping $T(\cdot, e) : \Omega \to E$ is measurable. Throughout this study, we assume that (Ω, ξ, μ) is a complete probability measure space and E is a nonempty subset of a separable Banach space X.

The following definitions will be needed in this study

Definition 2.1 ([20]) A random variable $x : \Omega \to E$ is Bochner integrable if, for each

$$\omega \in \Omega, \quad \int_{\Omega} \|x(\omega)\| \, d\mu(\omega) < \infty, \tag{2.1}$$

i.e., $\|x(\omega)\| \in L^1(\Omega, \xi, \mu)$.

Note: The Bochner integral is a natural generalization of the Lebesgue integral for vector-valued set functions.

Proposition 2.1 ([20]) *A random variable $x : \Omega \to E$ is Bochner integrable if and only if there exists a sequence of random variables $\{x_n\}_{n=1}^{\infty}$ converging strongly to x almost surely such that*

$$\lim_{n\to\infty} \int_{\Omega} \|x_n(\omega) - x(\omega)\| \, d\mu(\omega) = 0. \tag{2.2}$$

Definition 2.2 ([37]) Let (Ω, ξ, μ) be a complete probability measure space and E be a nonempty subset of a separable Banach space X. Let $T : \Omega \times E \to E$ be a random operator. Denote by $\mathrm{RF}(T) = \{x^*(\omega) \in E : T(\omega, x^*(\omega)) = x^*(\omega), \omega \in \Omega\}$ the random fixed point set of T. For any given arbitrary measurable mapping $x_0 : \Omega \to E$, let $\{x_n(\omega)\}_{n=0}^{\infty}$ be a sequence of measurable mapping from Ω to E, and

$$x_{n+1}(\omega) = f(T, x_n(\omega)), \quad n = 0, 1, 2, \ldots, \tag{2.3}$$

where f is some function measurable in the second variable.

Let $x^*(\omega)$ be a random fixed point of T and Bochner integrable with respect to $\{x_n(\omega)\}_{n=0}^{\infty}$. Let $\{y_n(\omega)\}_{n=0}^{\infty} \subset \Omega \times E$ be an arbitrary sequence of measurable mapping. Denote

$$\varepsilon_n(\omega) = \|y_{n+1}(\omega) - f(T, y_n(\omega))\|,$$

and assume that $\|\varepsilon_n(\omega)\| \in L^1(\Omega, \xi, \mu)$, $n = 0, 1, 2, \ldots$ Then the iterative scheme (2.3) is T-stable almost surely (or the iterative scheme (2.3) is stable with respect to T almost surely) if and only if

$$\lim_{n\to\infty} \int_{\Omega} \|\varepsilon_n(\omega)\| \, d\mu(\omega) = 0$$

implies that $x^*(\omega)$ is Bochner integrable with respect to $\{y_n(\omega)\}_{n=0}^{\infty}$.

Definition 2.3 ([37]) Let (Ω, Σ, μ) be a complete probability measure space and E be a nonempty subset of a separable Banach space X. A random operator $T : \Omega \times E \to E$ is called a ϕ-weakly contractive type random operator if there exists a continuous and nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(t) > 0$ for each $t \in (0,\infty)$ and $\phi(0) = 0$ such that, for each $x, \varsigma \in E$, $\omega \in \Omega$, we have

$$\int_\Omega \bigl\| T(\omega,x) - T(\omega,\varsigma)\bigr\| \,d\mu(\omega) \le \int_\Omega \|x-\varsigma\|\,d\mu(\omega) - \phi\biggl(\int_\Omega \|x-\varsigma\|\,d\mu(\omega)\biggr). \tag{2.4}$$

Definition 2.4 ([27]) Let (Ω,ξ,μ) be a complete probability measure space and E be a nonempty subset of a separable Banach space X. A random operator $T : \Omega \times E \to E$ is the generalized ϕ-weakly contractive type if there exist $L(\omega) \ge 0$ and a continuous and nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(t) > 0$ for each $t \in (0,\infty)$ and $\phi(0) = 0$ such that, for each $x, \varsigma \in E$, $\omega \in \Omega$,

$$\begin{aligned}&\int_\Omega \bigl\| T(\omega,x) - T(\omega,\varsigma)\bigr\| \,d\mu(\omega)\\ &\quad\le e^{L(\omega)\|x-\varsigma\|}\biggl[\int_\Omega \|x-\varsigma\|\,d\mu(\omega) - \phi\biggl(\int_\Omega \|x-\varsigma\|\,d\mu(\omega)\biggr)\biggr].\end{aligned} \tag{2.5}$$

Definition 2.5 ([22]) A random operator $T : \Omega \times C \to C$ is generalized ϕ-weakly contractive type if there exist $L(\omega) \ge 0$ and a continuous and nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(t) > 0$ for each $t \in (0,\infty)$, $\phi(0) = 0$ and for each $x, y \in C$, $\omega \in \Omega$, we have

$$\begin{aligned}&\int_\Omega \bigl\| T(\omega,x) - T(\omega,y)\bigr\| \,d\mu(\omega)\\ &\quad\le e^{L(\omega)\|x-T(\omega,x)\|}\biggl[\int_\Omega \|x-y\|\,d\mu(\omega) - \phi\biggl(\int_\Omega \|x-y\|\,d\mu(\omega)\biggr)\biggr].\end{aligned} \tag{2.6}$$

Motivated by the results above, we introduce the following generalized ϕ-weakly contraction of the rational type.

Definition 2.6 A random operator $T : \Omega \times C \to C$ is a generalized ϕ-weakly contraction of the rational type if there exist $L(\omega), M(\omega) \ge 0$ and a continuous and nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(t) > 0$ for each $t \in (0,\infty)$, $\phi(0) = 0$, and for each $x, y \in C$, $\omega \in \Omega$, we have

$$\begin{aligned}\int_\Omega \bigl\| T(\omega,x) - T(\omega,y)\bigr\| \,d\mu(\omega) &\le e^{L(\omega)\|x-T(\omega,x)\|}\biggl[\int_\Omega \frac{\|x-y\|}{1+M(\omega)\|x-T(\omega,x)\|}\,d\mu(\omega)\\ &\quad - \phi\biggl(\int_\Omega \frac{\|x-y\|}{1+M(\omega)\|x-T(\omega,x)\|}\,d\mu(\omega)\biggr)\biggr].\end{aligned} \tag{2.7}$$

Remark 2.1 Observe that if $M(\omega) = 0$, then relation (2.7) reduces to relation (2.6). Clearly, the generalized ϕ-weakly contraction of the rational type (2.7) includes (2.4) and (2.6) as special cases.

Let $T : \Omega \times C \to C$ be a random operator where C is a nonempty convex subset of X. Suppose that $u_0 : \Omega \to C$ and $p_0 : \Omega \to C$ are arbitrary measurable mappings, the random versions of the CR iterative scheme [13] and the Karahan–Ozdemir iterative scheme [21] are given as follows:

$$\begin{cases} u_0(\omega) \in C, \\ u_{n+1}(\omega) = (1-\alpha_n)v_n(\omega) + \alpha_n T(\omega, v_n(\omega)), \\ v_n(\omega) = (1-\beta_n)T(\omega, u_n(\omega)) + \beta_n T(\omega, y_n(\omega)), \\ y_n(\omega) = (1-\gamma_n)u_n(\omega) + \gamma_n T(\omega, u_n(\omega)), \quad n \in \mathbb{N}, \end{cases} \tag{2.8}$$

$$\begin{cases} p_0(\omega) \in C, \\ p_{n+1}(\omega) = (1-\alpha_n)T(\omega, p_n(\omega)) + \alpha_n T(\omega, q_n(\omega)), \\ q_n(\omega) = (1-\beta_n)T(\omega, p_n(\omega)) + \beta_n T(\omega, r_n(\omega)), \\ r_n(\omega) = (1-\gamma_n)p_n(\omega) + \gamma_n T(\omega, p_n(\omega)), \quad n \in \mathbb{N}. \end{cases} \tag{2.9}$$

Definition 2.7 ([11]) Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of positive numbers that converge to a, respectively b. Assume that there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}. \tag{2.10}$$

1. If $l = 0$, then it is said that the sequence $\{a_n\}_{n=0}^{\infty}$ converges to a faster than the sequence $\{b_n\}_{n=0}^{\infty}$ to b;
2. If $0 < l < \infty$, then we say that the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Lemma 2.1 ([5]) *Let $\{\zeta_n\}$ and $\{\lambda_n\}$ be two sequences of nonnegative real numbers. Let $\{\sigma_n\}$ be a sequence of positive numbers satisfying the conditions $\sum_{n=1}^{\infty} \sigma_n = \infty$ and $\lim_{n\to\infty} \frac{\zeta_n}{\sigma_n} = 0$. If the following condition is satisfied*:

$$\lambda_{n+1} \leq \lambda_n - \sigma_n \phi(\lambda_n) + \zeta_n, \quad \forall n \geq 1,$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and strictly increasing function with $\phi(0) = 0$, then $\{\lambda_n\}$ converges to 0 as $n \to \infty$.

3 Convergence theorems in separable Banach spaces

We begin this section by proving some convergence and existence of fixed point results in separable Banach spaces. Our results unify, generalize, and extend several known deterministic fixed point theorems in stochastic version.

Theorem 3.1 *Let C be a nonempty closed and convex subset of a separable Banach space X, $T : \Omega \times C \to C$ be a random generalized ϕ-weakly contraction of the rational type satisfying condition (2.7) with $\mathrm{RF}(T) \neq \emptyset$. Suppose that $x^*(\omega)$ is the random fixed point of T and $\{u_n(\omega)\}$ is the random CR-iteration process defined by (2.8), where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in $(0,1)$ such that $\sum_{n=1}^{\infty} \beta_n \gamma_n = \infty$. Then the random fixed point $x^*(\omega)$ of T is Bochner integrable.*

Proof To prove that $x^*(\omega)$ is Bochner integrable, it suffices to prove that

$$\lim_{n\to\infty} \int_{\Omega} \left\| u_n(\omega) - x^*(\omega) \right\| d\mu(\omega) = 0. \tag{3.1}$$

Using relations (2.7) and (2.8), we have

$$
\begin{aligned}
&\int_{\Omega} \| u_{n+1}(\omega) - x^*(\omega) \| \, d\mu(\omega) \\
&\leq (1-\alpha_n) \int_{\Omega} \| v_n(\omega) - x^*(\omega) \| \, d\mu(\omega) \\
&\quad + \alpha_n \int_{\Omega} \| T(\omega, v_n(\omega)) - x^*(\omega) \| \, d\mu(\omega) \\
&= (1-\alpha_n) \int_{\Omega} \| v_n(\omega) - x^*(\omega) \| \, d\mu(\omega) \\
&\quad + \alpha_n \int_{\Omega} \| T(\omega, x^*(\omega)) - T(\omega, v_n(\omega)) \| \, d\mu(\omega) \\
&\leq (1-\alpha_n) \int_{\Omega} \| v_n(\omega) - x^*(\omega) \| \, d\mu(\omega) \\
&\quad + \alpha_n e^{L(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \biggl[\int_{\Omega} \frac{\|x^*(\omega) - v_n(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \\
&\quad - \phi\biggl(\int_{\Omega} \frac{\|x^*(\omega) - v_n(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \biggr) \biggr] \\
&= (1-\alpha_n) \int_{\Omega} \| v_n(\omega) - x^*(\omega) \| \, d\mu(\omega) \\
&\quad + \alpha_n e^{L(\omega)\|0\|} \biggl[\int_{\Omega} \frac{\|v_n(\omega) - x^*(\omega)\|}{1 + M(\omega)\|0\|} \, d\mu(\omega) - \phi\biggl(\int_{\Omega} \frac{\|v_n(\omega) - x^*(\omega)\|}{1 + M(\omega)\|0\|} \, d\mu(\omega) \biggr) \biggr] \\
&= (1-\alpha_n) \int_{\Omega} \| v_n(\omega) - x^*(\omega) \| \, d\mu(\omega) \\
&\quad + \alpha_n \int_{\Omega} \| v_n(\omega) - x^*(\omega) \| \, d\mu(\omega) - \phi\biggl(\int_{\Omega} \| v_n(\omega) - x^*(\omega) \| \, d\mu(\omega) \biggr) \\
&\leq \int_{\Omega} \| v_n(\omega) - x^*(\omega) \| \, d\mu(\omega) \\
&\leq (1-\beta_n) \int_{\Omega} \| T(\omega, u_n(\omega)) - x^*(\omega) \| \, d\mu(\omega) \\
&\quad + \beta_n \int_{\Omega} \| T(\omega, y_n(\omega)) - x^*(\omega) \| \, d\mu(\omega) \\
&\leq (1-\beta_n) e^{L(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \biggl[\int_{\Omega} \frac{\|x^*(\omega) - u_n(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \\
&\quad - \phi\biggl(\int_{\Omega} \frac{\|x^*(\omega) - u_n(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \biggr) \biggr] \\
&\quad + \beta_n e^{L(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \biggl[\int_{\Omega} \frac{\|x^*(\omega) - y_n(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \\
&\quad - \phi\biggl(\int_{\Omega} \frac{\|x^*(\omega) - y_n(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \biggr) \biggr] \\
&= (1-\beta_n) e^{L(\omega)\|0\|} \biggl[\int_{\Omega} \frac{\|u_n(\omega) - x^*(\omega)\|}{1 + M(\omega)\|0\|} \, d\mu(\omega) - \phi\biggl(\int_{\Omega} \frac{\|u_n(\omega) - x^*(\omega)\|}{1 + M(\omega)\|0\|} \, d\mu(\omega) \biggr) \biggr] \\
&\quad + \beta_n e^{L(\omega)\|0\|} \biggl[\int_{\Omega} \frac{\|y_n(\omega) - x^*(\omega)\|}{1 + M(\omega)\|0\|} \, d\mu(\omega) - \phi\biggl(\int_{\Omega} \frac{\|y_n(\omega) - x^*(\omega)\|}{1 + M(\omega)\|0\|} \, d\mu(\omega) \biggr) \biggr]
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\beta_n)\biggl[\int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega) - \phi\biggl(\int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega)\biggr)\biggr] \\
&\quad + \beta_n\biggl[\int_\Omega \bigl\|y_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega) - \phi\biggl(\int_\Omega \bigl\|y_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega)\biggr)\biggr] \\
&\leq (1-\beta_n)\int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega) + \beta_n\int_\Omega \bigl\|y_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega).
\end{aligned} \tag{3.2}
$$

Next, we obtain the following estimate:

$$
\begin{aligned}
&\int_\Omega \bigl\|y_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega) \\
&\quad\leq (1-\gamma_n)\int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega) \\
&\qquad + \gamma_n\int_\Omega \bigl\|T\bigl(\omega,u_n(\omega)\bigr)-x^*(\omega)\bigr\|\,d\mu(\omega) \\
&\quad\leq (1-\gamma_n)\int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega) \\
&\qquad + \gamma_n e^{L(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|}\biggl[\int_\Omega \frac{\|x^*(\omega)-u_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|}\,d\mu(\omega) \\
&\qquad - \phi\biggl(\int_\Omega \frac{\|x^*(\omega)-u_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|}\,d\mu(\omega)\biggr)\biggr] \\
&\quad= (1-\gamma_n)\int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega) + \gamma_n e^{L(\omega)\|0\|}\biggl[\int_\Omega \frac{\|u_n(\omega)-x^*(\omega)\|}{1+M(\omega)\|0\|}\,d\mu(\omega) \\
&\qquad - \phi\biggl(\int_\Omega \frac{\|u_n(\omega)-x^*(\omega)\|}{1+M(\omega)\|0\|}\,d\mu(\omega)\biggr)\biggr] \\
&\quad= (1-\gamma_n)\int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega) + \gamma_n\int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega) \\
&\qquad - \gamma_n\phi\biggl(\int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega)\biggr) \\
&\quad= \int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega) - \gamma_n\phi\biggl(\int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega)\biggr).
\end{aligned} \tag{3.3}
$$

Using relation (3.3) in (3.2), we have

$$
\begin{aligned}
&\int_\Omega \bigl\|u_{n+1}(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega) \\
&\quad\leq (1-\beta_n)\int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega) \\
&\qquad + \beta_n\int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega) - \beta_n\gamma_n\phi\biggl(\int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega)\biggr) \\
&\quad= \int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega) - \beta_n\gamma_n\phi\biggl(\int_\Omega \bigl\|u_n(\omega)-x^*(\omega)\bigr\|\,d\mu(\omega)\biggr).
\end{aligned} \tag{3.4}
$$

If we take $\lambda_n = \int_\Omega \|u_n(\omega)-x^*(\omega)\|\,d\mu(\omega)$, $\sigma_n = \beta_n\gamma_n$, and $\zeta_n = 0$ in Lemma 2.1, by the assumptions of Theorem 3.1, we see that all the conditions of Lemma 2.1 are satisfied. Therefore, we have

$$\lim_{n\to\infty}\int_{\Omega}\left\|u_n(\omega)-x^*(\omega)\right\|d\mu(\omega)=0. \tag{3.5}$$

The proof of Theorem 3.1 is completed. □

Theorem 3.2 *Let C be a nonempty closed and convex subset of a separable Banach space X, $T:\Omega\times C\to C$ be a random generalized ϕ-weakly contraction of the rational type satisfying condition (2.7) with $\mathrm{RF}(T)\neq\emptyset$. Suppose $x^*(\omega)$ is the random fixed point of T and $\{p_n(\omega)\}$ is the random Karahan–Ozdemir iteration process defined by (2.9), where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in $(0,1)$ such that $\beta_n(1-\gamma_n)\leq\beta_n$, $\alpha_n\beta_n\gamma_n\leq\gamma_n$, and $\sum_{n=1}^{\infty}\beta_n\gamma_n=\infty$. Then the random fixed point $x^*(\omega)$ of T is Bochner integrable.*

Proof To prove that $x^*(\omega)$ is Bochner integrable, it suffices to prove that

$$\lim_{n\to\infty}\int_{\Omega}\left\|p_n(\omega)-x^*(\omega)\right\|d\mu(\omega)=0. \tag{3.6}$$

Using relations (2.7) and (2.9), we have

$$\begin{aligned}
&\int_{\Omega}\left\|p_{n+1}(\omega)-x^*(\omega)\right\|d\mu(\omega)\\
&\quad\leq(1-\alpha_n)\int_{\Omega}\left\|T\big(\omega,p_n(\omega)\big)-x^*(\omega)\right\|d\mu(\omega)\\
&\qquad+\alpha_n\int_{\Omega}\left\|T\big(\omega,q_n(\omega)\big)-x^*(\omega)\right\|d\mu(\omega)\\
&\quad=(1-\alpha_n)\int_{\Omega}\left\|T\big(\omega,x^*(\omega)\big)-T\big(\omega,p_n(\omega)\big)\right\|d\mu(\omega)\\
&\qquad+\alpha_n\int_{\Omega}\left\|T\big(\omega,x^*(\omega)\big)-T\big(\omega,q_n(\omega)\big)\right\|d\mu(\omega)\\
&\quad\leq(1-\alpha_n)e^{L(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|}\left[\int_{\Omega}\frac{\|x^*(\omega)-p_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|}\,d\mu(\omega)\right.\\
&\qquad\left.-\phi\left(\int_{\Omega}\frac{\|x^*(\omega)-p_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|}\,d\mu(\omega)\right)\right]\\
&\qquad+\alpha_n e^{L(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|}\left[\int_{\Omega}\frac{\|x^*(\omega)-q_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|}\,d\mu(\omega)\right.\\
&\qquad\left.-\phi\left(\int_{\Omega}\frac{\|x^*(\omega)-q_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|}\,d\mu(\omega)\right)\right]\\
&\quad\leq(1-\alpha_n)e^{L(\omega)\|0\|}\int_{\Omega}\frac{\|p_n(\omega)-x^*(\omega)\|}{1+M(\omega)\|0\|}\,d\mu(\omega)\\
&\qquad+\alpha_n e^{L(\omega)\|0\|}\int_{\Omega}\frac{\|q_n(\omega)-x^*(\omega)\|}{1+M(\omega)\|0\|}\,d\mu(\omega)\\
&\quad=(1-\alpha_n)\int_{\Omega}\left\|p_n(\omega)-x^*(\omega)\right\|d\mu(\omega)+\alpha_n\int_{\Omega}\left\|q_n(\omega)-x^*(\omega)\right\|d\mu(\omega).
\end{aligned} \tag{3.7}$$

Next, using the assumption that $\beta_n(1-\gamma_n) \leq \beta_n$, we have

$$\begin{aligned}
&\int_\Omega \|q_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\leq (1-\beta_n)\int_\Omega \|T(\omega, p_n(\omega)) - x^*(\omega)\| \, d\mu(\omega) \\
&\quad + \beta_n \int_\Omega \|T(\omega, r_n(\omega)) - x^*(\omega)\| \, d\mu(\omega) \\
&= (1-\beta_n)\int_\Omega \|T(\omega, x^*(\omega)) - T(\omega, p_n(\omega))\| \, d\mu(\omega) \\
&\quad + \beta_n \int_\Omega \|T(\omega, x^*(\omega)) - T(\omega, r_n(\omega))\| \, d\mu(\omega) \\
&\leq (1-\beta_n)e^{L(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|}\left[\int_\Omega \frac{\|x^*(\omega)-p_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \, d\mu(\omega)\right. \\
&\quad \left. - \phi\left(\int_\Omega \frac{\|x^*(\omega)-p_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \, d\mu(\omega)\right)\right] \\
&\quad + \beta_n e^{L(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|}\left[\int_\Omega \frac{\|x^*(\omega)-r_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \, d\mu(\omega)\right. \\
&\quad \left. - \phi\left(\int_\Omega \frac{\|x^*(\omega)-r_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \, d\mu(\omega)\right)\right] \\
&\leq (1-\beta_n)e^{L(\omega)\|0\|}\int_\Omega \frac{\|p_n(\omega)-x^*(\omega)\|}{1+M(\omega)\|0\|} \, d\mu(\omega) \\
&\quad + \beta_n e^{L(\omega)\|0\|}\int_\Omega \frac{\|r_n(\omega)-x^*(\omega)\|}{1+M(\omega)\|0\|} \, d\mu(\omega) \\
&= (1-\beta_n)\int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) + \beta_n \int_\Omega \|r_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\leq (1-\beta_n)\int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\quad + \beta_n\left[(1-\gamma_n)\int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega)\right. \\
&\quad \left. + \gamma_n \int_\Omega \|T(\omega, p_n(\omega)) - x^*(\omega)\| \, d\mu(\omega)\right] \\
&= (1-\beta_n)\int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\quad + \beta_n(1-\gamma_n)\int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\quad + \beta_n\gamma_n \int_\Omega \|T(\omega, x^*(\omega)) - T(\omega, p_n(\omega))\| \, d\mu(\omega) \\
&\leq (1-\beta_n)\int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\quad + \beta_n \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\quad + \beta_n\gamma_n \int_\Omega \|T(\omega, x^*(\omega)) - T(\omega, p_n(\omega))\| \, d\mu(\omega)
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) + \beta_n\gamma_n \int_\Omega \| T(\omega, x^*(\omega)) - T(\omega, p_n(\omega)) \| \, d\mu(\omega) \\
&\le \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\quad + \beta_n\gamma_n e^{L(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \biggl[\int_\Omega \frac{\|x^*(\omega) - p_n(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \\
&\quad - \phi\biggl(\int_\Omega \frac{\|x^*(\omega) - p_n(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \biggr) \biggr] \\
&= \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) + \beta_n\gamma_n \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&- \beta_n \phi\biggl(\int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \biggr).
\end{aligned} \tag{3.8}
$$

Using (3.8) in (3.7) and the assumption that $\alpha_n\beta_n\gamma_n \le \gamma_n$, we obtain

$$
\begin{aligned}
&\int_\Omega \|p_{n+1}(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\quad\le (1 - \alpha_n) \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\qquad + \alpha_n \biggl[\int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) + \beta_n\gamma_n \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\qquad - \beta_n \phi\biggl(\int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \biggr) \biggr] \\
&\quad= (1 - \alpha_n) \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\qquad + \alpha_n \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) + \alpha_n\beta_n\gamma_n \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\qquad - \alpha_n\beta_n \phi\biggl(\int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \biggr) \\
&\quad= \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) + \alpha_n\beta_n\gamma_n \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\qquad - \alpha_n\beta_n \phi\biggl(\int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \biggr) \\
&\quad\le \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) - \alpha_n\beta_n \phi\biggl(\int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \biggr) \\
&\qquad + \gamma_n \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega).
\end{aligned} \tag{3.9}
$$

If we take $\lambda_n = \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega)$, $\sigma_n = \alpha_n\beta_n$, and $\zeta_n = \gamma_n \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega)$ in Lemma 2.1, by the assumptions of Theorem 3.2, we see that all the conditions of Lemma 2.1 are satisfied. Therefore, we have

$$
\lim_{n\to\infty} \int_\Omega \|p_n(\omega) - x^*(\omega)\| \, d\mu(\omega) = 0. \tag{3.10}
$$

The proof of Theorem 3.2 is completed. □

Next, we obtain the following theorem which is a generalization of the results of Zhang *et al.* [37] among others.

Theorem 3.3 *Let C be a nonempty closed and convex subset of a separable Banach space X, $T : \Omega \times C \to C$ be a random generalized ϕ-weakly contraction of the rational type satisfying condition (2.7) with $\mathrm{RF}(T) \neq \emptyset$. Suppose that $x^*(\omega)$ is the random fixed point of T and $\{x_n(\omega)\}$ is the random Mann-iteration process defined by*

$$\begin{cases} x_0(\omega) \in C, \\ x_{n+1}(\omega) = (1-\alpha_n)x_n(\omega) + \alpha_n T(\omega, x_n(\omega)), \quad n \in \mathbb{N}, \end{cases} \tag{3.11}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then the random fixed point $x^*(\omega)$ of T is Bochner integrable.

Proof The proof of Theorem 3.3 follows similar lines as in the proof of Theorem 3.1. □

Remark 3.1 Theorem 3.1, Theorem 3.2, and Theorem 3.3 generalize several known results in the literature, including the results of Khan *et al.* [22], Okeke and Abbas [27], and Zhang *et al.* [37]. Moreover, our results extend and generalize several deterministic fixed point theorems in stochastic version, including the results of Chugh *et al.* [13] and Karahan and Ozdemir [21] among others.

Next, we prove the following existence results in separable Banach spaces.

Theorem 3.4 *Suppose that X is a separable Banach space and (Ω, Σ, μ) is a complete probability measure space. Let $T : \Omega \times X \to X$ be a continuous random operator such that*

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq \|x_1 - x_2\| - \phi(\|x_1 - x_2\|) \tag{3.12}$$

almost surely for all $x_1, x_2 \in X$, where ϕ is a continuous and nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(t) > 0$ for each $t \in (0,\infty)$ and $\phi(0) = 0$ almost surely. Then T has a unique random fixed point.

Proof Suppose

$$A = \{\omega \in \Omega : T(\omega, x) \text{ is a continuous function of } x\}, \tag{3.13}$$

$$C_{x_1,x_2} = \{\omega \in \Omega : \|T(\omega, x_1) - T(\omega, x_2)\| \leq \|x_1 - x_2\| - \phi(\|x_1 - x_2\|)\} \tag{3.14}$$

and

$$\begin{aligned} B = \{&\phi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ such that } \phi \text{ is continuous and nondecreasing, with} \\ &\phi(t) > 0, \forall t \in (0,\infty) \text{ and } \phi(0) = 0\}. \end{aligned} \tag{3.15}$$

Suppose that H is a countable dense subset of X. Then we show that

$$\bigcap_{x_1,x_2 \in X} (C_{x_1,x_2} \cap A \cap B) = \bigcap_{h_1,h_2 \in H} (C_{h_1,h_2} \cap A \cap B). \tag{3.16}$$

Now, we show that

$$\bigcap_{h_1,h_2\in H} C_{h_1,h_2} \cap A \cap B \subset \bigcap_{x_1,x_2\in X} C_{x_1,x_2} \cap A \cap B. \tag{3.17}$$

Let $\omega \in \bigcap_{h_1,h_2\in H}(C_{h_1,h_2} \cap A \cap B)$, then for each $h_1, h_2 \in H$, we have

$$\bigl\| T(\omega,h_1) - T(\omega,h_2)\bigr\| \le \|h_1 - h_2\| - \phi\bigl(\|h_1 - h_2\|\bigr). \tag{3.18}$$

Suppose $x_1, x_2 \in X$, we obtain

$$\begin{aligned}
&\bigl\| T(\omega,x_1) - T(\omega,x_2)\bigr\| \\
&\quad\le \bigl\| T(\omega,x_1) - T(\omega,h_1)\bigr\| + \bigl\| T(\omega,h_1) - T(\omega,h_2)\bigr\| + \bigl\| T(\omega,h_2) - T(\omega,x_2)\bigr\| \\
&\quad\le \bigl\| T(\omega,x_1) - T(\omega,h_1)\bigr\| + \bigl\| T(\omega,h_2) - T(\omega,x_2)\bigr\| + \|h_1 - h_2\| - \phi\bigl(\|h_1 - h_2\|\bigr) \\
&\quad\le \bigl\| T(\omega,x_1) - T(\omega,h_1)\bigr\| + \bigl\| T(\omega,h_2) - T(\omega,x_2)\bigr\| + \|h_1 - x_1\| \\
&\qquad + \|x_1 - x_2\| + \|x_2 - h_2\| - \phi\bigl(\|x_1 - x_2\|\bigr).
\end{aligned} \tag{3.19}$$

Since for each $\omega \in \Omega$, $T(\omega,x)$ is a continuous function of x, this means that for arbitrary $\epsilon > 0$, there exists $\delta_i(x_i) > 0$ $(i = 1,2)$ such that $\|T(\omega,x_1) - T(\omega,h_1)\| < \frac{\epsilon}{2}$ whenever $\|x_1 - h_1\| < \delta_1(x_1)$ and $\|T(\omega,h_2) - T(\omega,x_2)\| < \frac{\epsilon}{2}$ whenever $\|h_2 - x_2\| < \delta_2(x_2)$.

Now choose $\delta_1 = \min\{\delta_1(x_1), \frac{\epsilon}{2}\}$ and $\delta_2 = \min\{\delta_2(x_2), \frac{\epsilon}{2}\}$. Using the choice of δ_1, δ_2, we see that (3.19) becomes

$$\begin{aligned}
\bigl\| T(\omega,x_1) - T(\omega,x_2)\bigr\| &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} + \frac{\epsilon}{2} + \|x_1 - x_2\| + \frac{\epsilon}{2} - \phi\bigl(\|x_1 - x_2\|\bigr) \\
&\le \epsilon + \|x_1 - x_2\| - \phi\bigl(\|x_1 - x_2\|\bigr).
\end{aligned} \tag{3.20}$$

Since $\epsilon > 0$ is arbitrary, it follows from (3.20) that

$$\bigl\| T(\omega,x_1) - T(\omega,x_2)\bigr\| \le \|x_1 - x_2\| - \phi\bigl(\|x_1 - x_2\|\bigr). \tag{3.21}$$

This means that $\omega \in \bigcap_{x_1,x_2\in X} C_{x_1,x_2} \cap A \cap B$, this implies that

$$\bigcap_{h_1,h_2\in H} C_{h_1,h_2} \cap A \cap B \subset \bigcap_{x_1,x_2\in X} C_{x_1,x_2} \cap A \cap B. \tag{3.22}$$

Similarly, we can easily show that

$$\bigcap_{x_1,x_2\in X} C_{x_1,x_2} \cap A \cap B \subset \bigcap_{h_1,h_2\in H} C_{h_1,h_2} \cap A \cap B. \tag{3.23}$$

Hence, by (3.22) and (3.23), we have

$$\bigcap_{x_1,x_2\in X} C_{x_1,x_2} \cap A \cap B = \bigcap_{h_1,h_2\in H} C_{h_1,h_2} \cap A \cap B. \tag{3.24}$$

Suppose

$$M' = \bigcap_{h_1,h_2\in H} C_{h_1,h_2} \cap A \cap B, \tag{3.25}$$

then $\mu(M') = 1$.

Therefore, for all $\omega \in M'$, $T(\omega,x)$ is a deterministic operator satisfying relation (3.12). Hence T has a unique random fixed point in X. The proof of Theorem 3.4 is completed. □

Example 3.1 Suppose that $\Omega = [0,1]$ and Σ is the sigma algebra of the Lebesgue measurable subsets of Ω. Let $X = \mathbb{R}$, $C = [0,2]$ and define the generalized ϕ-weakly contraction of the rational type operator $T : \Omega \times C \to C$ as $T(\omega,x) = \frac{\omega-x}{6}$. Then the measurable mapping $x^* : \Omega \to X$ defined by $x^*(\omega) = \frac{\omega}{7}$ for every $\omega \in \Omega$ is a random fixed point of T. Let $\phi(t) = \frac{t}{3}$ for each $t \in (0,\infty)$, $L(\omega) = 7$, and $M(\omega) = 5$, then we have

$$\begin{aligned}
&\int_\Omega \bigl\| T(\omega,x) - T(\omega,y)\bigr\| \, d\mu(\omega) \\
&\quad = \biggl[\int_\Omega \frac{\|x-y\|}{1+5\|x-T(\omega,x)\|}\, d\mu(\omega) - \phi\biggl(\int_\Omega \frac{\|x-y\|}{1+5\|x-T(\omega,x)\|}\, d\mu(\omega)\biggr)\biggr] \\
&\quad \le e^{7\|x-T(\omega,x)\|}\biggl[\int_\Omega \frac{\|x-y\|}{1+5\|x-T(\omega,x)\|}\, d\mu(\omega) \\
&\qquad - \frac{1}{3}\int_\Omega \frac{\|x-y\|}{1+5\|x-T(\omega,x)\|}\, d\mu(\omega)\biggr].
\end{aligned} \tag{3.26}$$

Clearly, T satisfies condition (2.7). Choose the prototype sequences $\alpha_n = \beta_n = \frac{n^2}{1+n^2}$, $\gamma_n = \frac{n^3}{1+n^3}$. Then $\sum_{n=1}^{\infty} \beta_n\gamma_n = \sum_{n=1}^{\infty} \frac{n^5}{(1+n^2)(1+n^3)} = \infty$. Hence, all the conditions of Theorem 3.1 are satisfied; therefore, the random fixed point $x^*(\omega) = \frac{\omega}{7}$ of $T(\omega,x)$ is Bochner integrable.

4 Stability theorems in separable Banach spaces

In this section, we establish some stability results in separable Banach spaces. Our results unify, generalize, and extend several known deterministic fixed point theorems in stochastic version. Furthermore, we give a numerical example to demonstrate the applicability of our analytical results.

Theorem 4.1 *Let C be a nonempty closed and convex subset of a separable Banach space X, $T : \Omega \times C \to C$ be a random generalized ϕ-weakly contraction of the rational type satisfying condition* (2.7) *with* $\mathrm{RF}(T) \neq \emptyset$. *Suppose that $x^*(\omega)$ is the random fixed point of T and $\{u_n(\omega)\}$ is the random CR-iteration process defined by* (2.8), *where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in $(0,1)$ such that $0 < \alpha \le \alpha_n$, $0 < \beta \le \beta_n$, and $0 < \gamma \le \gamma_n$. Then $\{u_n(\omega)\}$ is T-stable almost surely.*

Proof Suppose that $\{h_n(\omega)\}$ is an arbitrary sequence of random variables in E and

$$\bigl\|\varepsilon_n(\omega)\bigr\| = \bigl\|h_{n+1}(\omega) - (1-\alpha_n)m_n(\omega) + \alpha_n T\bigl(\omega, m_n(\omega)\bigr)\bigr\|, \tag{4.1}$$

where

$$\begin{cases} m_n(\omega) = (1-\beta_n)T(\omega, h_n(\omega)) + \beta_n T(\omega, s_n(\omega)), \\ s_n(\omega) = (1-\gamma_n)h_n(\omega) + \gamma_n T(\omega, h_n(\omega)) \end{cases} \tag{4.2}$$

and $\lim_{n\to\infty}\int_\Omega \|\varepsilon_n(\omega)\|\,d\mu(\omega) = 0$.

Next, we prove that $x^*(\omega)$ is Bochner integrable with respect to the sequence $\{h_n(\omega)\}$. It follows from (4.1) that

$$\begin{aligned}
&\int_\Omega \|h_{n+1}(\omega) - x^*(\omega)\|\,d\mu(\omega) \\
&\quad\le \int_\Omega \big\|h_{n+1}(\omega) - (1-\alpha_n)m_n(\omega) - \alpha_n T\big(\omega, m_n(\omega)\big)\big\|\,d\mu(\omega) \\
&\qquad + (1-\alpha_n)\int_\Omega \|m_n(\omega) - x^*(\omega)\|\,d\mu(\omega) \\
&\qquad + \alpha_n \int_\Omega \big\|T\big(\omega, m_n(\omega)\big) - x^*(\omega)\big\|\,d\mu(\omega) \\
&\quad= \int_\Omega \|\varepsilon_n(\omega)\|\,d\mu(\omega) + (1-\alpha_n)\int_\Omega \|m_n(\omega) - x^*(\omega)\|\,d\mu(\omega) \\
&\qquad + \alpha_n \int_\Omega \big\|T\big(\omega, x^*(\omega)\big) - T\big(\omega, m_n(\omega)\big)\big\|\,d\mu(\omega) \\
&\quad\le \int_\Omega \|\varepsilon_n(\omega)\|\,d\mu(\omega) + (1-\alpha_n)\int_\Omega \|m_n(\omega) - x^*(\omega)\|\,d\mu(\omega) \\
&\qquad + \alpha_n e^{L(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|}\biggl[\int_\Omega \frac{\|x^*(\omega)-m_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|}\,d\mu(\omega) \\
&\qquad - \phi\biggl(\int_\Omega \frac{\|x^*(\omega)-m_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|}\,d\mu(\omega)\biggr)\biggr] \\
&\quad\le \int_\Omega \|\varepsilon_n(\omega)\|\,d\mu(\omega) + (1-\alpha_n)\int_\Omega \|m_n(\omega) - x^*(\omega)\|\,d\mu(\omega) \\
&\qquad + \alpha_n e^{L(\omega)\|0\|}\int_\Omega \frac{\|m_n(\omega)-x^*(\omega)\|}{1+M(\omega)\|0\|}\,d\mu(\omega) \\
&\quad= \int_\Omega \|\varepsilon_n(\omega)\|\,d\mu(\omega) + (1-\alpha_n)\int_\Omega \|m_n(\omega) - x^*(\omega)\|\,d\mu(\omega) \\
&\qquad + \alpha_n \int_\Omega \|m_n(\omega) - x^*(\omega)\|\,d\mu(\omega) \\
&\quad= \int_\Omega \|\varepsilon_n(\omega)\|\,d\mu(\omega) + \int_\Omega \|m_n(\omega) - x^*(\omega)\|\,d\mu(\omega) \\
&\quad\le \int_\Omega \|\varepsilon_n(\omega)\|\,d\mu(\omega) + (1-\beta_n)\int_\Omega \big\|T\big(\omega, h_n(\omega)\big) - x^*(\omega)\big\|\,d\mu(\omega) \\
&\qquad + \beta_n \int_\Omega \big\|T\big(\omega, s_n(\omega)\big) - x^*(\omega)\big\|\,d\mu(\omega) \\
&\quad= \int_\Omega \|\varepsilon_n(\omega)\|\,d\mu(\omega) + (1-\beta_n)\int_\Omega \big\|T\big(\omega, x^*(\omega)\big) - T\big(\omega, h_n(\omega)\big)\big\|\,d\mu(\omega) \\
&\qquad + \beta_n \int_\Omega \big\|T\big(\omega, x^*(\omega)\big) - T\big(\omega, s_n(\omega)\big)\big\|\,d\mu(\omega)
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\Omega} \|\varepsilon_n(\omega)\| \, d\mu(\omega) \\
&\quad + (1-\beta_n) e^{L(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \biggl[\int_{\Omega} \frac{\|x^*(\omega)-h_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \, d\mu(\omega) \\
&\quad - \phi\biggl(\int_{\Omega} \frac{\|x^*(\omega)-h_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \, d\mu(\omega) \biggr) \biggr] \\
&\quad + \beta_n e^{L(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \biggl[\int_{\Omega} \frac{\|x^*(\omega)-s_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \, d\mu(\omega) \\
&\quad - \phi\biggl(\int_{\Omega} \frac{\|x^*(\omega)-s_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \, d\mu(\omega) \biggr) \biggr] \\
&\leq \int_{\Omega} \|\varepsilon_n(\omega)\| \, d\mu(\omega) + (1-\beta_n) e^{L(\omega)\|0\|} \int_{\Omega} \frac{\|h_n(\omega)-x^*(\omega)\|}{1+M(\omega)\|0\|} \, d\mu(\omega) \\
&\quad + \beta_n e^{L(\omega)\|0\|} \int_{\Omega} \frac{\|s_n(\omega)-x^*(\omega)\|}{1+M(\omega)\|0\|} \, d\mu(\omega) \\
&= \int_{\Omega} \|\varepsilon_n(\omega)\| \, d\mu(\omega) + (1-\beta_n) \int_{\Omega} \|h_n(\omega)-x^*(\omega)\| \, d\mu(\omega) \\
&\quad + \beta_n \int_{\Omega} \|s_n(\omega)-x^*(\omega)\| \, d\mu(\omega).
\end{aligned}
\tag{4.3}
$$

Next, we have the following estimate:

$$
\begin{aligned}
&\int_{\Omega} \|s_n(\omega)-x^*(\omega)\| \, d\mu(\omega) \\
&\quad \leq (1-\gamma_n) \int_{\Omega} \|h_n(\omega)-x^*(\omega)\| \, d\mu(\omega) + \gamma_n \int_{\Omega} \|T(\omega,h_n(\omega))-x^*(\omega)\| \, d\mu(\omega) \\
&\quad = (1-\gamma_n) \int_{\Omega} \|h_n(\omega)-x^*(\omega)\| \, d\mu(\omega) \\
&\qquad + \gamma_n \int_{\Omega} \|T(\omega,x^*(\omega))-T(\omega,h_n(\omega))\| \, d\mu(\omega) \\
&\quad \leq (1-\gamma_n) \int_{\Omega} \|h_n(\omega)-x^*(\omega)\| \, d\mu(\omega) \\
&\qquad + \gamma_n e^{L(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \biggl[\int_{\Omega} \frac{\|x^*(\omega)-h_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \, d\mu(\omega) \\
&\qquad - \phi\biggl(\int_{\Omega} \frac{\|x^*(\omega)-h_n(\omega)\|}{1+M(\omega)\|x^*(\omega)-T(\omega,x^*(\omega))\|} \, d\mu(\omega) \biggr) \biggr] \\
&\quad = (1-\gamma_n) \int_{\Omega} \|h_n(\omega)-x^*(\omega)\| \, d\mu(\omega) + \gamma_n e^{L(\omega)\|0\|} \biggl[\int_{\Omega} \frac{\|h_n(\omega)-x^*(\omega)\|}{1+M(\omega)\|0\|} \, d\mu(\omega) \\
&\qquad - \phi\biggl(\int_{\Omega} \frac{\|h_n(\omega)-x^*(\omega)\|}{1+M(\omega)\|0\|} \, d\mu(\omega) \biggr) \biggr] \\
&\quad = (1-\gamma_n) \int_{\Omega} \|h_n(\omega)-x^*(\omega)\| \, d\mu(\omega) + \gamma_n \int_{\Omega} \|h_n(\omega)-x^*(\omega)\| \, d\mu(\omega) \\
&\qquad - \gamma_n \phi\biggl(\int_{\Omega} \|h_n(\omega)-x^*(\omega)\| \, d\mu(\omega) \biggr) \\
&\quad = \int_{\Omega} \|h_n(\omega)-x^*(\omega)\| \, d\mu(\omega) - \gamma_n \phi\biggl(\int_{\Omega} \|h_n(\omega)-x^*(\omega)\| \, d\mu(\omega) \biggr).
\end{aligned}
\tag{4.4}
$$

Substituting (4.4) in (4.3), we have

$$\begin{aligned}
&\int_\Omega \|h_{n+1}(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\quad\leq \int_\Omega \|\varepsilon_n(\omega)\| \, d\mu(\omega) + (1-\beta_n) \int_\Omega \|h_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\qquad + \beta_n \biggl[\int_\Omega \|h_n(\omega) - x^*(\omega)\| \, d\mu(\omega) - \gamma_n \phi \biggl(\int_\Omega \|h_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \biggr) \biggr] \\
&\quad= \int_\Omega \|\varepsilon_n(\omega)\| \, d\mu(\omega) + (1-\beta_n) \int_\Omega \|h_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\qquad + \beta_n \int_\Omega \|h_n(\omega) - x^*(\omega)\| \, d\mu(\omega) - \beta_n \gamma_n \phi \biggl(\int_\Omega \|h_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \biggr) \\
&\quad= \int_\Omega \|\varepsilon_n(\omega)\| \, d\mu(\omega) + \int_\Omega \|h_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\qquad - \beta_n \gamma_n \phi \biggl(\int_\Omega \|h_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \biggr).
\end{aligned} \tag{4.5}$$

Using the assumptions that $\lim_{n\to\infty} \int_\Omega \|\varepsilon_n(\omega)\| \, d\mu(\omega) = 0$, $0 < \alpha \leq \alpha_n$, $0 < \beta \leq \beta_n$, and $0 < \gamma \leq \gamma_n$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \frac{\int_\Omega \|\varepsilon_n(\omega)\| \, d\mu(\omega)}{\beta_n \gamma_n} \leq \lim_{n\to\infty} \frac{\int_\Omega \|\varepsilon_n(\omega)\| \, d\mu(\omega)}{\beta\gamma} = 0. \tag{4.6}$$

If in Lemma 2.1 we take $\lambda_n = \int_\Omega \|h_n(\omega) - x^*(\omega)\| \, d\mu(\omega)$, $\sigma_n = \beta_n \gamma_n$, and $\zeta_n = \int_\Omega \|\varepsilon_n(\omega)\| \, d\mu(\omega)$, we see that all the conditions of Lemma 2.1 are satisfied. Therefore, we have

$$\lim_{n\to\infty} \int_\Omega \|h_n(\omega) - x^*(\omega)\| \, d\mu(\omega) = 0. \tag{4.7}$$

Conversely, suppose that $x^*(\omega)$ is Bochner integrable with respect to the sequence $\{h_n(\omega)\}$, then we have

$$\begin{aligned}
&\int_\Omega \|\varepsilon_n(\omega)\| \, d\mu(\omega) \\
&\quad= \int_\Omega \|h_{n+1}(\omega) - (1-\alpha_n) m_n(\omega) - \alpha_n T(\omega, m_n(\omega))\| \, d\mu(\omega) \\
&\quad\leq \int_\Omega \|h_{n+1}(\omega) - x^*(\omega)\| \, d\mu(\omega) + (1-\alpha_n) \int_\Omega \|x^*(\omega) - m_n(\omega)\| \, d\mu(\omega) \\
&\qquad + \alpha_n \int_\Omega \|x^*(\omega) - T(\omega, m_n(\omega))\| \, d\mu(\omega) \\
&\quad\leq \int_\Omega \|h_{n+1}(\omega) - x^*(\omega)\| \, d\mu(\omega) + (1-\alpha_n) \int_\Omega \|m_n(\omega) - x^*(\omega)\| \, d\mu(\omega) \\
&\qquad + \alpha_n e^{L(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \biggl[\int_\Omega \frac{\|x^*(\omega) - m_n(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \\
&\qquad - \phi \biggl(\int_\Omega \frac{\|x^*(\omega) - m_n(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \biggr) \biggr]
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \big\| h_{n+1}(\omega) - x^*(\omega) \big\| \, d\mu(\omega) + (1-\alpha_n) \int_\Omega \big\| m_n(\omega) - x^*(\omega) \big\| \, d\mu(\omega) \\
&\quad + \alpha_n e^{L(\omega)\|0\|} \biggl[\int_\Omega \frac{\|m_n(\omega) - x^*(\omega)\|}{1 + M(\omega)\|0\|} \, d\mu(\omega) - \phi\biggl(\int_\Omega \frac{\|m_n(\omega) - x^*(\omega)\|}{1 + M(\omega)\|0\|} \, d\mu(\omega) \biggr) \biggr] \\
&\le \int_\Omega \big\| h_{n+1}(\omega) - x^*(\omega) \big\| \, d\mu(\omega) + (1-\alpha_n) \int_\Omega \big\| m_n(\omega) - x^*(\omega) \big\| \, d\mu(\omega) \\
&\quad + \alpha_n \int_\Omega \big\| m_n(\omega) - x^*(\omega) \big\| \, d\mu(\omega) \\
&= \int_\Omega \big\| h_{n+1}(\omega) - x^*(\omega) \big\| \, d\mu(\omega) + \int_\Omega \big\| m_n(\omega) - x^*(\omega) \big\| \, d\mu(\omega) \\
&\le \int_\Omega \big\| h_{n+1}(\omega) - x^*(\omega) \big\| \, d\mu(\omega) + (1-\beta_n) \int_\Omega \big\| T\big(\omega, h_n(\omega)\big) - x^*(\omega) \big\| \, d\mu(\omega) \\
&\quad + \beta_n \int_\Omega \big\| T\big(\omega, s_n(\omega)\big) - x^*(\omega) \big\| \, d\mu(\omega) \\
&\le \int_\Omega \big\| h_{n+1}(\omega) - x^*(\omega) \big\| \, d\mu(\omega) \\
&\quad + (1-\beta_n) e^{L(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \biggl[\int_\Omega \frac{\|x^*(\omega) - h_n(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \\
&\quad - \phi\biggl(\int_\Omega \frac{\|x^*(\omega) - h_n(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \biggr) \biggr] \\
&\quad + \beta_n e^{L(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \biggl[\int_\Omega \frac{\|x^*(\omega) - s_n(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \\
&\quad - \phi\biggl(\int_\Omega \frac{\|x^*(\omega) - s_n(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \biggr) \biggr] \\
&\le \int_\Omega \big\| h_{n+1}(\omega) - x^*(\omega) \big\| \, d\mu(\omega) + (1-\beta_n) \int_\Omega \big\| h_n(\omega) - x^*(\omega) \big\| \, d\mu(\omega) \\
&\quad + \beta_n \int_\Omega \big\| s_n(\omega) - x^*(\omega) \big\| \, d\mu(\omega) \\
&\le \int_\Omega \big\| h_{n+1}(\omega) - x^*(\omega) \big\| \, d\mu(\omega) + (1-\beta_n) \int_\Omega \big\| h_n(\omega) - x^*(\omega) \big\| \, d\mu(\omega) \\
&\quad + \beta_n \biggl[(1-\gamma_n) \int_\Omega \big\| h_n(\omega) - x^*(\omega) \big\| \, d\mu(\omega) + \gamma_n \int_\Omega \big\| T\big(\omega, h_n(\omega)\big) - x^*(\omega) \big\| \, d\mu(\omega) \biggr] \\
&\le \int_\Omega \big\| h_{n+1}(\omega) - x^*(\omega) \big\| \, d\mu(\omega) + (1-\beta_n) \int_\Omega \big\| h_n(\omega) - x^*(\omega) \big\| \, d\mu(\omega) \\
&\quad + \beta_n \int_\Omega \big\| h_n(\omega) - x^*(\omega) \big\| \, d\mu(\omega) \\
&\quad + \beta_n \gamma_n e^{L(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \biggl[\int_\Omega \frac{\|h_n(\omega) - x^*(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \\
&\quad - \phi\biggl(\int_\Omega \frac{\|h_n(\omega) - x^*(\omega)\|}{1 + M(\omega)\|x^*(\omega) - T(\omega, x^*(\omega))\|} \, d\mu(\omega) \biggr) \biggr] \\
&= \int_\Omega \big\| h_{n+1}(\omega) - x^*(\omega) \big\| \, d\mu(\omega) + \int_\Omega \big\| h_n(\omega) - x^*(\omega) \big\| \, d\mu(\omega) \\
&\quad + \beta_n \gamma_n \int_\Omega \big\| h_n(\omega) - x^*(\omega) \big\| \, d\mu(\omega) - \beta_n \gamma_n \phi\biggl(\int_\Omega \big\| h_n(\omega) - x^*(\omega) \big\| \, d\mu(\omega) \biggr). \qquad (4.8)
\end{aligned}
$$

Therefore, we obtain

$$\lim_{n\to\infty}\int_{\Omega}\big\|\varepsilon_n(\omega)\big\|\,d\mu(\omega)=0. \tag{4.9}$$

This means that the random CR-iteration process $\{u_n(\omega)\}$ defined by (2.8) is T-stable almost surely. The proof of Theorem 4.1 is completed. □

Theorem 4.2 *Let C be a nonempty closed and convex subset of a separable Banach space X, $T:\Omega\times C\to C$ be a random generalized ϕ-weakly contraction of the rational type satisfying condition (2.7) with $\mathrm{RF}(T)\neq\emptyset$. Suppose that $x^*(\omega)$ is the random fixed point of T and $\{p_n(\omega)\}$ is the random Karahan–Ozdemir iteration process defined by (2.9), where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in $(0,1)$ such that $0<\alpha\le\alpha_n$, $0<\beta\le\beta_n$, and $0<\gamma\le\gamma_n$. Then $\{p_n(\omega)\}$ is T-stable almost surely.*

Proof The proof of Theorem 4.2 follows similar lines as in the proof of Theorem 4.1. □

Remark 4.1 Theorem 4.1 and Theorem 4.2 generalize several known results in the literature, including the results of Khan *et al.* [22], Okeke and Abbas [27], and Zhang *et al.* [37]. Moreover, our results extend and generalize several deterministic fixed point theorems in stochastic version, including the results of Chugh *et al.* [13] and Karahan and Ozdemir [21] among others.

Next, we give the following numerical example to demonstrate the applicability of our results.

Example 4.1 Suppose that $\Omega=[0,1]$ and Σ is the sigma algebra of the Lebesgue measurable subsets of Ω. Let $X=\mathbb{R}$, $C=[0,2]$ and define the generalized ϕ-weakly contraction of the rational type operator $T:\Omega\times C\to C$ as $T(\omega,x)=\frac{\omega-x}{6}$. Then the measurable mapping $x^*:\Omega\to X$ defined by $x^*(\omega)=\frac{\omega}{7}$ for every $\omega\in\Omega$ is a random fixed point of T. Let $\phi(t)=\frac{t}{3}$ for each $t\in(0,\infty)$, $L(\omega)=7$, and $M(\omega)=5$, then we have

$$\begin{aligned}
&\int_{\Omega}\big\|T(\omega,x)-T(\omega,y)\big\|\,d\mu(\omega)\\
&\quad=\biggl[\int_{\Omega}\frac{\|x-y\|}{1+5\|x-T(\omega,x)\|}\,d\mu(\omega)-\phi\biggl(\int_{\Omega}\frac{\|x-y\|}{1+5\|x-T(\omega,x)\|}\,d\mu(\omega)\biggr)\biggr]\\
&\quad\le e^{7\|x-T(\omega,x)\|}\biggl[\int_{\Omega}\frac{\|x-y\|}{1+5\|x-T(\omega,x)\|}\,d\mu(\omega)\\
&\qquad-\frac{1}{3}\int_{\Omega}\frac{\|x-y\|}{1+5\|x-T(\omega,x)\|}\,d\mu(\omega)\biggr].
\end{aligned} \tag{4.10}$$

Clearly, T satisfies condition (2.7). Suppose that $\{u_n(\omega)\}$ is the random CR-iteration process defined by (2.8), where $\alpha_n=\beta_n=\frac{n^2}{1+n^2}$, $\gamma_n=\frac{n^3}{1+n^3}$. Then $0<\alpha\le\alpha_n$, $0<\beta\le\beta_n$, and $0<\gamma\le\gamma_n$. Let $\{y_n(\omega)\}=\frac{\omega}{n^2}$ be an arbitrary sequence of measurable mapping in C. Hence, all the conditions of Theorem 4.1 are satisfied. Therefore, $\{u_n(\omega)\}$ is T-stable almost surely.

5 Application to random nonlinear integral equation of the Hammerstein type

In this section, we shall apply Theorem 3.4 to prove the existence of a solution in a Banach space of a random nonlinear integral equation of the form

$$x(t;\omega) = h(t;\omega) + \int_S k(t,s;\omega) f\bigl(s, x(s;\omega)\bigr)\, d\mu_0(s), \tag{5.1}$$

where

(i) S is a locally compact metric space with a metric d on $S \times S$ equipped with a complete σ-finite measure μ_0 defined on the collection of Borel subsets of S;
(ii) $\omega \in \Omega$, where ω is a supporting element of a set of probability measure space (Ω, β, μ);
(iii) $x(t;\omega)$ is the unknown vector-valued random variable for each $t \in S$;
(iv) $h(t;\omega)$ is the stochastic free term defined for $t \in S$;
(v) $k(t,s;\omega)$ is the stochastic kernel defined for t and s in S and
(vi) $f(t,x)$ is a vector-valued function of $t \in S$ and x.

The integral Eq. (5.1) is interpreted as a Bochner integral (see Padgett [31]). Furthermore, we shall assume that S is the union of a countable family of compact sets $\{C_n\}$ having the properties that $C_1 \subset C_2 \subset \cdots$ and that for any other compact set S there is C_i which contains it (see Arens [6]).

Definition 5.1 ([14]) We define the space $C(S, L_2(\Omega, \beta, \mu))$ to be the space of all continuous functions from S into $L_2(\Omega, \beta, \mu)$ with the topology of uniform convergence on compacta, i.e., for each fixed $t \in S$, $x(t;\omega)$ is a vector-valued random variable such that

$$\bigl\| x(t;\omega) \bigr\|^2_{L_2(\Omega,\beta,\mu)} = \int_\Omega \bigl| x(t;\omega) \bigr|^2 \, d\mu(\omega) < \infty.$$

Note that $C(S, L_2(\Omega, \beta, \mu))$ is a locally convex space, whose topology is defined by a countable family of semi-norms (see Yosida [36]) given by

$$\bigl\| x(t;\omega) \bigr\|_n = \sup_{t \in C_n} \bigl\| x(t;\omega) \bigr\|_{L_2(\Omega,\beta,\mu)}, \quad n = 1, 2, \ldots.$$

Moreover, $C(S, L_2(\Omega, \beta, \mu))$ is complete relative to this topology, since $L_2(\Omega, \beta, \mu)$ is complete.

We define $\mathrm{BC} = \mathrm{BC}(S, L_2(\Omega, \beta, \mu))$ to be the Banach space of all bounded continuous functions from S into $L_2(\Omega, \beta, \mu)$ with the norm

$$\bigl\| x(t;\omega) \bigr\|_{\mathrm{BC}} = \sup_{t \in S} \bigl\| x(t;\omega) \bigr\|_{L_2(\Omega,\beta,\mu)}.$$

The space $\mathrm{BC} \subset C$ is the space of all second order vector-valued stochastic process defined on S, which is bounded and continuous in mean square. We will consider the function $h(t;\omega)$ and $f(t, x(t;\omega))$ to be in the space $C(S, L_2(\Omega, \beta, \mu))$ with respect to the stochastic kernel. We assume that for each pair (t,s), $k(t,s;\omega) \in L_\infty(\Omega, \beta, \mu)$ and denote the norm by

$$\|k(t,s;\omega)\| = \|k(t,s;\omega)\|_{L_\infty(\Omega,\beta,\mu)} = \mu - \text{ess}\sup_{\omega\in\Omega}|k(t,s;\omega)|.$$

Suppose that $k(t,s;\omega)$ is such that $|\|k(t,s;\omega)\||.\|x(s;\omega)\|_{L_2(\Omega,\beta,\mu)}$ is μ_0-integrable with respect to s for each $t \in S$ and $x(s;\omega)$ in $C(S, L_2(\Omega,\beta,\mu))$, and there exists a real-valued function G defined μ_0-a.e. on S, so that $G(S)\|x(s;\omega)\|_{L_2(\Omega,\beta,\mu)}$ is μ_0-integrable and for each pair $(t,s) \in S \times S$,

$$\big|\|k(t,u;\omega) - k(s,u;\omega)\|\big|.\|x(u,\omega)\|_{L_2(\Omega,\beta,\mu)} \le G(u)\|x(u,\omega)\|_{L_2(\Omega,\beta,\mu)}$$

μ_0-a.e. Furthermore, for almost all $s \in S$, $k(t,s;\omega)$ will be continuous in t from S into $L_\infty(\Omega,\beta,\mu)$.

Now, we define the random integral operator T on $C(S, L_2(\Omega,\beta,\mu))$ by

$$(Tx)(t;\omega) = \int_S k(t,s;\omega)x(s;\omega)\,d\mu_0(s), \tag{5.2}$$

where the integral is a Bochner integral. Moreover, we have that for each $t \in S$, $(Tx)(t;\omega) \in L_2(\Omega,\beta,\mu)$ and that $(Tx)(t;\omega)$ is continuous in mean square by Lebesgue dominated convergence theorem. So $(Tx)(t;\omega) \in C(S, L_2(\Omega,\beta,\mu))$.

Definition 5.2 ([1, 24]) Let B and D be two Banach spaces. The pair (B,D) is said to be admissible with respect to a random operator $T(\omega)$ if $T(\omega)(B) \subset D$.

Lemma 5.1 ([20]) *The linear operator T defined by (5.2) is continuous from $C(S, L_2(\Omega,\beta,\mu))$ into itself.*

Lemma 5.2 ([20, 24]) *If T is a continuous linear operator from $C(S, L_2(\Omega,\beta,\mu))$ into itself and $B,D \subset C(S, L_2(\Omega,\beta,\mu))$ are Banach spaces stronger than $C(S, L_2(\Omega,\beta,\mu))$ such that (B,D) is admissible with respect to T, then T is continuous from B into D.*

Remark 5.1 ([31]) The operator T defined by (5.2) is a bounded linear operator from B into D. It is to be noted that a random solution of Eq. (5.1) will mean a function $x(t;\omega)$ in $C(S, L_2(\Omega,\beta,\mu))$ which satisfies Eq. (5.1) μ-a.e.

We now prove the following theorem.

Theorem 5.1 *We consider the stochastic integral Eq. (5.1) subject to the following conditions:*

(a) *B and D are Banach spaces stronger than $C(S, L_2(\Omega,\beta,\mu))$ such that (B,D) is admissible with respect to the integral operator defined by (5.2);*

(b) *$x(t;\omega) \to f(t, x(t;\omega))$ is an operator from the set*

$$Q(\rho) = \{x(t;\omega) : x(t;\omega) \in D, \|x(t;\omega)\|_D \le \rho\}$$

into the space B satisfying

$$\begin{aligned}&\|f(t,x_1(t;\omega))-f(t,x_2(t;\omega))\|_B\\&\quad\le\|x_1(t;\omega)-x_2(t;\omega)\|_D-\phi(\|x_1(t;\omega)-x_2(t;\omega)\|_D)\end{aligned}\tag{5.3}$$

for all $x_1(t;\omega),x_2(t;\omega)\in Q(\rho)$, *where* $\phi:\mathbb{R}^+\to\mathbb{R}^+$ *is a continuous and nondecreasing function such that* $\phi(t)>0$ *for each* $t\in(0,\infty)$ *and* $\phi(0)=0$.

(c) $h(t;\omega)\in D$.

Then there exists a unique random solution of (5.1) *in* $Q(\rho)$, *provided*

$$\|h(t;\omega)\|_D+c(\omega)\|f(t;0)\|_B\le\rho(1-c(\omega)),\tag{5.4}$$

where $c(\omega)$ *is the norm of* $T(\omega)$.

Proof We define the operator $U(\omega)$ from $Q(\rho)$ into D as follows:

$$(Ux)(t;\omega)=h(t;\omega)+\int_S k(t,s;\omega)f(s,x(s;\omega))\,d\mu_0(s).\tag{5.5}$$

Next we have

$$\begin{aligned}\|(Ux)(t;\omega)\|_D&\le\|h(t;\omega)\|_D+c(\omega)\|f(t,x(t;\omega))\|_B\\&\le\|h(t;\omega)\|_D+c(\omega)\|f(t;0)\|_B+c(\omega)\|f(t,x(t;\omega))-f(t;0)\|_B.\end{aligned}\tag{5.6}$$

Using the condition of (5.3), we have

$$\begin{aligned}\|f(t,x(t;\omega))-f(t;0)\|_B&\le\|x(t;\omega)\|_D-\phi(\|x(t;\omega)\|_D)\\&\le\rho-\phi(\rho)\\&\le\rho.\end{aligned}\tag{5.7}$$

Using (5.7) in (5.6), we have

$$\begin{aligned}\|(Ux)(t;\omega)\|_D&\le\|h(t;\omega)\|_D+c(\omega)\|f(t;0)\|_B+c(\omega)\rho\\&\le\rho.\end{aligned}\tag{5.8}$$

This means that $(Ux)(t;\omega)\in Q(\rho)$. Then, for each $x_1(t;\omega),x_2(t;\omega)\in Q(\rho)$, we have by using assumption (b) that

$$\begin{aligned}&\|(Ux_1)(t;\omega)-(Ux_2)(t;\omega)\|_D\\&\quad=\left\|\int_S k(t,s;\omega)[f(s,x_1(s;\omega))-f(s,x_2(s;\omega))]\,d\mu_0(s)\right\|_D\\&\quad\le\|x_1(t;\omega)-x_2(t;\omega)\|_D-\phi(\|x_1(t;\omega)-x_2(t;\omega)\|_D).\end{aligned}\tag{5.9}$$

Since $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous and nondecreasing function such that $\phi(t)>0$ for each $t\in(0,\infty)$ and $\phi(0)=0$, it follows that $U(\omega)$ is a nonlinear contractive operator on

$Q(\rho)$. Therefore, by Theorem 3.4 there exists a unique random fixed point $x^*(t,\omega)$ of $U(\omega)$, which is the random solution of Eq. (5.1). The proof of Theorem 5.1 is completed. □

The following example demonstrates the applicability of Theorem 5.1.

Example 5.1 We consider the following nonlinear stochastic integral equation:

$$x(t;\omega) = \int_0^\infty \frac{e^{-t-s}}{16(1+|x(s;\omega)|)}\,ds - \frac{1}{2}\int_0^\infty \frac{e^{-t-s}}{16(1+|x(s;\omega)|)}\,ds. \tag{5.10}$$

By comparing relation (5.10) with (5.1), we observe that

$$h(t;\omega) = 0, \qquad k(t,s;\omega) = \frac{1}{4}e^{-t-s}, \qquad f\bigl(s,x(s;\omega)\bigr) = \frac{1}{4(1+|x(s;\omega)|)}. \tag{5.11}$$

By the usual computation, we clearly see that (5.3) is satisfied with $\phi(j) = \frac{j}{2}$ and $\phi(0) = \frac{0}{2} = 0$ for all $j \in (0,\infty)$. Hence, all the conditions of Theorem 5.1 are satisfied. Therefore, there exists a unique random fixed point $x^*(t,\omega)$ of the integral operator T satisfying (5.2).

6 Conclusion

In this research, we defined a new random operator called the generalized ϕ-weakly contraction of the rational type. This new random operator includes those studied by Khan *et al.* [22] and Zhang *et al.* [37] as special cases. We also introduced the random versions of some known faster fixed point iterative schemes (see [13, 21]). We proved some convergence and stability results for our newly introduced random operator via these random fixed point iterations. An existence result was also established for a generalized random operator. Moreover, we produced some numerical examples to demonstrate the applicability of our analytical results. Furthermore, we applied our results in proving the existence of a solution of a nonlinear integral equation of the Hammerstein type. Our results generalize several known results in the literature, including the results of Khan *et al.* [22] and Zhang *et al.* [37]. Moreover, our results unify, extend, and generalize several deterministic fixed point theorems in stochastic version, including the results of Akewe *et al.* [4], Akewe and Okeke [3], Chugh *et al.* [13], and Karahan and Ozdemir [21] among others.

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References
1. Achari, J.: On a pair of random generalized non-linear contractions. Int. J. Math. Math. Sci. **6**(3), 467–475 (1983)
2. Agarwal, R.P., O'Regan, D., Sahu, D.R.: Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. J. Nonlinear Convex Anal. **8**, 61–79 (2007)
3. Akewe, H., Okeke, G.A.: Convergence and stability theorems for the Picard–Mann hybrid iterative scheme for a general class of contractive-like operators. Fixed Point Theory Appl. **2015**, 66 (2015) 8 pages
4. Akewe, H., Okeke, G.A., Olayiwola, A.F.: Strong convergence and stability of Kirk-multistep-type iterative schemes for contractive-type operators. Fixed Point Theory Appl. **2014**, 45 (2014) 24 pages
5. Alber, Y.I., Guerre-Delabriere, S.: Principle of weakly contractive maps in Hilbert spaces. In: Gohberg, I., Lyubich, Y. (eds.) New Results in Operator Theory and Its Applications, pp. 7–22. Birkhäuser, Basel (1997)
6. Arens, R.F.: A topology for spaces of transformations. Ann. Math. **47**(2), 480–495 (1946)
7. Beg, I., Abbas, M.: Equivalence and stability of random fixed point iterative procedures. J. Appl. Math. Stoch. Anal. **2006**, Article ID 23297 (2006). https://doi.org/10.1155/JAMSA/2006/23297
8. Beg, I., Abbas, M.: Random fixed point theorems for Caristi type random operators. J. Appl. Math. Comput. **25**(1–2), 425–434 (2007)
9. Beg, I., Abbas, M., Azam, A.: Periodic fixed points of random operators. Ann. Math. Inform. **37**, 39–49 (2010)
10. Beg, I., Aleomraninejad, S.M.A.: Random fixed points of multifunctions on metric spaces. J. Nonlinear Funct. Anal. **2017**, Article ID 37 (2017)
11. Berinde, V.: Iterative Approximation of Fixed Points. Lecture Notes in Mathematics. Springer, Berlin (2007)
12. Bharucha-Reid, A.T.: Fixed point theorems in probabilistic analysis. Bull. Am. Math. Soc. **82**, 641–657 (1976)
13. Chugh, R., Kumar, V., Kumar, S.: Strong convergence of a new three step iterative scheme in Banach spaces. Am. J. Comput. Math. **2**, 345–357 (2012)
14. Dey, D., Saha, M.: Application of random fixed point theorems in solving nonlinear stochastic integral equation of the Hammerstein type. Malaya J. Mat. **2**(1), 54–59 (2013)
15. Getmanova, E., Obukhovskii, V.: A note on random equilibrium points of two multivalued maps. J. Nonlinear Var. Anal. **2**(3), 269–272 (2018)
16. Graef, J.R., Henderson, J., Ouahab, A.: Some Krasnosel'skii type random fixed point theorems. J. Nonlinear Funct. Anal. **2017**, Article ID 46 (2017)
17. Hans, O.: Random operator equations. In: Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Vol. II, Part I, pp. 185–202. University of California Press, California (1961)
18. Ishikawa, S.: Fixed points by a new iteration method. Proc. Am. Math. Soc. **44**, 147–150 (1974)
19. Itoh, S.: Random fixed point theorems with an application to random differential equations in Banach spaces. J. Math. Anal. Appl. **67**(2), 261–273 (1979)
20. Joshi, M.C., Bose, R.K.: Some Topics in Nonlinear Functional Analysis. Wiley Eastern, New Delhi (1985)
21. Karahan, I., Ozdemir, M.: A general iterative method for approximation of fixed points and their applications. Adv. Fixed Point Theory **3**(3), 510–526 (2013)
22. Khan, A.R., Kumar, V., Narwal, S., Chugh, R.: Random iterative algorithms and almost sure stability in Banach spaces. Filomat **31**(12), 3611–3626 (2017)
23. Kongban, C., Kumam, P., Martinez-Moreno, J., Lopez de Hierro, A.F.R.: On random fixed point for generalizations of Suzuki's type with application to stochastic dynamic programming. Res. Fixed Point Theory Appl. **2019**, Article ID 2018026 (2019)
24. Lee, A.C.H., Padgett, W.J.: On random nonlinear contraction. In: Mathematical Systems Theory ii, pp. 77–84 (1977)
25. Mann, W.R.: Mean value methods in iteration. In: Proceedings of the American Mathematical Society, vol. 4, pp. 506–510 (1953)
26. Noor, M.A.: New approximation schemes for general variational inequalities. J. Math. Anal. Appl. **251**(1), 217–229 (2000)
27. Okeke, G.A., Abbas, M.: Convergence and almost sure T-stability for a random iterative sequence generated by a generalized random operator. J. Inequal. Appl. **2015**, 146 (2015) 11 pages
28. Okeke, G.A., Eke, K.S.: Convergence and almost sure T-stability for random Noor-type iterative scheme. Int. J. Pure Appl. Math. **107**(1), 1–16 (2016)
29. Okeke, G.A., Kim, J.K.: Convergence and summable almost T-stability of the random Picard-Mann hybrid iterative process. J. Inequal. Appl. **2015**, 290 (2015) 14 pages
30. Okeke, G.A., Kim, J.K.: Convergence and (S,T)-stability almost surely for random Jungck-type iteration processes with applications. Cogent Math. **3**, 1258768 (2016) 15 pages
31. Padgett, W.J.: On a nonlinear stochastic integral equation of the Hammerstein type. Proc. Am. Math. Soc. **38**(3), 625–631 (1973)
32. Pheungrattana, W., Suantai, S.: On the rate of convergence of Mann, Ishikawa, Noor and SP iterations for continuous on an arbitrary interval. J. Comput. Appl. Math. **235**(9), 3006–3914 (2011)
33. Rashwan, R.A., Hammad, H.A.: Random fixed point theorems for random mappings. Pac.-Asian J. Math. **3**(2), 114–135 (2016)

34. Rashwan, R.A., Hammad, H.A.: A solution of random nonlinear integral equation via random hybrid iterative scheme. J. Fixed Point Theory **2018**, 4 (2018) 16 pages
35. Spacek, A.: Zufallige gleichungen. Czechoslov. Math. J. **5**, 462–466 (1955)
36. Yosida, K.: Functional Analysis. Academic Press, New York (1965)
37. Zhang, S.S., Wang, X.R., Liu, M.: Almost sure T-stability and convergence for random iterative algorithms. Appl. Math. Mech. **32**(6), 805–810 (2011)