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New algorithms for approximating zeros of inverse strongly monotone maps and J-fixed points

Charles E. Chidume[1*] and Chinedu G. Ezea[1,2]

*Correspondence:
cchidume@aust.edu.ng
[1] African University of Science and Technology, Abuja, Nigeria
Full list of author information is available at the end of the article

Abstract

Let E be a real Banach space with dual space E^*. A new class of *relatively weak J-nonexpansive maps*, $T: E \to E^*$, is introduced and studied. An algorithm to approximate a common element of J-fixed points for a countable family of relatively weak J-nonexpansive maps and zeros of a countable family of inverse strongly monotone maps in a 2-uniformly convex and uniformly smooth real Banach space is constructed. Furthermore, assuming existence, the sequence of the algorithm is proved to converge strongly. Finally, a numerical example is given to illustrate the convergence of the sequence generated by the algorithm.

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1 Introduction

Let E be a real Banach space with dual space E^*. A mapping $J: E \to 2^{E^*}$ defined by $J(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|, \forall x \in E\}$, is called the *normalized duality map* on E, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the elements of E and E^*. A mapping $A: E \to E$ is said to be *accretive* if for each $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that the following inequality holds: $\langle Ax - Ay, j(x-y) \rangle \geq 0$. A mapping $A: E \to E^*$ is said to be *monotone* if for each $x, y \in E$, the following inequality holds: $\langle x - y, Ax - Ay \rangle \geq 0$. In a real Hilbert space, accretive operators are called *monotone*.

It is known that many physically significant problems can be modeled in the form of the following evolution equation: $\frac{du}{dt} + Au = 0$, where $A: E \to E$ is an *accretive-type map*. This equation describes any system that generates energy over time. Observe that at equilibrium, u is independent of time so that the equation reduces to

$$Au = 0, \tag{1.1}$$

whose solutions correspond to the equilibrium state of the system described by equation $\frac{du}{dt} + Au = 0$.

Approximation of solutions of Eq. (1.1) has been studied extensively by various authors (see, e.g., Aoyama *et al.* [4], Blum and Oettli [6], Censor, Gibali, Reich and Sabach [12], Censor, Gibali and Reich [9–11], Chidume [14], Chidume *et al.* [15, 16, 18, 23, 25, 26], Gibali, Reich and Zalas [27], Iiduka and Takahashi [29], Iiduka *et al.* [31], Kassay, Reich and Sabach [33], Kinderlehrer and Stampacchia [34], Lions and Stampacchia [36], Liu [37], Liu and Nashed [38], Ofoedu and Malonza [43], Osilike *et al.* [44], Reich and Sabach [48], Reich [46], Rockafellar [49], Su and Xu [51], Zegeye *et al.* [58], Zegeye and Shahzad [57], and the references therein).

For approximating a solution of Eq. (1.1) in a real Hilbert space H, where $A : H \to H$ is *monotone*, assuming existence, Browder [7] introduced an operator $T := I - A$, where I is the identity map on H. He called such an operator a *pseudocontraction*. It is trivial to observe that zeros of A correspond to fixed points of T. Interest in *pseudocontractive-type map* stems mainly from this firm connection with the *accretive-type maps*. Hence, approximating fixed points of pseudocontractive maps has become a flourishing area of interest to researchers in nonlinear operator theory (see, e.g., the monographs of Alber [1], Berinde [5], Chidume [14], Goebel and Reich [28], and the references therein).

Let a function $f : E \to \mathbb{R} \cup \{\infty\}$ be convex and proper. The *subdifferential* of f, $\partial f : E \to 2^{E^*}$, is defined for each $x \in E$ by

$$\partial f(x) := \left\{x^* \in E^* : f(y) - f(x) \geq \langle y - x, x^* \rangle, \forall y \in E\right\}.$$

It is known that ∂f is a monotone map on E, and $0 \in \partial f(v)$ if and only if v is a minimizer of f. In general, if $A := \partial f : E \to 2^{E^*}$ is a *monotone-type map* defined on an appropriate real normed space E, *solutions of equation*

$$0 \in Au, \tag{1.2}$$

in this case *where A is of monotone type, correspond to minimizers of some convex functional defined on E*. This is one of the motivations for studying the equation $Au = 0$, where $A : E \to E^*$ is monotone.

Clearly, Browder's fixed point technique for the equation $Au = 0$, where $A : E \to E$ *is of accretive-type*, is not applicable to the equation $Au = 0$, where $A : E \to E^*$ is monotone, for an arbitrary real normed space E more general than Hilbert spaces.

As has been rightly observed by Hazewinkle, a Series Editor of Kluwer Academic Publishers,

> "…many, and probably most, mathematical objects and models do not naturally live in Hilbert space."

They live generally in real Banach spaces more general than Hilbert spaces.

Let E be a smooth real normed space, and let $T : E \to E^*$ be a map. If $J : E \to E^*$ is the normalized duality map, then $x \in E$ is called a *J-fixed point* of T if $Tx = Jx$ (see, e.g., Chidume *et al.* [23–25] and Chidume and Idu [20]). This concept was introduced by Zegeye [55] in 2008, who called such a fixed point *a semi-fixed point*. The concept was later, in 2012, called *duality fixed point* by Cheng *et al.* [13], Liu [37], and Su and Xu [51]. The notion of J-fixed point has been found to have numerous applications, and provides for monotone maps $T : E \to E^*$ the analog of Browder's pseudocontractive maps for accretive maps, $T: E \to E$. For more on J-fixed points, the reader may consult any of the following references: Chidume and Monday [21, 22].

A map $A: E \to E^*$ is called *inverse strongly monotone* if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2, \quad \text{for all } x, y \in E.$$

Chidume *et al.* [23] called a map $T: E \to E^*$ *strictly J-pseudocontractive* if for every $x, y \in E$, there exists $\alpha > 0$ such that the following inequality holds:

$$\langle Tx - Ty, x - y\rangle \leq \langle Jx - Jy, x - y\rangle - \alpha \left\|(Jx - Tx) - (Jy - Ty)\right\|^2. \tag{1.3}$$

Remark 1 Liu [37] called this concept α-strongly duality.

By setting $A := J - T: E \to E^*$, where E is a real normed space with dual space E^* and T is a strictly J-pseudocontractive map, we list some properties of A and T (Chidume *et al.* [23]):

(i) $x \in E$ is a J-fixed point of T if and only if x is a zero of A,

(ii) A is *inverse strongly monotone* if and only if T is strictly J-pseudocontractive.

We recall the following definitions.

Definition 1.1 Let C be a nonempty closed and convex subset of E; let $T: C \to E$ be a map. A point $x^* \in C$ is called a *fixed point* of T if $Tx^* = x^*$. The set of fixed points of T is denoted by $F(T)$. We say that $(I - T)$ is *demiclosed at zero* whenever a sequence $\{x_n\}$ in C converges *weakly* to x and $\{x_n - Tx_n\}$ converges *strongly* to 0, then $x \in F(T)$. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges *weakly* to p and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

Definition 1.2 (Chidume and Idu [20]) Let E be a smooth real normed space with dual space E^*. Let $T: E \to E^*$ be any map. A point $x \in E$ is called a *J-fixed point* of T if $Tx = Jx$.

The set of J-fixed point of a map T is denoted by $F_J(T) = \{x \in E : Tx = Jx\}$. We now give some examples of J-fixed points.

Example 1 Let H be a real Hilbert space; let $T: H \to H$ be any map with $F(T) := \{x \in H : Tx = x\} \neq \emptyset$. Then, $F(T) = F_J(T)$.

Example 2 (Chidume and Idu [20]) It is known that in l_p spaces, $1 < p < \infty$,

$$Jx = \|x\|_{l_p}^{2-p} \bigl(|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, \ldots\bigr),$$

for any $x = (x_1, x_2, \ldots) \in l_p$ (see, e.g., Alber [3], p. 36). For $1 < q < p$, we set $\gamma_p := (1 + \frac{1}{2^p})^{\frac{2-p}{p}}$ and define $T: l_p \to l_q\ (\subset l_p)$ by

$$T(x_1, x_2, x_3, \ldots) = \left(\gamma_p x_1, \frac{\gamma_p}{2^{p-1}} x_2, 0, 0, \ldots\right).$$

For any $\lambda \in \mathbb{R}$, let $x_\lambda = (\lambda, \frac{\lambda}{2}, 0, 0, \ldots)$. Then, $x_\lambda \in F_J(T)$.

Let E be a smooth real Banach space with dual space E^*. A function $\phi : E \times E \to \mathbb{R}$, defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x,y \in E, \tag{1.4}$$

where J is the normalized duality mapping from E into E^* will play a central role in what follows. It was introduced by Alber and has been studied by Alber and Ryazantseva [3], Alber and Guerre-Delabriere [2], Chidume *et al.* [16, 17, 23], Chidume and Idu [20], Kamimura and Takahashi [32], Reich [47], Takahashi and Zembayashi [53, 54], Zegeye [55], and a host of other authors.

If $E = H$, a real Hilbert space, Eq. (1.4) reduces to $\phi(x,y) = \|x-y\|^2$, $\forall x,y \in H$.

Definition 1.3 A map $T : C \to E$ is said to be *relatively nonexpansive* if the following conditions hold (see, e.g., Matsushita and Takahashi [40] and Reich [45]):

(1) $F(T) \neq \emptyset$,

(2) $\phi(p, Tx) \leq \phi(p,x)$, $\forall x \in C$ and $p \in F(T)$,

(3) $\widehat{F}(T) = F(T)$.

Definition 1.4 A point $p \in C$ is said to be a *strong asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges *strongly* to p and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ (see, e.g., Reich [45]). The set of *strongly* asymptotic fixed points of T is denoted by $\widetilde{F}(T)$.

Definition 1.5 A map $T : C \to E$ is said to be *relatively weak nonexpansive* if the following conditions hold (see, e.g., Liu [39] and Zegeye and Shahzad [56]):

(1) $F(T) \neq \emptyset$,

(2) $\phi(p, Tx) \leq \phi(p,x)$, $\forall x \in C$ and $p \in F(T)$,

(3) $\widetilde{F}(T) = F(T)$.

If E is a strictly convex and reflexive real Banach space and $A : E \to E^*$ is a continuous monotone map with $A^{-1}(0) \neq \varnothing$, it is known that $J_r := (J + rA)^{-1}J$, for $r > 0$, is relatively weak nonexpansive (see, e.g., Kohasaka [35]). Clearly, every relatively nonexpansive map is relatively weak nonexpansive. Let $T : C \to E$ be a map; we have that $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. It follows that $F(T) = \widetilde{F}(T) = \widehat{F}(T)$ for any relatively nonexpansive map.

An example of a relatively weak nonexpansive map which is not a relatively nonexpansive map is given in Zhang *et al.* [59]. In the following definitions, we assume that the space E is reflexive, strictly convex, and smooth real Banach space with dual space E^*.

Definition 1.6 (Chidume *et al.* [15]) Let $T : E \to E^*$ be a map. A point $x^* \in E$ is called an *asymptotic J-fixed point of* T if there exists a sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup x^*$ and $\|Jx_n - Tx_n\| \to 0$ as $n \to \infty$. We shall denote the set of asymptotic J-fixed points of T by $\widehat{F}_J(T)$.

Definition 1.7 (Chidume *et al.* [15]) A map $T : E \to E^*$ is said to be *relatively J-nonexpansive* if

(i) $\widehat{F}_J(T) = F_J(T) \neq \emptyset$,

(ii) $\phi(p, J^{-1}Tx) \leq \phi(p,x)$, $\forall x \in E$, $p \in F_J(T)$.

In 2008, Iiduka and Takahashi [30] introduced an iterative algorithm for finding a zero of an inverse strongly monotone map A in a 2-uniformly convex and uniformly smooth real Banach space. They proved a strong convergence theorem to some element of $A^{-1}(0)$.

For appropriating zeros of inverse strongly monotone maps and fixed points of *relatively weak nonexpansive maps*, Zegeye and Shahzad [56] in 2009 introduced a generalized projection algorithm and proved that the sequence generated by their algorithm converges strongly to a common element of the sets of zeros for inverse strongly monotone maps and fixed points of relatively weak nonexpansive maps

For finding an element in the set of solutions of zeros for an inverse strongly monotone map and fixed points for a *countable family of relatively weak nonexpansive maps* in a 2-uniformly convex and uniformly smooth real Banach space, Chidume *et al.* [19] proved the following theorem.

Theorem 1.8 *Let E be a 2-uniformly convex and uniformly smooth real Banach space with dual space E^*. Let $A : E \to E^*$ be an α-inverse strongly monotone map, and let $T_i : E \to E$, $i = 1, 2, \dots$, be a countable family of relatively weak nonexpansive maps. Assume that $W := \bigcap_{i=1}^{\infty} F(T_i) \cap A^{-1}0 \neq \emptyset$, where $A^{-1}0 = \{u \in E : Au = 0\} \neq \emptyset$. For arbitrary $x_1 \in E$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by*

$$\begin{cases} x_1 \in E := C_1, \\ u_n = J^{-1}(Jx_n - \lambda A x_n), \\ y_n = J^{-1}(\sum_{i=1}^{\infty} \alpha_i J T_i u_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $J : E \to E^*$ is the normalized duality map, $\lambda \in (0, \frac{\alpha}{2L})$, $L > 0$ denotes the Lipschitz constant of J^{-1}, and $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence in $(0, 1)$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Then, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W := \bigcap_{i=1}^{\infty} F(T_i) \cap A^{-1}0$.

Remark 2 This theorem is a significant improvement on the result of Zegeye and Shahzad [56]. We observe that the *relatively weak nonexpansive maps* studied in Zegeye and Shahzad [56] and Chidume *et al.* [19] are maps *from a real normed space to itself.*

It is our purpose in this paper to introduce a new class of maps called *relatively weak J-nonexpansive* maps *from a real normed space E to its dual space E^**. We first give some properties of this class of maps, and we then construct an algorithm to approximate a common element of J-fixed points for a countable family of relatively weak J-nonexpansive maps and zeros of a countable family of inverse strongly monotone maps in a 2-uniformly convex and uniformly smooth real Banach space. We prove a strong convergence theorem. Finally, we give a numerical example to illustrate the convergence of the sequence generated by the algorithm.

2 Preliminaries

It is obvious from the definition of the function ϕ that

$$\big(\|x\| - \|y\|\big)^2 \leq \phi(x, y) \leq \big(\|x\| + \|y\|\big)^2, \quad \forall x, y \in E. \tag{2.1}$$

Remark 3 It is known that if E is a reflexive, strictly convex and smooth real Banach space, then, for all $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$ (see, e.g., Zhou *et al.* [60]).

Define a map $V : E \times E^* \to \mathbb{R}$ by $V(x,x^*) = \|x\|^2 - 2\langle x,x^*\rangle + \|x^*\|^2$. Then, it is easy to see that $V(x,x^*) = \phi(x,J^{-1}(x^*))$, $\forall x \in E$, $x^* \in E^*$. Let C be a nonempty closed and convex subset of a smooth, strictly convex, and reflexive real Banach space E. The *generalized projection* map introduced by Alber [1], is a map $\Pi_C : E \to C$ such that, for any $x \in E$, there corresponds a unique element $x_0 := \Pi_C(x) \in C$ such that $\phi(x_0,x) = \min_{y\in C} \phi(y,x)$. We note that the existence and uniqueness of the generalized projection map Π_C follows from the strict monotonicity and properties of the Lyapunov functional ϕ (see, e.g., Takahashi [52], Alber [1], and Alber and Ryazantseva [3]). If E is a real Hilbert space, we remark that the generalized projection Π_C coincides with the metric projection from E onto C.

The following lemmas are needed in the sequel.

Lemma 2.1 (Matsushita and Takahashi [41]) *Let E be a strictly convex and smooth Banach space; let C be a closed convex subset of E, and let T be a relatively nonexpansive mapping from C into itself. Then, $F(T)$ is a closed convex subset of C.*

Lemma 2.2 (Schu [50]) *Let E be a uniformly convex Banach space and C be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of C into itself. Then, $(I - T)$ is demiclosed at zero.*

Lemma 2.3 (Chidume *et al.* [19]) *Let C be a closed convex subset of a uniformly convex and uniformly smooth real Banach space E, and let $T_i : C \to E$, $i = 1,2,\ldots$, be a countable family of relatively weak nonexpansive maps. Assume that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Let the map $T : C \to E$ be defined by $Tx = J^{-1}(\sum_{i=1}^{\infty} \alpha_i JT_i x)$ for each $x \in C$. Then, T is relatively weak nonexpansive and $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Lemma 2.4 (Bruck [8]) *Suppose E is strictly convex and $\{T_n\}$ is a sequence of nonexpansive mappings $T_n : C \to E$. Then there exists a nonexpansive mapping $T : C \to E$ such that $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Remark 4 It was proved in Bruck [8] that the map $T : C \to E$ defined by $Tx = \sum_{i=1}^{\infty} \alpha_i T_i x$, for each $x \in C$, where $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$, is nonexpansive.

3 Main results

Definition 3.1 Let E be a reflexive, strictly convex, and smooth real Banach space with dual space E^*. Let $T : E \to E^*$ be a map. A point $p \in E$ will be called a *strong asymptotic J-fixed point* of T if E contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges *strongly* to p and $\lim_{n\to\infty} \|Tx_n - Jx_n\| = 0$. The set of *strongly* asymptotic J-fixed points of T will be denoted by $\widetilde{F}_J(T)$.

Definition 3.2 Let E be a reflexive, strictly convex, and smooth real Banach space with dual space E^*. A map $T : E \to E^*$ will be called *relatively weak J-nonexpansive* if the following conditions hold:

(1) $F_J(T) \neq \emptyset$,

(2) $\phi(p,J^{-1}Tx) \leq \phi(p,x)$, $\forall x \in E$ and $p \in F_J(T)$,

(3) $\widetilde{F}_J(T) = F_J(T)$.

An example of a relatively *weak* J-nonexpansive map that is *not* relatively J-nonexpansive (see Zhang *et al.* [59]). In addition, we provide the following example.

Example 3 Let $\ell_p(\mathbb{R})$ be the sequence space, for $1 < p < \infty$. Let the sequence $\{x_n\}_{n=1}^{\infty} \in \ell_p(\mathbb{R})$ be defined by $x_0 = (1,0,0,0,\ldots)$, $x_1 = (1,1,0,0,0,\ldots)$, $x_2 = (1,0,1,0,0,0,\ldots)$, $x_3 = (1,0,0,1,0,0,0,\ldots)$, $\ldots$, $x_n = (\zeta_{n,1},\zeta_{n,2},\zeta_{n,3},\ldots,\zeta_{n,k+1},0,\ldots)$, $\ldots$, where

$$\zeta_{n,k} = \begin{cases} 1, & \text{if } k = 1, n+1; \\ 0, & \text{if } k \neq 1, k \neq n+1. \end{cases}$$

Let $T : \ell_p(\mathbb{R}) \to \ell_q(\mathbb{R})$ be a map defined as follows:

$$T(x) = \begin{cases} \frac{n}{n+1} Jx_n, & \text{if } x = x_n, \\ -Jx, & \text{if } x \neq x_n, \end{cases} \tag{3.1}$$

where $J : \ell_p(\mathbb{R}) \to \ell_q(\mathbb{R})$ is the single valued normalized duality map and $\frac{1}{p} + \frac{1}{q} = 1$. It is easy to see that the sequence $\{x_n\}_{n=1}^{\infty}$ converges weakly to x_0.

Let $f = (\alpha_1,\alpha_2,\ldots,\alpha_{n+1},\ldots) \in \ell_q(\mathbb{R})$, for $1 < q < \infty$; we have that $f(x_n - x_0) = \langle f, x_n - x_0\rangle = \alpha_{n+1} \to 0$ as $n \to \infty$. Therefore, $\{x_n\}_{n=1}^{\infty}$ converges weakly to x_0. Clearly, $\{x_n\}_{n=1}^{\infty}$ is not Cauchy since $\|x_n - x_m\| = \sqrt{2}$, for $n \neq m$, and the J-fixed point of T is zero, i.e., $F_J(T) = \{0\}$. Since $\{x_n\}_{n=1}^{\infty}$ converges weakly to x_0 and

$$\|Tx_n - Jx_n\| = \left\| \frac{n}{n+1} Jx_n - Jx_n \right\| = \frac{1}{n+1} \|Jx_n\| \to 0,$$

we see that $\widehat{F}_J(T) = \{x_0\}$, i.e., x_0 is an asymptotic J-fixed point of T. Hence, $\widehat{F}_J(T) \neq F_J(T)$, i.e., T is not a relatively J-nonexpansive map.

We now show that zero is a unique strong asymptotic J-fixed point of T. Let $\{y_n\}_{n=1}^{\infty} \in \ell_p(\mathbb{R})$ such that $y_n \to y^*$ and $\|Ty_n - Jy_n\| \to 0$ as $n \to \infty$. Since $\{x_n\}_{n=1}^{\infty}$ is not Cauchy, there exists sufficiently large N such that $y_n \neq x_m$, for $n, m > N$. Now, $Ty_n = -Jy_n$, for $n > N$; this implies that $Ty_n - Jy_n = -2Jy_n$, for $n > N$. It follows that $\|Ty_n - Jy_n\| = 2\|Jy_n\| \to 0$, and thus, $y_n \to y^* = 0$. Consequently, $\widetilde{F}_J(T) = F_J(T)$.

We now show that T is a relatively weak J-nonexpansive map. Based on the definition of T, we obtain

$$\phi\big(0, J^{-1}Tx\big) = \|Tx\|^2 \leq \|x\|^2 = \|0\|^2 + \langle 0, Jx\rangle + \|x\|^2 = \phi(0,x), \quad \forall x \in \ell_p(\mathbb{R}).$$

It follows that T is a relatively weak J-nonexpansive map. Hence, the map T is an example of a relatively weak J-nonexpansive map, which is not a relatively J-nonexpansive map.

Lemma 3.3 *Let E be a reflexive, strictly convex, and smooth real Banach space with dual space E^*. Let $T : E \to E^*$ be a map. Then, $F_J(T) \subset \widetilde{F}_J(T) \subset \widehat{F}_J(T)$.*

Proof Let $p \in F_J(T)$. Then, $T(p) = Jp$. Set $x_n = p$, $\forall n \geq 1$, so $Tx_n = Tp = Jp$. Therefore, $F_J(T) \subset \widetilde{F}_J(T)$. Clearly, $\widetilde{F}_J(T) \subset \widehat{F}_J(T)$. Hence, $F_J(T) \subset \widetilde{F}_J(T) \subset \widehat{F}_J(T)$. □

Theorem 3.4 *Let E be a reflexive, strictly convex, and smooth real Banach space with dual space E^*. Let $T : E \to E^*$ be a relatively J-nonexpansive map. Then, T is relatively weak J-nonexpansive.*

Proof Since T is a relatively J-nonexpansive map, we get that

(1) $\widehat{F}_J(T) = F_J(T) \neq \emptyset$,

(2) $\phi(p, J^{-1}Tx) \leq \phi(p, x)$, $\forall x \in E$ and $p \in F_J(T)$.

Using Lemma 3.3 and the fact that $\widehat{F}_J(T) = F_J(T)$, we see that $F_J(T) \subset \widetilde{F}_J(T) \subset \widehat{F}_J(T) = F_J(T)$. This implies that $F_J(T) = \widetilde{F}_J(T) = \widehat{F}_J(T)$. Therefore, T is relatively weak J-nonexpansive. □

Theorem 3.5 *Let E be a reflexive, strictly convex, and smooth real Banach space with dual space E^*. Let $T : E \to E^*$ be a relatively weak J-nonexpansive map. Then, the J-fixed point set of T, $F_J(T)$, is closed and convex.*

Proof We first show that $F_J(T)$ is closed. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence in $F_J(T)$ such that $x_n \to x \in E$ as $n \to \infty$. Using the fact that T is relatively weak J-nonexpansive and definition of ϕ, we see that $\phi(x_n, J^{-1}Tx) \leq \phi(x_n, x) \to 0$ as $n \to \infty$. Thus, $\phi(x_n, J^{-1}Tx) \to 0$. We observe that $\phi(x_n, J^{-1}Tx) = \|x_n\|^2 - 2\langle x_n, J(J^{-1}Tx)\rangle + \|JJ^{-1}Tx\|^2$, and so $\lim_{n\to\infty} \phi(x_n, J^{-1}Tx) = \phi(x, J^{-1}Tx) = 0$. From Remark 3, this implies that $J^{-1}Tx = x$, so $Tx = Jx$. Therefore, $x \in F_J(T)$. It follows that $F_J(T)$ is closed.

We now show that the set $F_J(T)$ is convex. Let $x_1, x_2 \in F_J(T)$, $t \in (0, 1)$, and set $x_3 = tx_1 + (1 - t)x_2$; we show that $x_3 \in F_J(T)$. From the definition of ϕ and the fact that T is relatively weak J-nonexpansive, we see that

$$\begin{aligned}
\phi\big(x_3, J^{-1}Tx_3\big) &= \|x_3\|^2 - 2\langle x_3, Tx_3\rangle + \big\|JJ^{-1}Tx_3\big\|^2 \\
&= \|x_3\|^2 - 2\big\langle tx_1 + (1-t)x_2, Tx_3\big\rangle + \big\|JJ^{-1}Tx_3\big\|^2 \\
&= \|x_3\|^2 - 2t\langle x_1, Tx_3\rangle - 2(1-t)\langle x_2, Tx_3\rangle + \big\|JJ^{-1}Tx_3\big\|^2 \\
&= \|x_3\|^2 + t\phi\big(x_1, J^{-1}Tx_3\big) + (1-t)\phi\big(x_2, J^{-1}Tx_3\big) - t\|x_1\|^2 \\
&\quad - (1-t)\|x_2\|^2 \\
&\leq \|x_3\|^2 + t\phi(x_1, x_3) + (1-t)\phi(x_2, x_3) - t\|x_1\|^2 - (1-t)\|x_2\|^2 \\
&= \|x_3\|^2 - 2\langle x_3, Jx_3\rangle + \|x_3\|^2 = 0,
\end{aligned}$$

so $\phi(x_3, J^{-1}Tx_3) = 0$. Therefore, we see that $J^{-1}Tx_3 = x_3$ from Remark 3, and this implies that $Tx_3 = Jx_3$, *i.e.*, $x_3 \in F_J(T)$. Therefore, $F_J(T)$ is convex. Hence, the J-fixed point set of T, $F_J(T)$, is closed and convex. □

Lemma 3.6 *Let E be a uniformly smooth and uniformly convex real Banach space with dual space E^*. Then, the map $T : E \to E^*$ is relatively weak J-nonexpansive map if and only if the map $J^{-1}T : E \to E$ is relatively weak nonexpansive map. Moreover, $F_J(T) = F(J^{-1}T)$ and $\widetilde{F}(J^{-1}T) = \widetilde{F}_J(T)$.*

Proof We first show that $F_J(T) = F(J^{-1}T)$. $y^* \in F_J(T)$ iff $Ty^* = Jy^*$ iff $J^{-1}Ty^* = y^*$ iff $y^* \in F(J^{-1}T)$. Therefore, $F_J(T) = F(J^{-1}T)$. We now show that $\widetilde{F}(J^{-1}T) = \widetilde{F}_J(T)$. $p \in \widetilde{F}_J(T)$

if and only if there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq E$ which converges *strongly* to p and $\lim_{n\to\infty} \|Tx_n - Jx_n\| = 0$ if and only if there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq E$ which converges *strongly* to p and $\lim_{n\to\infty} \|J^{-1}Tx_n - x_n\| = 0$ if and only if $p \in \widetilde{F}(J^{-1}T)$.

Therefore, $\widetilde{F}(J^{-1}T) = \widetilde{F}_J(T)$.

Now, T is a relatively weak J-nonexpansive map if

(1) $\widetilde{F}_J(T) = F_J(T) \neq \emptyset$,

(2) $\phi(p, J^{-1}Tx) \leq \phi(p, x)$, $\forall x \in E$ and $p \in F_J(T)$.

In addition, $J^{-1}T$ is a relatively weak nonexpansive map if

(1) $\widetilde{F}(J^{-1}T) = F(J^{-1}T) \neq \emptyset$,

(2) $\phi(p, J^{-1}Tx) \leq \phi(p, x)$, $\forall x \in E$ and $p \in F(J^{-1}T)$.

To show that $T : E \to E^*$ is relatively weak J-nonexpansive map if and only if the map $J^{-1}T : E \to E$ is relatively weak nonexpansive map, it suffices to show from the definitions above that $F_J(T) = F(J^{-1}T)$ and $\widetilde{F}(J^{-1}T) = \widetilde{F}_J(T)$. Moreover, we have already shown that $F_J(T) = F(J^{-1}T)$ and $\widetilde{F}(J^{-1}T) = \widetilde{F}_J(T)$. Hence, $T : E \to E^*$ is a relatively weak J-nonexpansive map if and only if the map $J^{-1}T : E \to E$ is relatively weak nonexpansive. □

We now prove the following lemma.

Lemma 3.7 *Let E be a uniformly smooth and 2-uniformly convex real Banach space with dual space E^*. Let $A : E \to E^*$ be an α-inverse strongly monotone map, and let $T : E \to E^*$ be a relatively weak J-nonexpansive map. Assume that $W := F_J(T) \cap A^{-1}0 \neq \emptyset$. For arbitrary $x_1 \in C_1$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by*

$$\begin{cases} x_1 \in E := C_1, \\ u_n = J^{-1}(Jx_n - \lambda Ax_n), \\ y_n = J^{-1}Tu_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{3.2}$$

where $J : E \to E^*$ is the normalized duality map, $\lambda \in (0, \frac{\alpha}{2L})$ and $L > 0$ denotes a Lipschitz constant of J^{-1}. Then, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W$.

Proof Since T is a relatively weak J-nonexpansive map, it follows from Lemma 3.6 that $J^{-1}T$ is relatively weak nonexpansive and $F_J(T) = F(J^{-1}T)$. Using Theorem 1.8, we have that the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W$. □

4 Strong convergence theorems for countable families of maps

Lemma 4.1 *Let E be a uniformly smooth and uniformly convex real Banach space with dual space E^*. Let $T_i : E \to E^*$, for each $i = 1, 2, \ldots$, be a countable family of relatively weak J-nonexpansive maps such that $\bigcap_{i=1}^{\infty} F_J(T_i) \neq \emptyset$. Let a map $T : E \to E^*$ be defined by $Tx = \sum_{i=1}^{\infty} \delta_i T_i x$, for each $x \in E$, where $\{\delta_i\}_{i=1}^{\infty}$ is a sequence in $(0, 1)$ such that $\sum_{i=1}^{\infty} \delta_i = 1$. Then, T is relatively weak J-nonexpansive and $F_J(T) = \bigcap_{i=1}^{\infty} F_J(T_i)$.*

Proof Given that the map T is defined by $Tx = \sum_{i=1}^{\infty} \delta_i T_i x$, for each $x \in E$. This implies that $J^{-1}Tx = J^{-1}(\sum_{i=1}^{\infty} \delta_i J(J^{-1}T_i)x)$. Using the fact that T_i is relatively weak J-nonexpansive for each i, it follows from Lemma 3.6 that $J^{-1}T_i$, for each i, is relatively weak nonexpansive and $F_J(T_i) = F(J^{-1}T_i)$. By applying Lemma 2.3, we obtain that $J^{-1}T$ is relatively weak nonexpansive and $F(J^{-1}T) = \bigcap_{i=1}^{\infty} F(J^{-1}T_i)$. It follows from Lemma 3.6 that T is relatively weak J-nonexpansive, and $F_J(T) = F(J^{-1}T) = \bigcap_{i=1}^{\infty} F(J^{-1}T_i) = \bigcap_{i=1}^{\infty} F_J(T_i)$. □

Lemma 4.2 *Let E be a uniformly convex and uniformly smooth real Banach space with dual space E^*. Let $A_i : E \to E^*$, for each $i = 1, 2, \ldots$, be a countable family of α_i-inverse strongly monotone maps such that $\alpha := \inf_{i\geq 1} \alpha_i > 0$ and $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$. Let a map $A : E \to E^*$ be defined by $Ax = \sum_{i=1}^{\infty} \beta_i A_i x$, for each $x \in E$, where $\{\beta_i\}_{i=1}^{\infty}$ is a sequence in $(0, 1)$ such that $\sum_{i=1}^{\infty} \beta_i = 1$. Then,* (i) *$A$ is well defined,* (ii) *A is α-inverse strongly monotone,* (iii) *$A^{-1}(0) = \bigcap_{i=1}^{\infty} A_i^{-1}(0)$.*

Proof (i) Let $x \in E$ and $x^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. Since A_i, for each $i = 1, 2, \ldots$, is Lipschitz, we obtain $\|\beta_i A_i x\| = \|\beta_i (A_i x - A_i x^*)\| \leq \frac{1}{\alpha} \|x - x^*\|$. Hence, for each $x \in E$, the series $\sum_{i=1}^{\infty} \beta_i A_i x$ converges absolutely. This shows that the map A is well defined.

(ii) Let $x, y \in E$. Then, using the fact that A_i, for each $i = 1, 2, \ldots$, is α_i-inverse strongly monotone and a result of Nilsrakoo and Saejung [42], we have that

$$\begin{aligned} \langle x - y, Ax - Ay\rangle &= \sum_{i=1}^{\infty} \beta_i \langle x - y, A_i x - A_i y\rangle \\ &\geq \sum_{i=1}^{\infty} \beta_i \alpha_i \|A_i x - A_i y\|^2 \\ &\geq \alpha \left\| \sum_{i=1}^{\infty} \beta_i A_i x - \sum_{i=1}^{\infty} \beta_i A_i y \right\|^2 \\ &= \alpha \|Ax - Ay\|^2. \end{aligned}$$

This yields that A is α-inverse strongly monotone.

(iii) It is obvious that $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \subseteq A^{-1}(0)$. We now show that $A^{-1}(0) \subseteq \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. Let $x \in A^{-1}(0)$. We show that $x \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. Let $y_0 \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. This implies that $y_0 \in A^{-1}(0)$. From the definition of A, we get

$$0 = \langle x - y_0, Ax - Ay_0\rangle = \sum_{i=1}^{\infty} \beta_i \langle x - y_0, A_i x - A_i y_0\rangle. \tag{4.1}$$

By applying the fact that A_i, for each $i = 1, 2, \ldots$, is monotone and $\sum_{i=1}^{\infty} \beta_i = 1$, it follows from Eq. (4.1) that

$$\langle x - y_0, A_i x - A_i y_0\rangle = 0, \quad \text{for each } i = 1, 2, \ldots. \tag{4.2}$$

Using the fact that A_i, for each $i = 1, 2, \ldots$, is α_i-inverse strongly monotone and $y_0 \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$, we obtain

$$0 = \langle x - y_0, A_i x - A_i y_0\rangle \geq \alpha_i \|A_i x - A_i y_0\|^2 \geq \alpha \|A_i x\|^2.$$

This implies that $A_i x = 0$, for each $i = 1, 2, \ldots$, so $x \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. Thus, $A^{-1}(0) \subseteq \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. Hence, $A^{-1}(0) = \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. This completes the proof. □

We now prove the following theorem.

Theorem 4.3 *Let E be a uniformly smooth and 2-uniformly convex real Banach space with dual space E^*. Let $A_i : E \to E^*$, for each $i = 1, 2, \dots$, be a countable family of α_i-inverse strongly monotone maps such that $\alpha := \inf_{i\geq 1} \alpha_i > 0$ and $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$. Let $T_i : E \to E^*$, for each $i = 1, 2, \dots$, be a countable family of relatively weak J-nonexpansive maps such that $\bigcap_{i=1}^{\infty} F_J(T_i) \neq \emptyset$. Let $\{\beta_i\}_{i=1}^{\infty}$ and $\{\delta_i\}_{i=1}^{\infty}$ be sequences in $(0,1)$ such that $\sum_{i=1}^{\infty} \beta_i = 1$, $\sum_{i=1}^{\infty} \delta_i = 1$, and $W := F_J(\sum_{i=1}^{\infty} \delta_i T_i) \cap (\sum_{i=1}^{\infty} \beta_i A_i)^{-1} 0 \neq \emptyset$. For arbitrary $x_1 \in C_1$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by*

$$\begin{cases} x_1 \in E := C_1, \\ u_n = J^{-1}(Jx_n - \lambda(\sum_{i=1}^{\infty} \beta_i A_i)x_n), \\ y_n = J^{-1}(\sum_{i=1}^{\infty} \delta_i T_i)u_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{4.3}$$

where $J : E \to E^*$ is the normalized duality map, $\lambda \in (0, \frac{\alpha}{2L})$, and $L > 0$ denotes a Lipschitz constant of J^{-1}. Then, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W$.

Proof We observe from Lemma 4.1 that the map $T : E \to E^*$ defined by $Tx := \sum_{i=1}^{\infty} \delta_i T_i x$, for each $x \in E$, where $\{\delta_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty} \delta_i = 1$, is relatively weak J-nonexpansive and $F_J(T) = \bigcap_{i=1}^{\infty} F_J(T_i)$. Also, consider the map $A : E \to E^*$ defined by $Ax = \sum_{i=1}^{\infty} \beta_i A_i x$, for each $x \in E$ where $\{\beta_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty} \beta_i = 1$. Then, we have the following by Lemma 4.2: (i) A is well defined, (ii) A is α-inverse strongly monotone, (iii) $A^{-1}(0) = \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. It follows by Lemma 3.7 that the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W := F_J(\sum_{i=1}^{\infty} \delta_i T_i) \cap (\sum_{i=1}^{\infty} \beta_i A_i)^{-1} 0 \neq \emptyset$. □

Corollary 4.4 *Let $E = L_p, \ell_p$, and W_m^p, $1 < p \leq 2$. Let $A_i : E \to E^*$, for each $i = 1, 2, \dots$, be a countable family of α_i-inverse strongly monotone maps such that $\alpha := \inf_{i\geq 1} \alpha_i > 0$ and $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$. Let $T_i : E \to E^*$, for each $i = 1, 2, \dots$, be a countable family of relatively weak J-nonexpansive maps such that $\bigcap_{i=1}^{\infty} F_J(T_i) \neq \emptyset$. Let $\{\beta_i\}_{i=1}^{\infty}$ and $\{\delta_i\}_{i=1}^{\infty}$ be sequences in $(0,1)$ such that $\sum_{i=1}^{\infty} \beta_i = 1$, $\sum_{i=1}^{\infty} \delta_i = 1$, and $W := F_J(\sum_{i=1}^{\infty} \delta_i T_i) \cap (\sum_{i=1}^{\infty} \beta_i A_i)^{-1} 0 \neq \emptyset$. For arbitrary $x_1 \in C_1$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by*

$$\begin{cases} x_1 \in E := C_1, \\ u_n = J^{-1}(Jx_n - \lambda(\sum_{i=1}^{\infty} \beta_i A_i)x_n), \\ y_n = J^{-1}(\sum_{i=1}^{\infty} \delta_i T_i)u_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{4.4}$$

where $J : E \to E^*$ is the normalized duality map, $\lambda \in (0, \frac{\alpha}{2L})$, and $L > 0$ denotes a Lipschitz constant of J^{-1}. Then, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W$.

Proof We observe that E is a 2-uniformly convex and uniformly smooth real Banach space. It follows from Theorem 4.3 that the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W := F_J(\sum_{i=1}^{\infty} \delta_i T_i) \cap (\sum_{i=1}^{\infty} \beta_i A_i)^{-1} 0 \neq \emptyset$. □

Corollary 4.5 *Let $E = H$ be a real Hilbert space. Let $A_i : H \to H$, for each $i = 1, 2, \dots$, be a countable family of α_i-inverse strongly monotone maps such that $\alpha := \inf_{i\geq 1} \alpha_i > 0$ and $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$. Let $T_i : H \to H$, for each $i = 1, 2, \dots$, be a countable family of nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{\beta_i\}_{i=1}^{\infty}$ and $\{\delta_i\}_{i=1}^{\infty}$ be sequences in $(0,1)$ such that $\sum_{i=1}^{\infty} \beta_i = 1$, $\sum_{i=1}^{\infty} \delta_i = 1$, and $W := F_J(\sum_{i=1}^{\infty} \delta_i T_i) \cap (\sum_{i=1}^{\infty} \beta_i A_i)^{-1}0 \neq \emptyset$. For arbitrary $x_1 \in C_1$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by*

$$\begin{cases} x_1 \in H := C_1, \\ u_n = x_n - \lambda(\sum_{i=1}^{\infty} \beta_i A_i)x_n, \\ y_n = (\sum_{i=1}^{\infty} \delta_i T_i)u_n, \\ C_{n+1} = \{v \in C_n : \|v - y_n\| \leq \|v - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{4.5}$$

where $\lambda \in (0, \frac{\alpha}{2})$ and $P_{C_{n+1}}$ denotes the projection map from H onto C_{n+1}. Then, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W$.

Proof The map $T : H \to H$ defined by $Tx = \sum_{i=1}^{\infty} \delta_i T_i x$, for each $x \in H$, where $\{\delta_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty} \delta_i = 1$, is nonexpansive by Remark 4. It suffices to show that if T is nonexpansive in a Hilbert space, then T is relatively weak J-nonexpansive. We only need to show that $\widehat{F}(T) \subset F(T)$ since for any map, T, $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. Let $p \in \widehat{F}(T)$. Then, E contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges *weakly* to p and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. The fact that T is nonexpansive map gives by Lemma 2.2 that $(I - T)$ is demiclosed at zero. Thus, we obtain $Tp = p$, i.e., $p \in F(T)$. Thus, $\widehat{F}(T) \subset F(T)$. Since H is a real Hilbert space and the map T is nonexpansive, it follows that $\phi(Tx, Ty) \leq \phi(x, y)$, for all $x, y \in H$. Hence, T is a relatively nonexpansive map, so T is a relatively weak nonexpansive map. This implies that T is relatively weak J-nonexpansive. It follows by Theorem 4.3 that the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to some $x^* \in W$. □

5 Numerical illustration

We now present a numerical example to illustrate the convergence of the sequence generated by our algorithm in Theorem 4.3.

Example 4 Let $E = \mathbb{R}$ and $C = [a, b]$, for $a, b \in \mathbb{R}$. Let $A_i : \mathbb{R} \to \mathbb{R}$, for each $i = 1, 2, \dots$, be defined by $A_i x = 2x$. Let a map $A : \mathbb{R} \to \mathbb{R}$ be defined by $Ax = \sum_{i=1}^{\infty} \beta_i A_i x = 2x$, for each $x \in \mathbb{R}$, where $\beta_i = \frac{1}{2^i}$ is a sequence in $(0,1)$ and $\sum_{i=1}^{\infty} \beta_i = 1$. Let $T_i : \mathbb{R} \to \mathbb{R}$, for each $i = 1, 2, \dots$, be defined by $T_i x = \frac{4}{7}x$. Let a map $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = \sum_{i=1}^{\infty} \delta_i T_i x = \frac{4}{7}x$, for each $x \in \mathbb{R}$, where $\delta_i = \frac{1}{2^i}$ is a sequence in $(0,1)$ and $\sum_{i=1}^{\infty} \delta_i = 1$. It is clear that A is $\frac{1}{2}$-inverse strongly monotone map and T is nonexpansive map with zero as its unique fixed point, so T is relatively nonexpansive map. We observe that the intersection of the set of zeros of A and the fixed points of T is zero. We consider two different initial values $x_1 = 1.8$ and $x_1 = -1.8$ with $\lambda = \frac{1}{8}$ and define a map

$$P_C x = \begin{cases} a, & \text{if } x < a, \\ x, & \text{if } x \in C, \\ b, & \text{if } x > b, \end{cases}$$

Figure 1 Convergence of the sequence $\{x_n\}$ with two different initial values $x_1 = 1.8, -1.8$

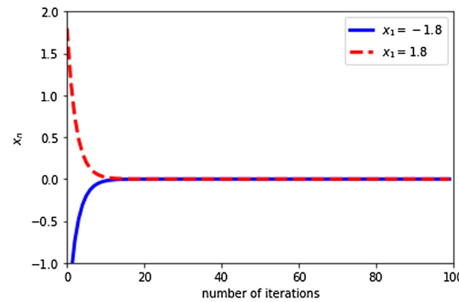
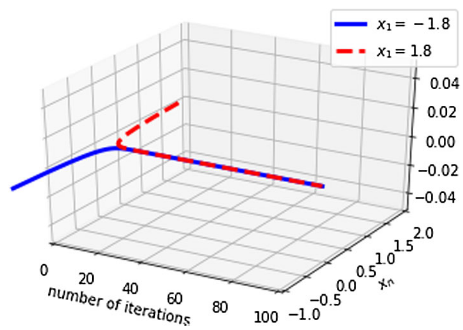


Figure 2 Convergence of the sequence $\{x_n\}$ with two different initial values $x_1 = 1.8, -1.8$



where P_C denotes the projection map from H onto C. It follows by Theorem 4.3 that the sequence generated by algorithm (3.2) converges to zero. The sketch of the numerical example is given in Figs. 1 and 2, where the y-axis represents the value of $x_n - 0$, while the x-axis represents the number of iterations n.

Conclusion. It is obvious that our algorithm (3.2) can be implemented from Figs. 1 and 2 and that the sequence $\{x_n\}$ converges to the solution we desire.

5.1 Analytical representations of duality maps in L_p, l_p, and W_m^p spaces, $1 < p < \infty$

The analytical representations of duality maps are known in a number of Banach spaces. In particular, they are known in L_p, l_p, and W_m^p, $1 < p < \infty$, (see, e.g., Alber and Ryazantseva [3], page 36).

6 Conclusion

In this paper, we introduced and studied a new class of maps called *relatively weak J-nonexpansive* maps *from a real normed space E to its dual space E**. An algorithm was constructed to approximate a common element of J-fixed points for a countable family of relatively weak J-nonexpansive maps and zeros of a countable family of inverse strongly monotone maps in a 2-uniformly convex and uniformly smooth real Banach space. We proved a strong convergence theorem and gave a numerical example to illustrate the convergence of the sequence generated by the algorithm.

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Author details
[1]African University of Science and Technology, Abuja, Nigeria. [2]Department of Mathematics, Nnamdi Azikiwe University, Awka, Nigeria.

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