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On monotone nonexpansive mappings in $CAT_p(0)$ spaces

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Abstract

In this paper, based on some geometrical properties of $CAT_p(0)$ spaces, for $p \geq 2$, we obtain two fixed point results for monotone multivalued nonexpansive mappings and proximally monotone nonexpansive mappings. Which under some assumptions, reduce to coincide and generalize a fixed point result for monotone nonexpansive mappings. This work is a continuity of the previous work of Ran and Reurings, Nieto and Rodríguez-López done for monotone contraction mappings.

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1 Introduction

Recently, a new direction has been discovered for the extension of the Banach contraction principle [6] to metric spaces endowed with a partial order. Ran and Reurings [20] have successfully carried out the first attempt; in particular, they showed, how this extension is useful when dealing with some special matrix equations. A similar approach was followed by Nieto and Rodríguez-López and used such arguments in solving some differential equations [18]. Bachar and Khamsi [5] studied the existence of fixed points of monotone nonexpansive mappings defined on partially ordered Banach spaces. Dehaish and Khamsi [10] have given an analogue of the fixed point theorem of Browder and Göhde for monotone nonexpansive mappings on a very general nonlinear domain. For a thorough discussion of monotone nonexpansive mappings, see also the paper by Uddin et al. [23]. Recently, Al-furaidan and Khamsi [3] have given an analogue of the fixed point theorem of Goebel and Kirk for monotone asymptotically nonexpansive mapping. In 2005, Sahu [21] introduced nearly asymptotically nonexpansive mapping and proved some fixed point theorems. Aggarwal et al. [1, 2] studied some convergence behaviors of monotone nearly asymptotically nonexpansive mappings in partially ordered hyperbolic metric space.

Some classical fixed point theorems for single-valued nonexpansive mappings have been extended to multivalued mappings. The first results in this direction were established by Markin [17] in a Hilbert space setting and by Browder [9] for spaces having a weakly continuous duality mapping. Lami Dozo [16] generalized these results to a Banach space satisfying Opial's condition.

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If the fixed point equation $Tx = x$ of a given mapping T does not have a solution, then it is of interest to find an approximate solution for it. In other words, we are in search of an element in the domain of the mapping, whose image is as close to it as possible. This situation motivated to develop the notion of “best proximity point” (see [11, 15]). The best proximity point theorems can be viewed as a generalization of fixed point theorems, since most fixed point theorems can be derived as corollaries of best proximity point theorems.

In this work, based on some geometrical properties of $CAT_p(0)$ spaces, for $p \geq 2$, we obtain two fixed point results for monotone multivalued nonexpansive mappings and proximally monotone nonexpansive mappings. Which under some assumptions, reduce to coincide and generalize a fixed point result for monotone nonexpansive mappings. This work is a continuity of the previous work of Ran and Reurings, Nieto and Rodríguez-López done for monotone contraction mappings.

2 $CAT_p(0)$ spaces

In this section, we introduce the basic notations and terminologies which we will use throughout this work. Let (M, d) be a metric space. A continuous mapping from the interval $[0, 1]$ to M is called a *path*. A path $\gamma : [0, 1] \rightarrow M$ is called a *geodesic* if $d(\gamma(s), \gamma(t)) = |s - t|d(\gamma(0), \gamma(1))$, for every $s, t \in [0, 1]$. We will say that (M, d) is a *geodesic space* if every two points $x, y \in M$ are connected by a geodesic, i.e., there exists a geodesic $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = x$ and $\gamma(1) = y$. In this case, we denote such geodesic by $[x, y]$. Note that in general such a geodesic is not uniquely determined by its endpoints. For a point $z \in [x, y]$, we will use the notation $z = (1 - t)x \oplus ty$, where $t = d(x, z)/d(x, y)$ assuming $x \neq y$. The metric space (M, d) is called *uniquely geodesic* if every two points of M are connected by a unique geodesic. In this case $[x, y]$ will denote the unique geodesic connecting x and y in M . A subset C of M is said to be convex whenever $[x, y] \subset C$ for any $x, y \in C$.

The most fundamental examples of geodesic metric spaces are normed vector spaces, As nonlinear examples, one can consider the $CAT(0)$ spaces [8].

A metric space M is said to be a $CAT(0)$ space (the term is due to M. Gromov-see, e.g., [8], page 159) if it is geodesically connected, and if every geodesic triangle in M is at least as “thin” as its comparison triangle in the Euclidean plane.

Recently, Khamsi and Shukri in [14], have extended the Gromov geometric definition of $CAT(0)$ spaces to the case when the comparison triangles belong to a general Banach space. In particular, the case when the Banach space is l_p , $p \geq 2$.

Recall that a *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (M, d) consists of three points x_1, x_2, x_3 in M (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (M, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Banach space l_p , for $p \geq 2$, such that $\|\bar{x}_i - \bar{x}_j\| = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A point $\bar{x} \in [\bar{x}_1, \bar{x}_2]$ is called a comparison point for $x \in [x_1, x_2]$ if $d(x_1, x) = d(\bar{x}_1, \bar{x})$.

Definition 2.1 ([14]) Let (M, d) be a geodesic metric space. M is said to be a $CAT_p(0)$ space, for $p \geq 2$, if, for any geodesic triangle Δ in M , there exists a comparison triangle $\bar{\Delta}$ in l_p such that the comparison axiom is satisfied, i.e., for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$, we have

$$d(x, y) \leq \|\bar{x} - \bar{y}\|.$$

It is obvious that $I_p, p > 2$, is a $CAT_p(0)$ space which is not a $CAT(0)$ space [14].

Let x, y_1, y_2 be in M , and $\frac{y_1 \oplus y_2}{2}$ be the midpoint of the geodesic $[y_1, y_2]$, then the comparison axiom implies

$$d^p\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^p(x, y_1) + \frac{1}{2}d^p(x, y_2) - \frac{1}{2^p}d^p(y_1, y_2), \tag{2.1}$$

This inequality is the (CN_p) inequality of Khamsi and Shukri [14].

3 Monotone multivalued nonexpansive mappings

The extension of the Banach contraction principle in metric spaces endowed with a partial order was initiated by Ran and Reurings [20]. In order to discuss such an extension, we will need to assume that the metric space (X, d) is endowed with a partial order \leq . We will say that $x, y \in X$ are comparable whenever $x \leq y$ or $y \leq x$. Recall that an order interval is any of the subsets $[a, \rightarrow) = \{x \in X; a \leq x\}$ and $(\leftarrow, b] = \{x \in X; x \leq b\}$, for any $a, b \in X$.

Next we give the definition of monotone mappings.

Definition 3.1 Let (M, d, \leq) be a metric space endowed with a partial order. Let $T : M \rightarrow M$ be a mapping. T is said to be monotone or order-preserving if

$$x \leq y \implies T(x) \leq T(y),$$

for any $x, y \in M$.

Next we give the definition of monotone nonexpansive mappings.

Definition 3.2 Let (M, d, \leq) be a metric space endowed with a partial order. Let $T : M \rightarrow M$ be a mapping. T is said to be monotone nonexpansive mapping if T is monotone and

$$d(T(x), T(y)) \leq d(x, y),$$

for any $x, y \in M$ such that x and y are comparable.

The definition of monotone multivalued nonexpansive mappings finds its roots in [13].

Definition 3.3 Let (M, d, \leq) be a metric space endowed with a partial order and C a nonempty subset of M . A multivalued mapping $T : C \rightarrow 2^C$ is said to be monotone increasing (resp. decreasing) nonexpansive if for any $x, y \in C$ with $x \leq y$ and any $u \in T(x)$ there exists $v \in T(y)$ such that

$$u \leq v \text{ (resp. } v \leq u) \text{ and } d(u, v) \leq d(x, y).$$

$x \in C$ is called a fixed point of a single-valued mapping T if and only if $T(x) = x$. For a multivalued mapping T , x is a fixed point if and only if $x \in T(x)$. The set of all fixed points of a mapping T is denoted by $\text{Fix}(T)$.

Let us discuss the behavior of type functions in $CAT_p(0)$ metric spaces, for $p \geq 2$. It is worth mentioning that these functions are very useful when one needs to prove the

existence of fixed points of mappings. Recall that $\tau : M \rightarrow \mathbb{R}_+$ is called a type if there exists a bounded sequence $\{x_n\}$ in M such that

$$\tau(x) = \limsup_{n \rightarrow \infty} d(x, x_n).$$

Theorem 3.1 ([14]) *Let (M, d) be a complete $CAT_p(0)$ metric space, with $p \geq 2$. Let C be any nonempty, closed, convex and bounded subset of M . Let τ be a type defined on C . Then any minimizing sequence $\{x_n\} \subset C$ of τ is convergent. Its limit x is the unique minimum of τ , which is called the asymptotic center of $\{x_n\}$, and satisfies*

$$\tau^p(x) + \frac{1}{2^{p-1}}d^p(x, z) \leq \tau^p(z), \tag{3.1}$$

for any $z \in C$.

Note that the inequality (3.1) is similar to Opial's condition defined in Banach spaces, which is introduced in [19], to give a characterization for weak convergent sequences. From this point of view, an analogue to the weak convergence in l_p spaces is introduced in complete $CAT_p(0)$ spaces, for $p \geq 2$, as follows.

Definition 3.4 We shall say that $\{x_n\} \subset M$ weakly converges to a point $x \in M$ if x is the asymptotic center of each subsequence of $\{x_n\}$. We use the notation $x_n \rightarrow^\omega x$.

Clearly, if $x_n \rightarrow x$, then $x_n \rightarrow^\omega x$. If there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow^\omega x$ for some $x \in M$, we say that x is a weak cluster point of the sequence $\{x_n\}$. It is obvious that each bounded sequence has a weak cluster point (see Proposition 3.1.2. of [4]). This situation is completely analogous to strong and weak convergences in l_p spaces, for $p \geq 2$.

Now we are ready to state our main result,

Theorem 3.2 *Let (M, d, \preceq) be a complete partially ordered $CAT_p(0)$ for $p \geq 2$, and assume that order intervals are convex and closed. Let C be a nonempty, closed, convex and bounded subset of M not reduced to one point. Set $\mathcal{C}(C)$ to be the set of all nonempty closed subsets of C . Let $T : C \rightarrow \mathcal{C}(C)$ be a monotone increasing multivalued nonexpansive mapping. If $C_T := \{x \in C; x \preceq y \text{ for some } y \in T(x)\}$ is not empty, then T has a fixed point.*

Proof For $x_0 \in C_T$. i.e., there exists $y_0 \in T(x_0)$ such that $x_0 \preceq y_0$. Set $x_1 = \frac{x_0 \oplus y_0}{2}$. Since order intervals are convex, we have $x_0 \preceq x_1 \preceq y_0$. Since T is monotone increasing multivalued nonexpansive mapping, there is $y_1 \in T(x_1)$ such that $y_0 \preceq y_1$ and $d(y_1, y_0) \leq d(x_1, x_0)$. Continuing in this manner we get an iteration sequence $\{x_n\}$ in C defined by

$$x_{n+1} = \frac{x_n \oplus y_n}{2}, \quad n \geq 0. \tag{3.2}$$

By induction, we will prove that

$$x_n \preceq x_{n+1} \preceq y_n \preceq y_{n+1}$$

and

$$d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n),$$

for any $n \geq 1$. Note that the sequence $\{d(x_n, y_n)\}$ is decreasing. Indeed,

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &\leq d(x_{n+1}, y_n) + d(y_{n+1}, y_n) \\ &= d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \\ &\leq d(x_{n+1}, x_n) + d(x_{n+1}, x_n) \\ &= 2d(x_{n+1}, x_n) \\ &= d(x_n, y_n). \end{aligned}$$

It is not difficult to see that

$$\begin{aligned} \left(1 + \frac{n}{2}\right)d(y_i, x_i) &\leq d(y_{i+n}, x_i) \\ &\quad + 2^n(d(y_i, x_i) - d(y_{i+n}, x_{i+n})), \end{aligned}$$

for any $i, n \in \mathbb{N}$ by an induction argument on i .

Set $r = \lim_{n \rightarrow +\infty} d(y_n, x_n)$. Then, if we let $i \rightarrow +\infty$ in the above inequality, we get $(1 + \frac{n}{2})r \leq \delta(C)$, for any $n \geq 1$, where $\delta(C) = \sup\{d(x, y); x, y \in C\} < +\infty$. This will obviously imply $r = 0$, i.e., $\lim_{n \rightarrow +\infty} d(x_n, y_n) = 0$. In particular, we have $\lim_{n \rightarrow \infty} \text{dist}(x_n, T(x_n)) = 0$, where

$$\text{dist}(x_n, T(x_n)) = \inf\{d(x_n, y); y \in T(x_n)\}.$$

i.e., T has an approximate fixed point sequence $\{x_n\} \in C$ of T .

Since C is bounded and closed, there exists $\omega \in C$, a weak cluster point of $\{x_n\}$, i.e., there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ weakly converging to $\omega \in C$.

Since $\{x_n\}$ is monotone increasing, together with the assumption that order intervals are closed, we have $x_{n_k} \leq \omega$, for any $n \geq 1$. Since T is monotone increasing multivalued non-expansive mapping, there exists $\omega_{n_k} \in T(\omega)$ such that $y_{n_k} \leq \omega_{n_k}$ and $d(y_{n_k}, \omega_{n_k}) \leq d(x_{n_k}, \omega)$, for any n . Assume $\{\omega_{n_k}\}$ converges to ω . Since $T(\omega)$ is closed, we conclude that $\omega \in T(\omega)$, i.e. ω is a fixed point of T . Indeed, if $\omega_{n_k} \not\rightarrow \omega$, then there exist a subsequence $\{z_{n_k}\}$ of $\{\omega_{n_k}\}$ and $\epsilon > 0$ such that $d(z_{n_k}, \omega) > \epsilon$. By the (CN_p) inequality, we observe that

$$d^p\left(x_{n_k}, \frac{z_{n_k} \oplus \omega}{2}\right) \leq \frac{1}{2}d^p(x_{n_k}, z_{n_k}) + \frac{1}{2}d^p(x_{n_k}, \omega) - \left(\frac{\epsilon}{2}\right)^p.$$

Taking $\limsup_{n_k \rightarrow \infty}$,

$$\begin{aligned} \limsup_{n_k \rightarrow \infty} d^p\left(x_{n_k}, \frac{z_{n_k} \oplus \omega}{2}\right) &\leq \frac{1}{2} \limsup_{n_k \rightarrow \infty} d^p(x_{n_k}, z_{n_k}) + \frac{1}{2} \limsup_{n_k \rightarrow \infty} d^p(x_{n_k}, \omega) - \left(\frac{\epsilon}{2}\right)^p \\ &\leq \frac{1}{2} \limsup_{n_k \rightarrow \infty} d^p(x_{n_k}, \omega_{n_k}) + \frac{1}{2} \limsup_{n_k \rightarrow \infty} d^p(x_{n_k}, \omega) - \left(\frac{\epsilon}{2}\right)^p \\ &= \frac{1}{2} \limsup_{n_k \rightarrow \infty} d^p(y_{n_k}, \omega_{n_k}) + \frac{1}{2} \limsup_{n_k \rightarrow \infty} d^p(x_{n_k}, \omega) - \left(\frac{\epsilon}{2}\right)^p \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2} \limsup_{n_k \rightarrow \infty} d^p(x_{n_k}, \omega) + \frac{1}{2} \limsup_{n_k \rightarrow \infty} d^p(x_{n_k}, \omega) - \left(\frac{\epsilon}{2}\right)^p \\ &= \limsup_{n_k \rightarrow \infty} d^p(x_{n_k}, \omega) - \left(\frac{\epsilon}{2}\right)^p. \end{aligned}$$

Finally, by Theorem 3.1, we conclude

$$\limsup_{n_k \rightarrow \infty} d^p(x_{n_k}, \omega) < \limsup_{n_k \rightarrow \infty} d^p\left(x_{n_k}, \frac{z_{n_k} \oplus \omega}{2}\right) \leq \limsup_{n_k \rightarrow \infty} d^p(x_{n_k}, \omega) - \left(\frac{\epsilon}{2}\right)^p.$$

This contradicts the fact that $\epsilon > 0$. Therefore, $\lim_{n_k \rightarrow \infty} \omega_{n_k} = \omega$. □

Corollary 3.1 *Let (M, d, \leq) be a complete partially ordered $CAT_p(0)$ for $p \geq 2$, and assume that order intervals are convex and closed. Let C be a nonempty, closed, convex and bounded subset of M not reduced to one point. Let $T : C \rightarrow C$ be a monotone increasing nonexpansive mapping. Assume there exists $x_0 \in C$ such that x_0 and $T(x_0)$ are comparable. Then T has a fixed point.*

4 Proximally monotone nonexpansive mappings

Let A, B be nonempty subsets of a metric space (M, d) . Then the proximity pair associated with the pair (A, B) , denoted by (A_0, B_0) , is defined by

$$A_0 = \{x \in A : d(x, y) = d(A, B); \text{ for some } y \in B\}$$

and

$$B_0 = \{y \in B : d(x, y) = d(A, B); \text{ for some } x \in A\},$$

where $d(A, B) = \inf\{d(x, y); (x, y) \in A \times B\}$. It is clear that A_0 is not empty if and only if B_0 is not empty.

If the fixed point equation $Tx = x$ of a given mapping T does not have a solution, then it is of interest to find an approximate solution for the fixed point equation.

Definition 4.1 Let A and B be nonempty subsets of a metric space (M, d) , and $T : A \rightarrow B$ be a mapping. A point $x \in A$ is said to be a best proximity point of T if

$$d(x, Tx) = d(A, B) = \inf\{d(a, b); a \in A, b \in B\}.$$

Note that, if $A \cap B \neq \emptyset$, then x is a best proximity point of T if $T(x) = x$, i.e., x is a fixed point of T .

The definition of proximally monotone mappings has roots in [7]. Let us define the concept of proximally monotone nonexpansive mappings on a partially ordered metric space.

Definition 4.2 Let A, B be nonempty subsets of M and let $T : A \rightarrow B$ be a mapping.

- (1) T is said to be proximally monotone if it satisfies the condition:

$$x \leq y, \quad d(u, Tx) = d(A, B) \quad \text{and} \quad d(v, Ty) = d(A, B) \quad \text{imply} \quad u \leq v$$

for all $x, y, u, v \in A$.

(2) T is said to be proximally monotone nonexpansive mapping if T is proximally monotone and

$$d(Tx, Ty) \leq d(x, y),$$

where $x, y \in A$ and x and y are comparable.

If $A = B$, then the above definition coincides with the definition of monotone nonexpansive mappings and the best proximity point x reduces to a fixed point of T .

Next, let us define the nearest point projection $P_C : M \rightarrow 2^C$ by

$$P_C(x) = \{c \in C; d(x, c) = \inf\{d(x, c) : c \in C\}\}.$$

If $P_C(x)$ is reduced to one point, for every x in M , then C is said to be a Chebyshev set. In this case, the mapping P_C is not seen as a multivalued mapping but a single-valued mapping, i.e., $P_C : M \rightarrow C$ defined by

$$d(x, P_C(x)) = \inf\{d(x, c) : c \in C\},$$

for any $x \in M$.

Next, we need the following lemma.

Lemma 4.1 *Let (M, d) be a complete $CAT_p(0)$ space, $p \geq 2$. Then any nonempty, closed and convex subset C of M is a Chebyshev subset.*

Proof We need to show that the function $c \mapsto d(x, c)$ on C has a unique minimizer. It is clear that instead of minimizing $d(x, \cdot)$ we can equivalently minimize the function $d^p(x, \cdot)$. Since the latter is strictly convex, it has at most one minimizer. To show the existence of a minimizer, we take a minimizing sequence $(c_n) \subset C$, that is,

$$d^p(x, c_n) \rightarrow \inf_C d^p(x, \cdot),$$

and denote $c_{mn} := \frac{c_m \oplus c_n}{2}$. Then by the convexity of C we have $c_{mn} \in C$, and the (CN_p) inequality yields

$$d^p(x, c_{mn}) \leq \frac{1}{2}d^p(x, c_m) + \frac{1}{2}d^p(x, c_n) - \frac{1}{2^p}d^p(c_m, c_n),$$

or

$$\frac{1}{2^p}d^p(c_m, c_n) \leq \frac{1}{2}d^p(x, c_m) + \frac{1}{2}d^p(x, c_n) - d^p(x, c_{mn}),$$

which, together with the fact that (c_n) is a minimizing sequence, implies that (c_n) is Cauchy. The limit point is clearly a minimizer and lies in C . □

Definition 4.3 ([12]) A uniquely geodesic metric space (M, d) is said to have the property (R) if any non-increasing sequence of nonempty, convex, bounded and closed sets, has a nonempty intersection.

A direct consequence of the (CN_p) inequality is that any complete $CAT_p(0)$ space, $p \geq 2$, has the property (R) [14].

Next, we need the following lemma.

Lemma 4.2 *Let (A, B) be a pair of nonempty, bounded and closed convex subsets of a complete $CAT_p(0)$ space, $p \geq 2$. Then the proximity pair (A_0, B_0) associated with the pair (A, B) is a pair of nonempty, bounded and closed convex subsets.*

Proof By Lemma 4.1, B is a Chebyshev subset. Let P_B be the nearest point projection onto B , consider the set

$$A_n = \left\{ x \in A; d(x, B) = d(x, P_B(x)) \leq d(A, B) + \frac{1}{n} \right\},$$

for any $n \geq 1$. From the definition of $d(A, B)$ and the continuity and the convexity of the function $x \rightarrow d(x, B)$, we know that A_n is a nonempty, bounded, closed and convex subset of A , for any $n \geq 1$. Obviously $\{A_n\}$ is decreasing. By the property (R) , we conclude that $A_\infty = \bigcap_{n \geq 1} A_n \neq \emptyset$. Let $u \in A_\infty$. Hence

$$d(u, B) = d(u, P_B(u)) \leq d(A, B) + \frac{1}{n},$$

for any $n \geq 1$, which implies that $d(u, B) = d(u, P_B(u)) \leq d(A, B)$. Since, by the definition of $d(A, B)$, we have $d(A, B) \leq d(u, P_B(u))$, we get $d(u, P_B(u)) = d(A, B)$, i.e., $u \in A_0$ and $P_B(u) \in B_0$. Therefore, A_0 is nonempty.

Finally, A_0 is a closed convex subset of A . Indeed, that A_0 is closed follows in a straightforward way from its definition and the fact that A is closed. The convexity of the set A_0 follows from the convexity of the metric of $CAT_p(0)$ spaces for $p \geq 2$. Since A_0 is nonempty and closed convex subset of A , it is bounded.

Similarly, we show that B_0 is a nonempty, bounded and closed convex subset of B . □

Now we are ready to obtain our main result for proximally monotone nonexpansive mappings as follows.

Theorem 4.1 *Let (A, B) be a pair of nonempty, bounded, closed, and convex subsets of a partially ordered $CAT_p(0)$ space for $p \geq 2$ (M, d, \leq) such that order intervals are closed and convex with A_0 not reducible to one point. Let $T : A \rightarrow B$ be a proximally monotone nonexpansive mapping such that $T(A_0) \subseteq B_0$. If there exist $x_0, x_1 \in A_0$ such that $x_0 \leq x_1$ and $d(x_1, Tx_0) = d(A, B)$, then T has a best proximity point x in A .*

Proof Note that, for any $y_0 \in B_0$, there exists a unique $x_0 \in A_0$ such that $d(x_0, y_0) = d(A, B)$. Hence

$$d(A, B) \leq d(A, y_0) \leq d(A_0, y_0) \leq d(x_0, y_0) = d(A, B).$$

That is, $d(A, y) = d(A_0, y) = d(A, B)$, for all $y \in B_0$.

Consider the restriction of the nearest point projection P_{A_0} to B_0 . Since P_{A_0} is an isometry, our assumption on the mapping T implies that the mapping $P_{A_0} \circ T : A_0 \rightarrow A_0$ is monotone nonexpansive.

Moreover, let $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$. By Lemma 4.1 A_0 is a Chebyshev subset, so $P_{A_0}(T(x_0)) = x_1$. Hence, $x_0 \preceq P_{A_0}(T(x_0))$.

By Lemma 4.2 A_0 is a nonempty, closed and convex subset of A , therefore A_0 is bounded. Taking this all together with Corollary 3.1, we deduce that the monotone nonexpansive mapping $P_{A_0} \circ T$ has a fixed point $x \in A_0$, i.e., $x = P_{A_0}(Tx)$. Then we have $x \in A_0$ and $Tx \in B_0$. Hence

$$d(x, Tx) = d(P_{A_0}(Tx), Tx) = d(Tx, A_0) = d(A, B).$$

Therefore, x is a best proximity point of T in A . □

Remark 4.1 If $A = B$ in Theorem 4.1, or T in Theorem 3.2 is assumed to be single-valued, then these results reduce to coincide with Corollary 3.1 and thus generalize a fixed point result for monotone nonexpansive mappings. This is considered to be a continuity of the previous work of Ran and Reurings [20], and Nieto and Rodríguez-López [18] done for monotone contraction mappings.

At the end, we set out to give an example to illustrate our results, and how the best proximity point theorems can be viewed as a generalization of fixed point theorems. The following example has its roots in [22].

Consider the real sequence space $X = l_p, p > 2$. Let $A = \{(x, 0, 0, \dots) : 0 \leq x \leq 1\}$ and $B = \{(x, 1, 0, 0, \dots) : 0 \leq x \leq 1\}$ be nonempty subsets of X . Clearly, the pair (A, B) is nonempty, bounded, closed, and convex in X . Moreover, $A_0 = A, B_0 = B$ and $d(A, B) = 1$.

Consider the product order \preceq on X , i.e. for $x = (x_i)$ and $y = (y_i)$, then $x \preceq y$ iff $x_i \leq y_i$. Clearly, order intervals are closed and convex.

Define a mapping $T : A \rightarrow B$ by

$$T(x, 0, 0, \dots) = (x, 1, 0, 0, \dots).$$

We now show that T is a proximally monotone. For $(x, 0, 0, \dots), (y, 0, 0, \dots), (u, 0, 0, \dots), (v, 0, 0, \dots) \in A$ with $x \leq y, \|(u, 0, 0, \dots) - T(x, 0, 0, \dots)\| = 1$ and $\|(v, 0, 0, \dots) - T(y, 0, 0, \dots)\| = 1$, we have $x = u$ and $y = v$. Hence, $u \leq v$, i.e., $(u, 0, 0, \dots) \preceq (v, 0, 0, \dots)$.

Moreover, since $\|T(x, 0, 0, \dots) - T(y, 0, 0, \dots)\| = \|(x, 1, 0, 0, \dots) - (y, 1, 0, 0, \dots)\| = |x - y| = \|(x, 0, 0, \dots) - (y, 0, 0, \dots)\|$, T is a nonexpansive mapping.

Now, $T(A_0) \subseteq B_0$. Let $x = y = 0$. Then $x \preceq y$ and $\|y - Tx\| = \|0 - T0\| = \|(0, 0, \dots) - (0, 1, 0, 0, \dots)\| = 1 = d(A, B)$. Therefore, Theorem 4.1 implies that T has a best proximity point x in A . In fact, any $x \in A$ is a best proximity point of T .

On the other hand, consider the restriction of the nearest point projection P_A to B . Since P_A is an isometry, our assumption on the mapping T implies that the mapping $P_A \circ T : A \rightarrow A$ is a monotone increasing nonexpansive mapping. Reflexivity of the partial order and Corollary 3.1 implies that $P_A \circ T$ has a fixed point x in A . Indeed, the mapping $P_A \circ T$ is the identity mapping I_A on A , thus any $x \in A$ is a fixed point of $P_A \circ T$.

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