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An inertial-type algorithm for approximation of solutions of Hammerstein integral inclusions in Hilbert spaces

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Abstract

Let H be a real Hilbert space. Let $F : H \to 2^H$ and $K : H \to 2^H$ be two maximal monotone and bounded operators. Suppose the Hammerstein inclusion $0 \in u + KFu$ has a solution. We construct an inertial-type algorithm and show its strong convergence to a solution of the inclusion. As far as we know, this is the first inertial-type algorithm for Hammerstein inclusions in Hilbert spaces. We also give numerical examples to compare the new algorithm with some existing ones in the literature.

Keywords: Maximal monotone maps; Hammerstein integral inclusion; Inertial algorithm

1 Introduction

Let Ω be a measurable bounded subset of $\mathbb{R}^n$. A nonlinear integral equation of Hammerstein type is of the form

$$u(x) + \int_{\Omega} k(x,y)f\big(y,u(y)\big)\,dy = w(x), \tag{1.1}$$

where dy is a σ-finite measure. The function $k : \Omega \times \Omega \to \mathbb{R}$ is the kernel of the equation, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable real-valued function. The function w and the unknown function u lie in a suitable Banach space of measurable real-valued functions, say, $\mathcal{F}(\Omega,\mathbb{R})$. If we define the operators $F : \mathcal{F}(\Omega,\mathbb{R}) \to \mathcal{F}(\Omega,\mathbb{R})$ and $K : \mathcal{F}(\Omega,\mathbb{R}) \to \mathcal{F}(\Omega,\mathbb{R})$ by

$$Fu(x) = f\big(x,u(x)\big) \quad \text{and} \quad Kv(x) = \int_{\Omega} k(x,y)v(y)\,dy, \quad x \in \Omega, \tag{1.2}$$

then (1.1) can be easily written as the abstract Hammerstein equation

$$u + KFu = 0, \tag{1.3}$$

where, without loss of generality, we have taken w to be the zero map in $\mathcal{F}(\Omega,\mathbb{R})$. Interest in Hammerstein equations stems mainly from their applications in various fields. For instance, (1.1) can be used to describe the final state of a spatially distributed population (see, e.g., [30] and [42]). Consider also the following nonlinear boundary value problem:

$$\begin{cases} -\Delta u = f(x, u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a smooth subset of $\mathbb{R}^n$. Define the operator $K : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ by $Kg = u$, where u is the unique solution of the corresponding linear boundary value problem

$$\begin{cases} -\Delta u = g, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.5}$$

Then (1.4) can be written in the form (1.3), where F is as defined in (1.2). Other areas of application of Hammerstein integral equations include differential equations (see, e.g., Pascali [38]), optimal control system, automation and network theory, and many other areas (see, e.g., Doležal [31]). If the operator F in (1.3) is multivalued and defined by

$$Fu(y) = \{v(y) : v \text{ is a selection of } f(\cdot, u(\cdot))\},$$

then (1.3) becomes the Hammerstein inclusion

$$u + KFu \ni 0. \tag{1.6}$$

Hammerstein inclusions are closely related to nonsmooth calculus of variations. For instance, consider the energy functional given by

$$Ju = \int_\Omega \bigl(h(u(t)) - f(s, u(s))\bigr)\, ds, \tag{1.7}$$

where h denotes the kinetic energy of the system, and f is the potential energy generator of the superposition operator. In general, the functional J is not differentiable in the classical sense. However, it admits generalized gradient or subgradient in the sense of, for instance, Clarke's generalized gradient (see, e.g., [29]). Consequently, the problem of minimizing the energy functional J leads to the Euler–Lagrange inclusion

$$Lu \in \partial Fu, \tag{1.8}$$

where L is a linear operator, and ∂F is the generalized Clarke gradient. Equation (1.8), in turn, is equivalent to (1.6) defined on a suitable Banach space of measurable real-valued functions. Let H be a Hilbert space. A map $A : H \to 2^H$ is called *monotone* if for all $x, y \in H$,

$$\langle \eta - \nu, x - y\rangle \geq 0, \quad \forall \eta \in Ax, \nu \in Ay. \tag{1.9}$$

Several existence results have been proved for (1.3) when the operators F and K are monotone. (See, e.g., Brézis and Browder [4–6], Browder [7, 8, 10, 11], Browder and De Figueiredo [11, 12], Chepanovich [14], Appel et al. [3], and Cardinali and Papageorgiou

[13]). In general, there is no closed-form solution for Hammerstein integral equations. Therefore developing algorithms for approximating such solutions is of great interest. Let $A : H \to H$ be a nonlinear operator. Then A is said to be *angle bounded* with angle $\beta > 0$ if

$$\langle Ax - Ay, z - y \rangle \le \beta \langle Ax - Ay, x - y \rangle \tag{1.10}$$

for all $x, y, z \in H$. For $y = z$, inequality (1.10) implies the monotonicity of A. A monotone *linear* operator $A : H \to H$ is said to be *angle bounded* with angle $\alpha > 0$ if

$$\bigl| \langle Ax, y \rangle - \langle Ay, x \rangle \bigr| \le 2\alpha \langle Ax, x \rangle^{\frac{1}{2}} \langle Ay, y \rangle^{\frac{1}{2}} \tag{1.11}$$

for all $x, y \in H$. Brézis and Browder [5] proved the following theorem for the approximation of solutions of Hammerstein equations with angle-bounded operators using a suitably defined Galerkin method.

Theorem 1.1 (Brézis and Browder [5]) *Let H be a separable Hilbert space, and let C be a closed subspace of H. Let $K : H \to C$ be a bounded continuous monotone operator, and let $F : C \to H$ be an angle-bounded and weakly compact mapping. For a given $f \in C$, consider the Hammerstein equation*

$$(I + KF)u = f \tag{1.12}$$

and its nth Galerkin approximation given by

$$(I + K_n F_n)u_n = P^* f, \tag{1.13}$$

where $K_n = P_n^* K P_n : H \to C$ and $F_n = P_n F P_n^* : C_n \to H$, with the symbols having their usual meanings (see, e.g., Pascali [38, Chap., p. 202]. Then, for each $n \in \mathbb{N}$, the Galerkin approximation (1.13) admits a unique solution u_n in C_n, and $\{u_n\}$ converges strongly in H to the unique solution $u \in C$ of equation (1.12).

Attempts have been made to develop iterative algorithms for approximating solutions of (1.3) (see, e.g., Mann [21, 35] and the references therein). However, most of these results require the inverse of the operator K not only to exist but also to be strongly monotone. These requirements do not only limit the class of operators involved but are also not convenient for implementation. The first satisfactory result for approximating solution of Hammerstein equation was given by Chidume and Zegeye [26–28]. They considered the product space $E = H \times H$ and defined the auxiliary operator $T : E \to E$ by

$$T[u, v] = [Fu - v, Kv + u], \quad u, v \in E. \tag{1.14}$$

We can easily see that u^* solves (1.3) if and only if $T[u^*, v^*] = 0$ with $v^* = Fu^*$. The auxiliary operator T gave an insight on how to develop a coupled algorithm for computing solutions of (1.3). The same authors (see [28]) defined the following coupled algorithm: for $u_0, v_0 \in X$, define the sequences $\{u_n\}$ and $\{v_n\}$ recursively by

$$u_{n+1} = u_n - \alpha_n (Fu_n - v_n), \quad n \ge 0, \tag{1.15}$$

$$v_{n+1} = v_n - \alpha_n(Kv_n + u_n), \quad n \geq 0, \tag{1.16}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying appropriate conditions. Many strong convergence results have been proved in Banach spaces by many authors using the coupled algorithm. (See, e.g., Chidume and Djitte [17–19], Chidume and Ofeodu [20], Chidume and Shehu [23], Chidume and Osilike [22], and Chidume and Bello [15].) Recently, Minjibir and Muhammad [36] proved a strong convergence result for *Hammerstein inclusion* under the setting of Hilbert spaces. They proved the following theorem with the sequences $\{\alpha_n\}, \{\theta_n\} \subset (0,1)$ satisfying some appropriate conditions.

Theorem 1.2 (Minjibir and Mohammad [36]) *Let H be a real Hilbert space, and let $F, K : H \to CB(H)$ be maps with $D(F) = D(K) = H$ such that the following conditions hold*:

(i) *F is monotone, continuous (relative to h), and bounded*;

(ii) *K is monotone, continuous (relative to h), and bounded.*

Let $\{u_n\}$ and $\{v_n\}$ be sequences generated iteratively from arbitrary $u_1, v_1 \in H$ by

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n(\xi_n - v_n) - \lambda_n\theta_n(u_n - u_1), \quad \xi_n \in Fu_n, n \geq 0, \\ v_{n+1} &= v_n - \lambda_n(\eta_n + u_n) - \lambda_n\theta_n(v_n - v_1), \quad \eta_n \in Kv_n, n \geq 0. \end{aligned} \tag{1.17}$$

Suppose that the inclusion $0 \in u + KFu$ has a solution in H. Then there exists a real constant $\gamma_0 > 0$ such that if $\lambda_n \leq \gamma_0\theta_n$ for some $n_0 \geq 1$, and then the sequence $\{u_n\}$ converges strongly to u^*, a solution of $0 \in u + KFu$.

The need to speed up the convergence of iterative algorithms has always been of great interest. One of the recent methods of speeding up the convergence is via addition of inertial terms to algorithms. The use of the term “inertial” can be traced back, at least, to Poljak [39], where he considered the following second-order system of differential equations:

$$\omega'' + \gamma\omega' + \nabla f(\omega) = 0, \quad \gamma > 0, \tag{1.18}$$

in the context of optimization. In two-dimensional case, system (1.18) describes, roughly, the motion of a heavy ball that rolls under its own inertial over the graph of f until it is impeded by friction. For results concerning inertial algorithms, see, for instance, Moudafi [37], Alvarez [2], and Maingé and Merabet [32]. In this paper, we introduce an inertial algorithm for approximating solution of *Hammerstein inclusion* in Hilbert spaces. As far as we know, this is the first inertial algorithm involving *Hammerstein inclusions*. Our algorithm converges much faster than the existing noninertial algorithms for Hammerstein inclusions. We give numerical examples to support this claim. Moreover, in our theorem, we make no assumption of continuity of the operators.

2 Preliminaries

In this section, we present some definitions and lemmas used in the proof of the main theorem. We further assume that H is a real Hilbert space and $E = H \times H$ is the Cartesian product of H. We define the norm $\|\cdot\|_E$ on E by

$$\|a\|_E = \left(\|a_1\|_H^2 + \|a_2\|_H^2\right)^{\frac{1}{2}}, \quad \forall a = (a_1, a_2) \in E. \tag{2.1}$$

Let W be any nonempty subset of H, and let $x \in H$ be fixed. Then the distance between x and W is given as $\operatorname{dist}(x, W) = \inf\{\|x - y\| : y \in W\}$.

Definition 2.1 Let CB(H) denote the set of closed bounded subsets of H. Then the *Hausdorff distance* between two nonempty closed bounded subsets V and W of H is given as

$$\mathcal{D}(V, W) = \max\Big\{\sup_{x\in V}\operatorname{dist}(x, W), \sup_{y\in W}\operatorname{dist}(y, V)\Big\}.$$

It is well known that $\mathcal{D}$ is a metric on CB(H). A multivalued mapping $T : D(T) \subset H \longrightarrow$ CB(H) with domain $D(T) = \{x \in H : Tx \neq \emptyset\}$ and range $R(T) = \cup\{Tx : x \in H\}$ is monotone if

$$\langle u - v, x - y\rangle_H \geq 0, \quad \forall x, y \in D(T), u \in Tx, v \in Ty,$$

where the function $\langle\cdot,\cdot\rangle_H$ is the inner product on H. A monotone map T is maximal monotone if its graph $Gr(T) = \{(x, y) \in E : x \in D(T), y \in Tx\}$ is not properly contained in the graph of any other monotone map. It is well known that if T is maximal monotone, then the range $R(I + \lambda T) = H$ for $\lambda > 0$. For $\lambda > 0$, the resolvent operator is given by $J_\lambda = (I + \lambda T)^{-1}$, where I is the identity map on H. The operator J_λ is always single-valued. (See, e.g., Browder [9, 33] for more detail.) The following inequality, which characterizes the monotone maps in Hilbert spaces, was given by Kato in [34]:

$$\|x - y\| \leq \big\|x - y + r(u - v)\big\| \tag{2.2}$$

for all $x, y \in D(T), u \in Tx, v \in Ty$, and $r > 0$ if and only if T is monotone. In what follows, we present the lemmas used in the proof of the main theorem.

Lemma 2.2 (Xu [43]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 0,$$

where (i) $\{\alpha_n\} \subset (0, 1), \sum\alpha_n = \infty$; (ii) $\limsup\sigma_n \leq 0$; and (iii) $\gamma_n \geq 0, n \geq 0, \sum_{n=0}^{\infty}\gamma_n < \infty$. Then $a_n \to 0$ as $n \to \infty$.

Lemma 2.3 (Reich [40]) *Let H be a real Hilbert space, and let $A : D(A) \subset H \longrightarrow 2^H$ be maximal monotone. If $0 \in R(A)$, then for each $x \in H$, the strong limit $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}0$, that is, $0 \in A(\lim_{t\to\infty} J_t x)$.*

Lemma 2.4 *Let H be a real Hilbert space, and let $F : H \to CB(H), K : H \to CB(H)$ be two maximal monotone and bounded multivalued maps. Then $T : E \to CB(E)$ defined by*

$$Tw = (Fu - v) \times (Kv + u) = \big\{(\xi - v, \eta + u) : \xi \in Fu, \eta \in Kv\big\} \tag{2.3}$$

is also maximal monotone and bounded.

Proof The boundedness of T follows from the boundedness of F and K. Likewise, the maximal monotonicity of T follows from that of F and K (see Alber and Ryazantseva [1, p. 280]). □

3 Main theorem

Theorem 3.1 *Let H be a real Hilbert space, and let $F,K : H \to CB(H)$ be maximal monotone and bounded maps. For arbitrary $u_1,v_1,u_2,v_2 \in H$, define the sequences $\{h_n\}$, $\{p_n\}$, $\{u_n\}$, and $\{v_n\}$ by*

$$\begin{cases} h_n = u_n + c_n(u_{n-1} - u_n), \\ p_n = v_n + c_n(v_{n-1} - v_n), \\ u_{n+1} = h_n - \lambda_n(\xi_n - p_n) - \lambda_n\theta_n h_n, \quad \xi_n \in Fh_n, n \geq 2, \\ v_{n+1} = p_n - \lambda_n(\eta_n + h_n) - \lambda_n\theta_n p_n, \quad \eta_n \in Kp_n, n \geq 2, \end{cases} \tag{3.1}$$

where $\{\theta_n\}$ and $\{\lambda_n\}$ are sequences in $(0,1)$, and $\{c_n\}$ is a sequence in $[0,1)$ satisfying the following conditions:

$$\text{(i)} \quad \lim_{n\to\infty}\theta_n = 0, \lim_{n\to\infty}\frac{\lambda_n}{\theta_n} = 0, \qquad \text{(ii)} \quad \sum_{n=1}^{\infty}\lambda_n\theta_n = \infty,$$

$$\text{(iii)} \quad \lim_{n\to\infty}\left(\frac{\frac{\theta_{n-1}}{\theta_n} - 1}{\lambda_n\theta_n}\right) = 0 \quad \textit{and} \quad \text{(iv)} \quad \sum_{n=1}^{\infty}c_n < \infty.$$

Suppose that the inclusion $0 \in u + KFu$ has a solution in H. Then there exists a real constant γ_0 such that $\lambda_n \leq \gamma_0\theta_n$ for all $n \geq n_0$, for some $n_0 \geq 2$, and the sequence $\{u_n\}$ converges strongly to u^*, a solution of $0 \in u + KFu$.

Proof Consider the inner product on $E = H \times H$ defined by

$$\langle a,b\rangle_E = \langle a_1,b_1\rangle_H + \langle a_2,b_2\rangle_H, \quad \forall a = (a_1,a_2), b = (b_1,b_2) \in E.$$

The norm induced by this inner product is given by (2.1). We define two sequences $\{w_n\}$ and $\{r_n\}$ in E by $w_n := (u_n,v_n)$ and $r_n := (h_n,p_n)$ for $n \in \mathbb{N}$. Let $u^* \in H$ be a solution of the inclusion $u + KFu \ni 0$ with $v^* \in Fu^*$ such that $u^* \in -Kv^*$. Now set $w^* = (u^*,v^*)$. To show that $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, it suffices to show that $\{w_n\}$ converges strongly to w^* in E. For any $w \in E$ and $t > 0$, we define the closed ball in E with center w and radius t as $\overline{B}(w,t) = \{x \in E : \|x - w\|_E \leq t\}$. Let $r > 0$ be such that

$$w^* \in \overline{B}\left(0,\frac{r}{2}\right) \quad \text{and} \quad w_i \in \overline{B}\left(w^*,r\right), \quad \forall 1 \leq i \leq n_0, \tag{3.2}$$

where n_0 is as defined in the theorem. Let

$$M_1 := \sup\left\{\left(\|\xi - y\|_H + \frac{7r}{2}\right)^2 : \xi \in Fx, (x,y) \in \overline{B}\left(w^*,r\right)\right\} < \infty,$$

$$M_2 := \sup\left\{\left(\|\eta + x\|_H + \frac{7r}{2}\right)^2 : \eta \in Ky, (x,y) \in \overline{B}\left(w^*,r\right)\right\} < \infty,$$

and $M := M_1 + M_2$. By the boundedness of F and K, M_1, M_2, and M are all finite. The first step is showing that $\{w_n\}$ is bounded, and then $\{u_n\}$ and $\{v_n\}$ will be necessarily bounded. By construction, $w_i \in \overline{B}(w^*,r)$, $1 \leq i = 1,2,\ldots,n_0$. We now show that $w_n \in \overline{B}(w^*,r)$ for all

$n \ge n_0$. To do this, we proceed by induction. Supposing that $w_n \in \overline{B}(w^*, r)$ up to some $n \ge n_0$, we show that $w_{n+1} \in \overline{B}(w^*, r)$. Using (2.1), we have

$$\|w_{n+1} - w^*\|_E^2 = \|u_{n+1} - u^*\|_H^2 + \|v_{n+1} - v^*\|_H^2.$$

Now

$$\begin{aligned}
&\|u_{n+1} - u^*\|_H^2 \\
&\quad = \|h_n - u^* - \lambda_n(\xi_n - p_n) - \lambda_n\theta_n h_n\|_H^2 \\
&\quad = \|h_n - u^*\|_H^2 - 2\lambda_n\langle \xi_n - p_n + \theta_n h_n, h_n - u^*\rangle_H + \lambda_n^2\|\xi_n - p_n + \theta_n h_n\|_H^2.
\end{aligned} \tag{3.3}$$

Since $v^* \in Fu^*, \xi_n \in Fh_n$, and F is monotone, we get that

$$\begin{aligned}
&\langle \xi_n - p_n + \theta_n h_n, h_n - u^*\rangle_H \\
&\quad = \langle \xi_n - v^* + v^* - p_n + \theta_n h_n, h_n - u^*\rangle_H \\
&\quad = \langle \xi_n - v^*, h_n - u^*\rangle_H + \langle v^* - p_n, h_n - u^*\rangle_H + \theta_n\langle h_n - u^* + u^*, h_n - u^*\rangle_H \\
&\quad \ge -\langle p_n - v^*, h_n - u^*\rangle_H + \theta_n\langle u^*, h_n - u^*\rangle_H + \theta_n\|h_n - u^*\|_H^2.
\end{aligned} \tag{3.4}$$

Also,

$$\begin{aligned}
\|\xi_n - p_n + \theta_n h_n\|_H^2 &\le \big(\|\xi_n - p_n\|_H + \theta_n\|u_n + c_n(u_{n-1} - u_n)\|_H\big)^2 \\
&\le \big(\|\xi_n - p_n\|_H + (1 + c_n)\|u_n - u^*\|_H + \|u^*\|_H + c_n\|u_{n-1} - u^*\|_H\big)^2 \\
&\le \big(\|\xi_n - p_n\|_H + 2\|u_n - u^*\|_H + \|u^*\|_H + \|u_{n-1} - u^*\|_H\big)^2 \\
&\le \left(\|\xi_n - p_n\|_H + \frac{7}{2}r\right)^2 \le M_1.
\end{aligned} \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we obtain

$$\begin{aligned}
&\|u_{n+1} - u^*\|_H^2 \\
&\quad \le (1 - 2\lambda_n\theta_n)\|h_n - u^*\|_H^2 + 2\lambda_n\langle p_n - v^*, h_n - u^*\rangle - 2\lambda_n\theta_n\langle u^*, h_n - u^*\rangle + \lambda_n^2 M_1.
\end{aligned} \tag{3.6}$$

Similarly,

$$\begin{aligned}
&\|v_{n+1} - v^*\|_H^2 \\
&\quad = \|p_n - v^* - \lambda_n(\eta_n + h_n) - \lambda_n\theta_n p_n\|_H^2 \\
&\quad = \|p_n - v^*\|_H^2 - 2\lambda_n\langle \eta_n + h_n + \theta_n p_n, p_n - v^*\rangle_H + \lambda_n^2\|\eta_n + h_n + \theta_n p_n\|_H^2,
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
\|\eta_n + h_n + \theta_n p_n\|_H^2 &\le \big(\|\eta_n + h_n\|_H + \theta_n\|v_n + c_n(v_{n-1} - v_n)\|_H\big)^2 \\
&\le \big(\|\eta_n + h_n\|_H + (1 + c_n)\|v_n - v^*\|_H + \|v^*\|_H + c_n\|v_{n-1} - v^*\|_H\big)^2
\end{aligned}$$

$$\leq \left(\|\eta_n + h_n\|_H + 2\|v_n - v^*\|_H + \|v^*\|_H + \|v_{n-1} - v^*\|_H\right)^2$$
$$\leq \left(\|\eta_n + h_n\|_H + \frac{7}{2}r\right)^2 \leq M_2. \tag{3.8}$$

Applying inequality (3.8) in (3.7), we have

$$\|v_{n+1} - v^*\|_H^2 \leq \|p_n - v^*\|_H^2 - 2\lambda_n\langle \eta_n + h_n + \theta_n p_n, p_n - v^*\rangle_H + \lambda_n^2 M_2. \tag{3.9}$$

Since K is monotone, $u^* \in -Kv^*$, and $\eta_n \in Kp_n$, we have

$$\begin{aligned}&\langle \eta_n + h_n + \theta_n p_n, p_n - v^*\rangle_H \\ &\quad = \langle \eta_n + u^*, p_n - v^*\rangle_H + \langle h_n - u^*, p_n - v^*\rangle_H + \theta_n\langle p_n - v^* + v^*, p_n - v^*\rangle_H \\ &\quad \geq \langle h_n - u^*, p_n - v^*\rangle_H + \theta_n\langle v^*, p_n - v^*\rangle_H + \theta_n\|p_n - v^*\|_H^2.\end{aligned} \tag{3.10}$$

Using inequality (3.10) in (3.9), we get

$$\begin{aligned}&\|v_{n+1} - v^*\|_H^2 \\ &\quad \leq (1 - 2\lambda_n\theta_n)\|p_n - v^*\|_H^2 - 2\lambda_n\langle p_n - v^*, h_n - u^*\rangle_H \\ &\qquad - 2\lambda_n\theta_n\langle v^*, p_n - v^*\rangle_H + \lambda_n^2 M_2.\end{aligned} \tag{3.11}$$

Adding (3.6) and (3.11), we obtain

$$\begin{aligned}\|w_{n+1} - w^*\|_E^2 &\leq (1 - 2\lambda_n\theta_n)\|r_n - w^*\|_E^2 - 2\lambda_n\theta_n\langle w^*, r_n - w^*\rangle_E + \lambda_n^2 M \\ &\leq (1 - 2\lambda_n\theta_n)\|r_n - w^*\|_E^2 - 2\lambda_n\theta_n\langle w^*, r_n - w^*\rangle_E + \lambda_n^2 M.\end{aligned} \tag{3.12}$$

Observe that

$$-2\langle w^*, r_n - w^*\rangle_E \leq 2\|w^*\|_E\|r_n - w^*\|_E \leq \|w^*\|_E^2 + \|r_n - w^*\|_E^2. \tag{3.13}$$

Substituting (3.13) into (3.12), we see that

$$\|w_{n+1} - w^*\|_E^2 \leq (1 - \lambda_n\theta_n)\|r_n - w^*\|_E^2 + \lambda_n\theta_n\|w^*\|_E^2 + \lambda_n^2 M. \tag{3.14}$$

Furthermore, we have the following estimate:

$$\begin{aligned}\|r_n - w^*\|_E &= \|(1 - c_n)w_n + c_n w_{n-1} - w^*\|_E \\ &= \|(1 - c_n)(w_n - w^*) - c_n(w^* - w_{n-1})\|_E \leq r.\end{aligned} \tag{3.15}$$

Now taking $\gamma_0 = \frac{r^2}{4M}$ and using the assumption that $\lambda_n \leq \gamma_0\theta_n$, by (3.15) we have that

$$\begin{aligned}\|w_{n+1} - w^*\|_E^2 &= (1 - \lambda_n\theta_n)r^2 + \frac{1}{4}\lambda_n\theta_n r^2 + \frac{1}{4}\lambda_n\theta_n r^2 \\ &= \left(1 - \frac{\lambda_n\theta_n}{2}\right)r^2 \leq r^2.\end{aligned} \tag{3.16}$$

Thus $\{w_n\}$ is bounded, which implies that $\{u_n\}$ and $\{v_n\}$ are bounded.

The next step is showing that there exists a sequence $\{z_n\}, z_n := (x_n, y_n) \in E, n \geq 1$, such that

$$\theta_n x_n + \hat{x}_n - y_n = 0, \quad \hat{x}_n \in Fx_n,$$
$$\theta_n y_n + \hat{y}_n + x_n = 0, \quad \hat{y}_n \in Ky_n,$$

$x_n \to x^*$, $y_n \to y^*$ with $y^* \in Fx^*$, and $0 \in x^* + KFx^*$. By Lemma 2.4 the map T defined in (1.14) is maximal monotone. Moreover, since $\theta_n \to 0$, by Lemma 2.3 we get that

$$\lim_{n\to\infty}\left(I + \frac{1}{\theta_n}T\right)^{-1}(0,0) \in T^{-1}(0,0). \tag{3.17}$$

Setting $(x_n, y_n) := (I + \frac{1}{\theta_n}T)^{-1}(0,0)$ for $n \geq 1$, we have

$$(0,0) \in \left(I + \frac{1}{\theta_n}T\right)(x_n, y_n), \quad \forall n \geq 1.$$

Thus we have

$$(x_n, y_n) + \frac{1}{\theta_n}(\hat{x}_n - y_n, \hat{y}_n + x_n) = (0,0)$$

for some $\hat{x}_n \in Fx_n$ and $\hat{y}_n \in Ky_n$. This implies that

$$\begin{cases} x_n + \frac{1}{\theta_n}(\hat{x}_n - y_n) = 0, & \hat{x}_n \in Fx_n, n \geq 1, \\ y_n + \frac{1}{\theta_n}(\hat{y}_n + x_n) = 0, & \hat{y}_n \in Ky_n, n \geq 1. \end{cases}$$

Consequently,

$$\begin{cases} \theta_n x_n + \hat{x}_n - y_n = 0, & \hat{x}_n \in Fx_n, n \geq 1, \\ \theta_n y_n + \hat{y}_n + x_n = 0, & \hat{y}_n \in Ky_n, n \geq 1. \end{cases} \tag{3.18}$$

Using (3.17), we have $z_n \to z^* = (x^*, y^*) \in T^{-1}(0,0)$. However, we have that

$$\begin{aligned} (x^*, y^*) \in T^{-1}(0,0) &\Leftrightarrow (0,0) \in T(x^*, y^*) \\ &\Leftrightarrow (0,0) \in (Fx^* - y^*) \times (Ky^* + x^*) \\ &\Leftrightarrow y^* \in Fx^* \quad \text{and} \quad 0 \in x^* + Ky^* \\ &\Rightarrow 0 \in x^* + KFx^*. \end{aligned}$$

Thus $x_n \to x^*$ and $y_n \to y^*$ with $y^* \in Fx^*$, and $0 \in x^* + KFx^*$.

The final step is showing that $w_n \to (u^*, v^*)$, where $0 \in u^* + KFu^*$ and $v^* \in Fu^*$. Hence it suffices to show that $w_{n+1} - z_n \to 0$ as $n \to \infty$. Now

$$\begin{aligned} \|u_{n+1} - x_n\|_H^2 &= \left\|h_n - x_n - \lambda_n(\xi_n - p_n) - \lambda_n\theta_n h_n\right\|_H^2 \\ &= \|h_n - x_n\|_H^2 - 2\lambda_n\langle \xi_n - p_n + \theta_n h_n, h_n - x_n\rangle_H + \lambda_n^2\|\xi_n - p_n + \theta_n h_n\|_H^2. \end{aligned}$$

Since, by (3.18), $\theta_n x_n = y_n - \hat{x}_n$, we observe that

$$\begin{aligned}
&\langle \xi_n - p_n + \theta_n h_n, h_n - x_n \rangle \\
&\quad = \langle \xi_n - \hat{x}_n + \hat{x}_n - p_n + \theta_n (h_n - x_n + x_n), h_n - x_n \rangle \\
&\quad = \langle \xi_n - \hat{x}_n, h_n - x_n \rangle + \langle \hat{x}_n - p_n, h_n - x_n \rangle + \theta_n \langle h_n - x_n + x_n, h_n - x_n \rangle \\
&\quad = \langle \xi_n - \hat{x}_n, h_n - x_n \rangle + \langle \hat{x}_n - p_n, h_n - x_n \rangle + \langle y_n - \hat{x}_n, h_n - x_n \rangle + \theta_n \| h_n - x_n \|_H^2 \\
&\quad = \langle \xi_n - \hat{x}_n, h_n - x_n \rangle + \langle y_n - p_n, h_n - x_n \rangle + \theta_n \| h_n - x_n \|_H^2
\end{aligned}$$

for $\xi_n \in Fh_n$ and some $\hat{x}_n \in Fx_n$, so that

$$\begin{aligned}
\| u_{n+1} - x_n \|_H^2 = {} & (1 - 2\lambda_n \theta_n) \| h_n - x_n \|_H^2 - 2\lambda_n \langle \xi_n - \hat{x}_n, h_n - x_n \rangle \\
& - 2\lambda_n \langle y_n - p_n, h_n - x_n \rangle + \lambda_n^2 \| \xi_n - p_n + \theta_n h_n \|_H^2.
\end{aligned}$$

Using the monotonicity and boundedness of F and employing the boundedness of the sequences $\{u_n\}$ and $\{v_n\}$, we get that there exists $M_3 > 0$ such that

$$\| u_{n+1} - x_n \|_H^2 \le (1 - 2\lambda_n \theta_n) \| h_n - x_n \|_H^2 - 2\lambda_n \langle y_n - p_n, h_n - x_n \rangle + \lambda_n^2 M_3. \tag{3.19}$$

From (3.18) we get that $\theta_n y_n = -(x_n + \hat{y}_n)$. Applying similar arguments, we obtain

$$\| v_{n+1} - y_n \|_H^2 \le (1 - 2\lambda_n \theta_n) \| p_n - y_n \|_H^2 + 2\lambda_n \langle y_n - p_n, h_n - x_n \rangle + \lambda_n^2 M_4 \tag{3.20}$$

for some constant $M_4 > 0$. Consequently, adding (3.19) and (3.20), we have

$$\| w_{n+1} - z_n \|_E^2 \le (1 - 2\lambda_n \theta_n) \| r_n - z_n \|_E^2 + \lambda_n^2 M_o, \tag{3.21}$$

where $M_o := M_3 + M_4$. Using the monotonicity of T and the Kato inequality (2.2), we have

$$\begin{aligned}
\| z_{n-1} - z_n \|_E^2 \le {} & \left\| x_{n-1} - x_n + \theta_n^{-1} (\hat{x}_{n-1} - y_{n-1} - \hat{x}_n + y_n) \right\|_H^2 \\
& + \left\| y_{n-1} - y_n + \theta_n^{-1} (\hat{y}_{n-1} + x_{n-1} - \hat{y}_n - x_n) \right\|_H^2,
\end{aligned} \tag{3.22}$$

where $\hat{x}_{n-1} \in Fx_{n-1}, \hat{x}_n \in Fx_n, \hat{y}_{n-1} \in Ky_{n-1}$, and $\hat{y}_n \in Ky_n$. Using (3.18), we have that

$$x_{n-1} - x_n + \theta_n^{-1} (\hat{x}_{n-1} - y_{n-1} - \hat{x}_n + y_n) = \frac{\theta_n - \theta_{n-1}}{\theta_n} x_{n-1}, \quad \hat{x}_{n-1} \in Fx_{n-1}, \hat{x}_n \in Fx_n \tag{3.23}$$

and

$$y_{n-1} - y_n + \theta_n^{-1} (\hat{y}_{n-1} + x_{n-1} - \hat{y}_n - x_n) = \frac{\theta_n - \theta_{n-1}}{\theta_n} y_{n-1}, \quad \hat{y}_{n-1} \in Ky_{n-1}, \hat{y}_n \in Ky_n. \tag{3.24}$$

Substituting (3.23) and (3.24) into (3.22), we have

$$\| z_{n-1} - z_n \|_E \le \frac{\theta_{n-1} - \theta_n}{\theta_n} \| z_{n-1} \|_E. \tag{3.25}$$

From (3.21) we have

$$
\begin{aligned}
&\|w_{n+1}-z_n\|_E^2\\
&\quad\le (1-2\lambda_n\theta_n)\|r_n-z_n\|_E^2+\lambda_n^2M_o\\
&\quad\le (1-\lambda_n\theta_n)\|r_n-z_n\|_E^2+\lambda_n^2M_o\\
&\quad= (1-\lambda_n\theta_n)\big\|w_n-z_n+c_n(w_{n-1}-w_n)\big\|_E^2+\lambda_n^2M_o\\
&\quad= (1-\lambda_n\theta_n)\big[\|w_n-z_n\|_E^2+2c_n\langle w_n-z_n,w_{n-1}-w_n\rangle_E+c_n^2\|w_{n-1}-w_n\|_E^2\big]+\lambda_n^2M_o\\
&\quad\le (1-\lambda_n\theta_n)\|w_n-z_n\|_E^2+2c_n\|w_n-z_n\|_E\|w_{n-1}-w_n\|_E+c_n^2\|w_{n-1}-w_n\|_E^2+M_o\lambda_n^2\\
&\quad= (1-\lambda_n\theta_n)\big[\|w_n-z_{n-1}\|_E^2+2\langle w_n-z_{n-1},z_{n-1}-z_n\rangle_E+\|z_{n-1}-z_n\|_E^2\big]\\
&\qquad+2c_n\|w_n-z_n\|_E\|w_{n-1}-w_n\|_E+c_n^2\|w_{n-1}-w_n\|_E^2+M_o\lambda_n^2\\
&\quad\le (1-\lambda_n\theta_n)\|w_n-z_{n-1}\|_E^2+2\|w_n-z_{n-1}\|_E\|z_{n-1}-z_n\|_E+\|z_{n-1}-z_n\|_E^2\\
&\qquad+c_n\big(2\|w_n-z_n\|_E\|w_{n-1}-w_n\|_E+\|w_{n-1}-w_n\|_E^2\big)+M_o\lambda_n^2.
\end{aligned}
\tag{3.26}
$$

Using inequality (3.25) in (3.26), we have that

$$
\begin{aligned}
&\|w_{n+1}-z_n\|_E^2\\
&\quad\le (1-\lambda_n\theta_n)\|w_n-z_{n-1}\|_E^2+c_n\big(2\|w_n-z_n\|_E\|w_{n-1}-w_n\|_E+\|w_{n-1}-w_n\|_E^2\big)\\
&\qquad+\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)\left(2\|w_n-z_{n-1}\|_E\|z_{n-1}\|_E+\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)\|z_{n-1}\|_E^2\right)+M_o\lambda_n^2.
\end{aligned}
\tag{3.27}
$$

Thus by the boundedness of $\{w_n\}$, $\{z_n\}$, and $\{((\theta_{n-1}-\theta_n)/\theta_n)\}$ there exist $C,\hat{C}>0$ such that

$$
\|w_{n+1}-z_n\|_E^2\le(1-\lambda_n\theta_n)\|w_n-z_{n-1}\|_E^2+\lambda_n\theta_nL\left[\frac{(\frac{\theta_{n-1}}{\theta_n}-1)}{\lambda_n\theta_n}+\frac{\lambda_n}{\theta_n}\right]+\hat{C}c_n,
\tag{3.28}
$$

where $L=\max\{C,M_o\}$. By Lemma 2.2 it follows that $w_{n+1}-z_n\to 0$. This implies that $u_n\to u^*$ and $v_n\to v^*$, where $u^*=x^*$ and $v^*=y^*$. This completes the proof. □

Corollary 3.2 *Let H be a real Hilbert space, and let $F:H\to CB(H), K:H\to H$ be maximal monotone and bounded maps. For arbitrary $u_1,v_1,u_2,v_2\in H$, define the sequences $\{h_n\}$, $\{p_n\}$, $\{u_n\}$, and $\{v_n\}$ by*

$$
\begin{cases}
h_n=u_n+c_n(u_{n-1}-u_n),\\
u_{n+1}=h_n-\lambda_n(\xi_n-p_n)-\lambda_n\theta_nh_n, \quad \xi_n\in Fh_n, n\ge 2,\\
p_n=v_n+c_n(v_{n-1}-v_n),\\
v_{n+1}=p_n-\lambda_n(Kp_n+h_n)-\lambda_n\theta_np_n, \quad n\ge 2,
\end{cases}
\tag{3.29}
$$

where $\{\theta_n\}$ and $\{\lambda_n\}$ are sequences in $(0,1)$, and $\{c_n\}$ is a sequence in $[0,1)$ satisfying the conditions of Theorem 3.1. Suppose that the inclusion $0\in u+KFu$ has a solution in H. Then there exists a real constant γ_0 such that $\lambda_n\le\gamma_0\theta_n$ for all $n\ge n_0$, for some $n_0\ge 2$, and the sequence $\{u_n\}$ converges strongly to u^*, a solution of $0\in u+KFu$.

Proof It follows from the proof of Theorem 3.1 when K is single-valued. □

Corollary 3.3 *Let H be a real Hilbert space, and let $F, K : H \to H$ be maximal monotone and bounded maps. For arbitrary $u_1, v_1, u_2, v_2 \in H$, define the sequences $\{h_n\}$, $\{p_n\}$, $\{u_n\}$, and $\{v_n\}$ by*

$$\begin{cases} h_n = u_n + c_n(u_{n-1} - u_n), \\ u_{n+1} = h_n - \lambda_n(Fh_n - p_n) - \lambda_n\theta_n h_n, \quad n \geq 2, \\ p_n = v_n + c_n(v_{n-1} - v_n), \\ v_{n+1} = p_n - \lambda_n(Kp_n + h_n) - \lambda_n\theta_n p_n, \quad n \geq 2, \end{cases} \tag{3.30}$$

where $\{\theta_n\}, \{\lambda_n\}$, and $\{c_n\}$ are sequences in $(0,1)$ satisfying the conditions of Theorem 3.1. Suppose that the inclusion $0 \in u + KFu$ has a solution in H. Then there exists a real constant γ_0 such that $\lambda_n \leq \gamma_0\theta_n$ for all $n \geq n_0$, for some $n_0 \geq 2$, and the sequence $\{u_n\}$ converges strongly to u^*, a solution of $0 \in u + KFu$.

Proof It follows from the proof of Theorem 3.1 when F and K are single-valued. □

Definition 3.4 (Hemicontinuity) An operator $A : H \longrightarrow H$ is said to be hemicontinuous at a point x_0 if for all $x \in H$, $A(x_0 + t_n x) \rightharpoonup A(x_0)$ as $t_n \to 0^+$.

Theorem 3.5 ([1, p. 29]) *Let X be a reflexive Banach space, and let X^* be its dual space. Then every monotone hemicontinuous operator $A : X \longrightarrow X^*$ is maximal monotone.*

Corollary 3.6 *Let H be a real Hilbert space, and let $F, K : H \to H$ be hemicontinuous, monotone, and bounded maps. For arbitrary $u_1, v_1, u_2, v_2 \in H$, define the sequences $\{h_n\}$, $\{p_n\}$, $\{u_n\}$, and $\{v_n\}$ by*

$$\begin{cases} h_n = u_n + c_n(u_{n-1} - u_n), \\ u_{n+1} = h_n - \lambda_n(Fh_n - p_n) - \lambda_n\theta_n h_n, \quad n \geq 2, \\ p_n = v_n + c_n(v_{n-1} - v_n), \\ v_{n+1} = p_n - \lambda_n(Kp_n + h_n) - \lambda_n\theta_n p_n, \quad n \geq 2, \end{cases} \tag{3.31}$$

where $\{\theta_n\}$ and $\{\lambda_n\}$ are sequences in $(0,1)$, and $\{c_n\}$ is a sequence in $[0,1)$ satisfying the conditions of Theorem 3.1. Suppose that the inclusion $0 \in u + KFu$ has a solution in H. Then there exists a real constant γ_0 such that $\lambda_n \leq \gamma_0\theta_n$ for all $n \geq n_0$, for some $n_0 \geq 2$, and the sequence $\{u_n\}$ converges strongly to u^*, a solution of $0 \in u + KFu$.

Proof The maps F and K are maximal monotone by Theorem 3.5. The rest of the proof follows from Corollary 3.3. □

Remark 1 Note that if $0 < b < a$ and $a + b < 1$, then for each natural number n, $\lambda_n = (n+1)^{-a}$ and $\theta_n = (n+1)^{-b}$ satisfy the hypotheses of Theorem 3.1 (see [16]).

4 Algorithms for comparison

Mainly for numerical comparison, we present the following algorithms.

Theorem 4.1 (Chidume and Shehu (CS12) [24]) *Let H be a real Hilbert space, and let $F, K : H \to H$ be bounded, continuous, and monotone mappings. Let $\{u_n\}$ and $\{v_n\}$ be sequences defined iteratively from arbitrary $u_1, v_1 \in H$ by*

$$\begin{aligned} u_{n+1} &= u_n - \beta_n(Fu_n - v_n) - \beta_n(u_n - u_1), \quad n \ge 0, \\ v_{n+1} &= v_n - \beta_n(Kv_n + u_n) - \beta_n(v_n - v_1), \quad n \ge 0, \end{aligned} \tag{4.1}$$

where $\{\beta_n\}_n^\infty$ is a real sequence in $(0,1)$ such that $\sum_{n=0}^\infty \beta_n^2 < \infty$ and $\sum_{n=0}^\infty \beta_n = \infty$. Then the sequence $\{u_n\}$ converges strongly to u^*, a solution of $u + KFu = 0$.

Theorem 4.2 (Chidume and Shehu (CS13) [25]) *For $q > 1$, let E be a q-uniformly smooth real Banach space. For each $i = 1, 2, \dots, m$, let $F_i, K_i : E \to E$ be bounded and accretive mappings. Let $\{u_n\}$ and $\{v_{i,n}\}_{n=1}^\infty$, $i = 1, 2, \dots, m$, be sequences defined iteratively from arbitrary $u_1, v_{i,1} \in E$ by*

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n\alpha_n\left(u_n + \sum_{i=1}^m K_i v_{i,n}\right) - \lambda_n\theta_n(u_n - u_1), \\ v_{1,n+1} &= v_{1,n} - \lambda_n\alpha_n(F_1 u_n - v_{1,n}) - \lambda_n\theta_n(v_{1,n} - v_{1,1}), \\ v_{2,n+1} &= v_{2,n} - \lambda_n\alpha_n(F_2 u_n - v_{2,n}) - \lambda_n\theta_n(v_{2,n} - v_{1,1}), \\ &\vdots \\ v_{m,n+1} &= v_{m,n} - \lambda_n\alpha_n(F_m u_n - v_{m,n}) - \lambda_n\theta_n(v_{m,n} - v_{m,1}), \quad n \ge 0, \end{aligned} \tag{4.2}$$

where $\{\lambda_n\}_n^\infty$, $\{\alpha_n\}_n^\infty$, $\{\theta_n\}_n^\infty$ are real sequences in $(0,1)$ such that $\lambda_n = o(\theta_n), \alpha_n = o(\theta_n)$, and $\sum_{n=0}^\infty \lambda_n\theta_n = \infty$. Then the sequence $\{u_n\}$ converges strongly to u^*, a solution of $u + \sum_{i=1}^m K_i F_i u = 0$.

Theorem 4.3 (Shehu (S14) [41]) *Let H be a real Hilbert space, and let $F : H \to H$ be a bounded, coercive, and maximal monotone mapping. Let $F : H \to H$ be a bounded and maximal monotone mapping. Suppose that both maps satisfy the range condition. Let $\{u_n\}$ and $\{v_n\}$ be sequences defined iteratively from arbitrary $u_1, v_1 \in H$ by*

$$\begin{aligned} u_{n+1} &= u_n - \beta_n^2(Fu_n - v_n) - \beta_n(u_n - u_1), \quad n \ge 0, \\ v_{n+1} &= v_n - \beta_n^2(Kv_n + u_n) - \beta_n(v_n - v_1), \quad n \ge 0, \end{aligned} \tag{4.3}$$

where $\{\beta_n\}_n^\infty$ is a real sequence in $(0,1)$ such that $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^\infty \beta_n^2 < \infty$, and $\sum_{n=0}^\infty \beta_n = \infty$. Suppose that u^* is a solution of $u + KFu = 0$. Then the sequence $\{u_n\}$ converges strongly to u^*.

5 Numerical examples

In this section, we present some numerical examples illustrating the advantages of the present algorithm (3.1) over MM (1.17), CS12 (4.1), CS13 (4.2), and S14 (4.3). We construct these examples on $\mathbb{R}^p, p \ge 1$, $\ell_2(\mathbb{R})$, and $\mathcal{L}^2_{\mathbb{R}}(\Omega)$, where Ω is a closed bounded interval of $\mathbb{R}$. Numerical experiments were carried out on MATLAB R2013a and R2020 versions.

Table 1 Parameters used during numerical experiment

BOY (3.1)	MM (1.17)	CS12 (4.1)	CS13 (4.2)	S14 (4.3)
$\lambda_n = (n+1)^{-\frac{1}{4}}$ $\theta_n = (n+1)^{-\frac{1}{5}}$	$\lambda_n = (n+1)^{-\frac{1}{4}}$ $\theta_n = (n+1)^{-\frac{1}{5}}$	$\beta_n = (n+1)^{-1}$	$\lambda_n = (n+1)^{-\frac{1}{4}}$ $\theta_n = (n+1)^{-\frac{1}{5}}$ $\alpha_n = (n+1)^{-\frac{1}{4}}$	$\beta_n = (n+1)^{-1}$

Table 2 Numerical results for Example 1

p	N	BOY (3.1)	MM (1.17)	CS12 (4.1)	CS13 (4.2)	S14 (4.3)
50	100	1.615651*e*–01	3.685202*e*+00	6.321833*e*+00	3.329286*e*+01	7.225393*e*+00
50	1000	9.002337*e*–02	2.254541*e*+00	6.321075*e*+00	1.821770*e*+01	7.099678*e*+00
50	10 000	5.025679*e*–02	1.029760*e*+00	6.321068*e*+00	1.184520*e*+01	7.075314*e*+00
500	100	1.615651*e*–01	1.430221*e*+01	2.161299*e*+01	9.710598*e*+01	2.293779*e*+01
500	1000	9.002337*e*–02	1.023801*e*+01	2.161070*e*+01	5.319094*e*+01	2.246502*e*+01
500	10 000	5.025679*e*–02	6.733823*e*+00	2.161068*e*+01	3.458890*e*+01	2.237599*e*+01
5000	100	1.615651*e*–01	4.787614*e*+01	6.996787*e*+01	2.989008*e*+02	7.262474*e*+01
5000	1000	9.002337*e*–02	3.548396*e*+01	6.996075*e*+01	1.637860*e*+02	7.105451*e*+01
5000	10 000	5.025679*e*–02	2.477166*e*+01	6.996068*e*+01	1.065108*e*+02	7.076099*e*+01

Table 3 Computation times for all algorithms in Table 2

p	N	BOY (3.1)	MM (1.17)	CS12 (4.1)	CS13 (4.2)	S14 (4.3)
50	100	1.562500*e*–02	0	0	1.562500*e*–02	0
50	1000	7.812500*e*–02	9.375000*e*–02	1.406250*e*–01	1.093750*e*–01	1.406250*e*–01
50	10 000	2.593750*e*+00	2.546875*e*+00	2.656250*e*+00	2.515625*e*+00	2.578125*e*+00
500	100	1.562500*e*–02	0	1.562500*e*–02	0	1.562500*e*–02
500	1000	4.218750*e*–01	2.656250*e*–01	3.593750*e*–01	3.593750*e*–01	3.281250*e*–01
500	10 000	4.878125*e*+01	4.646875*e*+01	4.451563*e*+01	4.451563*e*+01	4.439063*e*+01
5000	100	2.500000*e*–01	1.562500*e*–01	1.406250*e*–01	2.031250*e*–01	1.093750*e*–01
5000	1000	5.875000*e*+00	6.359375*e*+00	6.890625*e*+00	6.406250*e*+00	6.437500*e*+00
5000	10 000	4.935313*e*+02	4.780469*e*+02	4.635000*e*+02	4.800625*e*+02	4.840781*e*+02

Table 4 Numerical results for different choices of $\{c_n\}$ for BOY (3.1) in Example 1

p	N	$c_n = 0$	$c_n = n^{-3}$	$c_n = n^{-\frac{3}{2}}$	$c_n = n^{-2}$
50	100	1.615480*e*–01	1.615482*e*–01	1.615497*e*–01	1.615651*e*–01
50	1000	9.002328*e*–02	9.002328*e*–02	9.002328*e*–02	9.002337*e*–02
50	10 000	5.025678*e*–02	5.025678*e*–02	5.025678*e*–02	5.025679*e*–02
500	100	1.615480*e*–01	1.615482*e*–01	1.615497*e*–01	1.615651*e*–01
500	1000	9.002328*e*–02	9.002328*e*–02	9.002328*e*–02	9.002337*e*–02
500	10 000	5.025678*e*–02	5.025678*e*–02	5.025678*e*–02	5.025679*e*–02

All programs were run on PCs with Intel(R) Core(TM)2 Duo CPU and 3 GB RAM and Intel(R) Core(TM) i5-1035G1 CPU@1.00 GHz with 12 GB RAM.

The accompanying parameters of the aforementioned methods employed in the experiments are presented in Table 1, whereas their numerical results and computation times are illustrated in Tables 2–9 and Figs. 1–7 in *loglog* plots. In these tables, p represents the dimension of the real line in Example 1, s denotes the index defined in the sequences given in Example 2 (see, e.g., Eq. (5.2)), N is the maximum number of iterations, and n is the index of each sequence. For experimental purpose, n is looped from 0 to N. The term $\|u_n\|_2$ denotes the second norm at index n. In Example 3, j represents the number of partitions on $[0, 1]$.

Table 5 Computation times for BOY (3.1) in Table 4

p	N	$c_n = 0$	$c_n = n^{-3}$	$c_n = n^{-\frac{3}{2}}$	$c_n = n^{-2}$
50	100	1.562500e–02	0	0	0
50	1000	6.250000e–02	6.250000e–02	1.250000e–01	1.875000e–01
50	10 000	3.968750e+00	2.843750e+00	3.406250e+00	3.796875e+00
500	100	0	0	1.562500e–02	1.562500e–02
500	1000	4.218750e–01	4.218750e–01	3.906250e–01	3.906250e–01
500	10 000	5.395313e+01	5.226563e+01	5.281250e+01	5.184375e+01

Table 6 Numerical results for Example 2

s	N	BOY (3.1)	MM (1.17)	CS12 (4.1)	CS13 (4.2)	S14 (4.3)
50	10	3.192044e–01	5.330595e–01	4.163250e–01	3.814988e+00	1.151610e+00
50	100	1.615651e–01	2.501438e–04	3.994505e–01	6.786458e+00	1.252605e+00
50	1000	9.002337e–02	9.722273e–02	3.992966e–01	3.599628e+00	1.271957e+00
50	10 000	5.025679e–02	5.025675e–02	3.992950e–01	2.246459e+00	1.274467e+00
500	10	3.192128e–01	5.262828e–01	4.241019e–01	3.875973e+00	1.159661e+00
500	100	1.615651e–01	1.332668e–02	4.071295e–01	6.837405e+00	1.259847e+00
500	1000	9.002337e–02	2.171271e–01	4.069747e–01	3.628486e+00	1.278969e+00
500	10 000	5.025679e–02	5.025675e–02	4.069731e–01	2.267003e+00	1.281438e+00
5000	10	3.192135e–01	5.256572e–01	4.248883e–01	3.882197e+00	1.160474e+00
5000	100	1.615651e–01	1.443111e–02	4.079060e–01	6.842619e+00	1.260576e+00
5000	1000	9.002337e–02	2.170950e–01	4.077510e–01	3.631443e+00	1.279676e+00
5000	10 000	5.025679e–02	5.025675e–02	4.077495e–01	2.269110e+00	1.282140e+00

Table 7 Computation times for all algorithms in Table 6

s	N	BOY (3.1)	MM (1.17)	CS12 (4.1)	CS13 (4.2)	S14 (4.3)
50	10	0	0	0	4.687500e–02	0
50	100	0	0	0	1.562500e–02	0
50	1000	6.250000e–02	6.250000e–02	1.406250e–01	1.406250e–01	1.406250e–01
50	10 000	2.890625e+00	2.656250e+00	2.890625e+00	2.656250e+00	3.078125e+00
500	10	0	0	0	0	0
500	100	3.125000e–02	1.562500e–02	0	1.562500e–02	0
500	1000	3.125000e–01	2.500000e–01	2.343750e–01	2.656250e–01	2.343750e–01
500	10 000	4.171875e+01	3.965625e+01	3.954688e+01	3.937500e+01	3.950000e+01
5000	10	9.375000e–02	0	7.812500e–02	0	3.125000e–02
5000	100	2.187500e–01	2.343750e–01	1.875000e–01	1.562500e–01	1.406250e–01
5000	1000	1.184375e+01	5.156250e+00	5.312500e+00	5.453125e+00	5.640625e+00
5000	10 000	4.465469e+02	4.397813e+02	4.288750e+02	4.372031e+02	4.359844e+02

Table 8 Numerical results for Example 3

j	N	BOY (3.1)	MM (1.17)	CS12 (4.1)	CS13 (4.2)	S14 (4.3)
10	10	1.152918e–02	2.319396e–01	3.186175e–01	3.490227e–01	6.480232e–01
10	100	3.296270e–24	1.579375e–01	3.235187e–01	3.741739e–01	7.676897e–01
10	1000	6.196310e–136	1.045951e–01	3.234724e–01	4.003847e–01	7.960680e–01
100	10	1.167714e–02	2.315197e–01	3.183921e–01	3.487705e–01	6.480069e–01
100	100	1.547816e–24	1.577588e–01	3.232611e–01	3.739162e–01	7.679286e–01
100	1000	1.917870e–136	1.044657e–01	3.232151e–01	4.001309e–01	7.963726e–01

Table 9 Computation times for all algorithms in Table 8

j	N	BOY (3.1)	MM (1.17)	CS12 (4.1)	CS13 (4.2)	S14 (4.3)
10	10	0	0	0	0	0
10	100	0	0	0	0	0
10	1000	0	1.562500e–02	1.562500e–02	1.562500e–02	3.125000e–02
100	10	0	0	0	0	0
100	100	0	0	0	0	0
100	1000	6.250000e–02	3.125000e–02	4.687500e–02	4.687500e–02	4.687500e–02

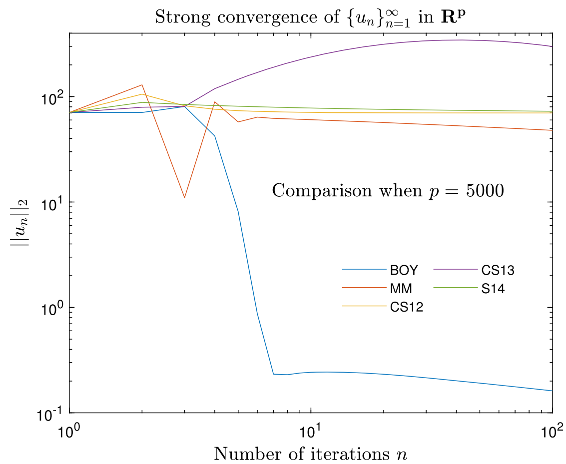

Figure 1 Strong convergence of u_n in $\mathbb{R}^p$

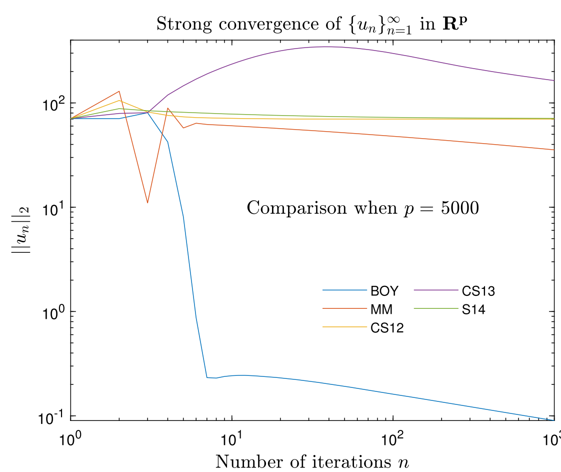

Figure 2 Strong convergence of u_n in $\mathbb{R}^p$

Example 1 Let $f : \mathbb{R}^p \longrightarrow \mathbb{R} : x \mapsto \|x\|$. Then the subdifferential of f at x (see, e.g., [36]) is given by

$$\partial f(x) = \begin{cases} \{\frac{x}{\|x\|}\}, & x \neq 0, \\ \{x \in \mathbb{R}^p : \|x\| \leq 1\}, & x = 0. \end{cases} \tag{5.1}$$

It is well known that $F = \partial f$ defined in (5.1) is maximal monotone. Also, define $K : \mathbb{R}^p \longrightarrow \mathbb{R}^p$ by $Kx = x$ for all $x \in \mathbb{R}^p$. Clearly, K is maximal monotone. Moreover, the only solution of the inclusion $0 \in u + KFu$ is $u^* = 0$. For BOY, the present algorithm, the initial points in $\mathbb{R}^p$ are $u_1, v_1, u_2, v_2 = (1, 1, \dots, 1)$, whereas $u_1, v_1 = (1, 1, \dots, 1)$ in the other algorithms.

As tabulated in Table 2 and illustrated in Figs. 1–3, the present algorithm appears to be more efficient at approximating the zero of Eq. (5.1) as compared to others. In loops the presence of an inertial term leads to more computational time. However, as depicted in

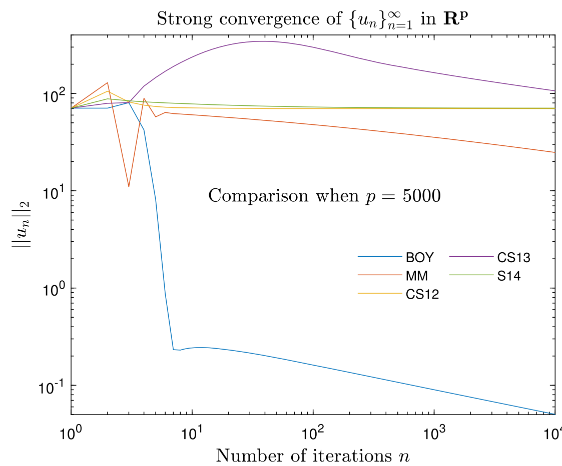

Figure 3 Strong convergence of u_n in $\mathbb{R}^p$

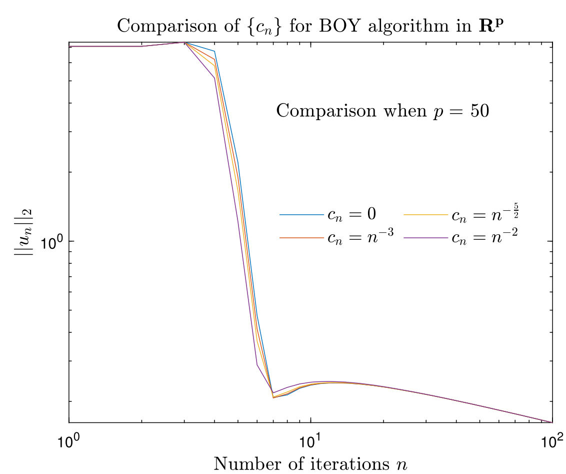

Figure 4 Comparison of c_n for BOY algorithm in $\mathbb{R}^p$

Table 3, the present algorithm—containing two inertial terms—still competes favorably with others.

Example 2 Define a map $f : \ell_2(\mathbb{R}) \longrightarrow \mathbb{R}$ by $f(x) = \|x\|$ for all $x \in \ell_2(\mathbb{R})$. Then the subdifferential of f at $x \in \ell_2(\mathbb{R})$ is given by

$$\partial f(x) = \begin{cases} \{\frac{x}{\|x\|}\}, & x \neq 0, \\ \overline{B}(0,1), & x = 0. \end{cases}$$

It is well known that $F := \partial f$ is maximal monotone. Also, define the map $K : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ by $Kx = 2x$ for all $x \in \ell_2(\mathbb{R})$. Obviously, K is monotone and bounded. Consider the points

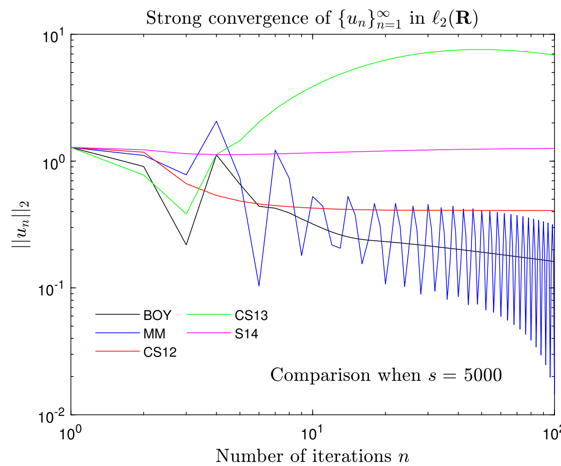

Figure 5 Strong convergence of u_n in $l_2(\mathbb{R})$

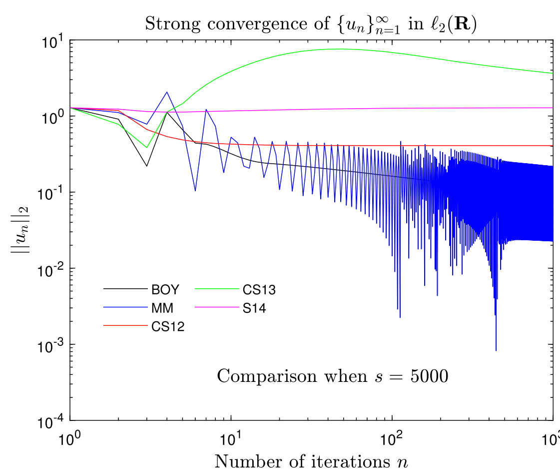

Figure 6 Strong convergence of u_n in $l_2(\mathbb{R})$

$u_1 := \{u_{1,j}\}_{j=1}^{\infty}$, $v_1 := \{v_{1,j}\}_{j=1}^{\infty}$, $u_2 := \{u_{2,j}\}_{j=1}^{\infty}$, and $v_2 := \{v_{2,j}\}_{j=1}^{\infty}$ defined by

$$u_{1,j} = \begin{cases} \frac{1}{j} & 1 \leq j \leq s, \\ 0 & \text{otherwise,} \end{cases} \qquad v_{1,j} = \begin{cases} \frac{1}{j+1} & 1 \leq j \leq s, \\ 0 & \text{otherwise,} \end{cases} \tag{5.2}$$

$$u_{2,j} = \begin{cases} \frac{1}{j+\sin^2 j} & 1 \leq j \leq \lceil \frac{s}{2} \rceil, \\ \frac{1}{j(j+1)} & \lceil \frac{s}{2} \rceil < j \leq s, \\ 0 & \text{otherwise,} \end{cases} \qquad v_{2,j} = \begin{cases} \frac{1}{j \ln j} & 3 \leq j \leq \lceil \frac{s}{2} \rceil, \\ \frac{1}{j^2+\cos j} & \lceil \frac{s}{2} \rceil < j \leq s, \\ 0 & \text{otherwise,} \end{cases} \tag{5.3}$$

where $s > 6$, $j \in \mathbb{N}$, and the ceiling $\lceil x \rceil = \min\{z \in \mathbb{Z} | z \geq x\}$. Algorithm (3.1) uses the initial points defined in (5.2) and (5.3), whereas other algorithms are initialized by (5.2) only. Obviously, u_1, u_2, v_1, and v_2 are elements of $\ell_2(\mathbb{R})$. Numerical results are displayed in Tables 6 and 7, whereas graphical illustrations are presented in Figs. 9–7. During the experiment, the algorithm MM (1.17) becomes unstable, as this can be seen from its results in Table 6

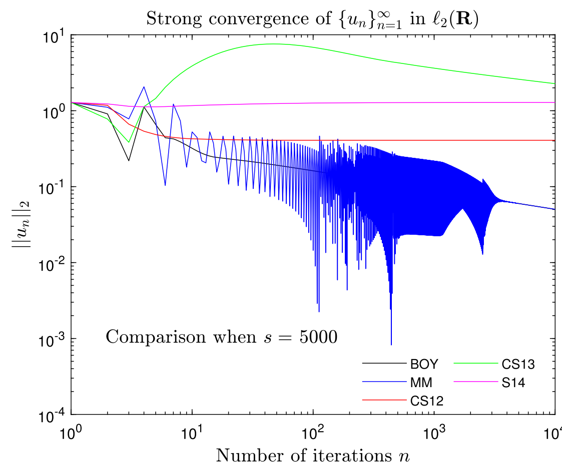

Figure 7 Strong convergence of u_n in $l_2(\mathbb{R})$

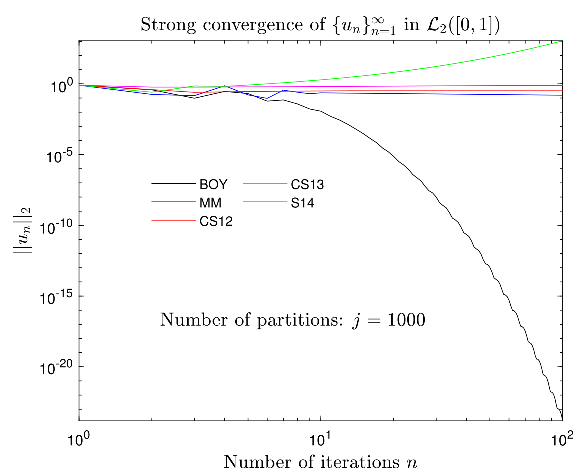

Figure 8 Strong convergence of u_n in $L_2([0,1])$

and the figures. However, BOY (3.1) remains stable and competes favorably against others in terms of computational time (see Table 7), thus gaining advantage over others.

Example 3 In this example, we set $\Omega = [0,1]$. Define $F, K : \mathcal{L}_2^{\mathbb{R}}([0,1]) \longrightarrow \mathcal{L}_2^{\mathbb{R}}([0,1])$ by $(Fu)(t) = (t+1)u(t)$ and $(Ku)(t) = u(t)$ for all $t \in [0,1]$. Then

$$\langle u, Fu\rangle = \int_0^1 u(t)(Fu)(t)\,dt = \int_0^1 (t+1)u^2(t)\,dt \geq 0 \quad \text{and}$$

$$\langle u, Ku\rangle = \int_0^1 u(t)(Ku)(t)\,dt = \int_0^1 u^2(t)\,dt \geq 0.$$

Therefore F and K are monotone. It is also clear that F and K are linear and bounded. Thus by Theorem 3.5 they are maximal monotone. In this example, we choose $u_1 = (1 + x^2)^{-1}, v_1 = x^2, u_2 = x^3$, and $v_2 = \sin x$. For the purpose of experiment, the interval $[0,1]$ is

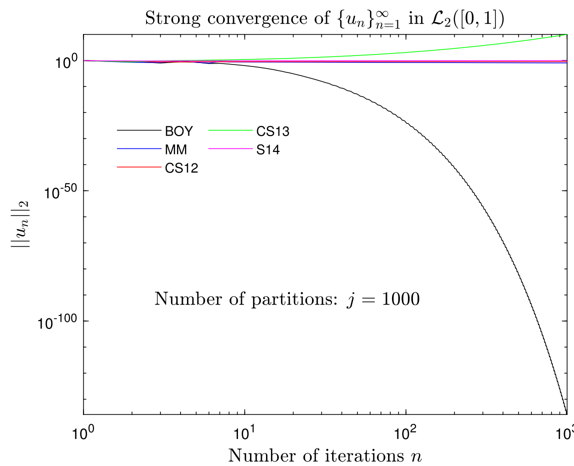

Figure 9 Strong convergence of u_n in $L_2([0,1])$

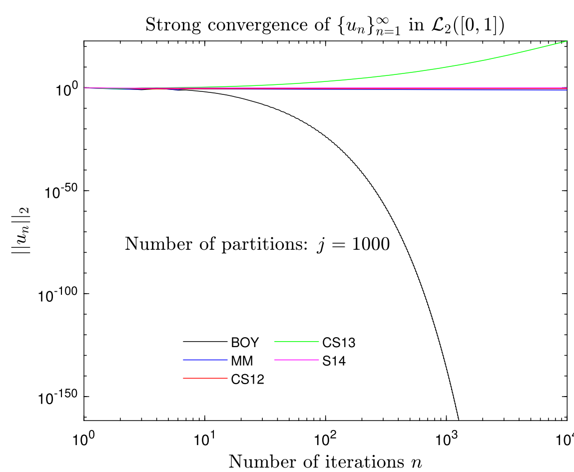

Figure 10 Strong convergence of u_n in $L_2([0,1])$

partitioned into j subintervals. The numerical output and computational time for different number of iterations N and subintervals j can be seen in Tables 8 and 9, respectively. As illustrated in those tables and Figs. 8–10, the sequence produced by BOY (3.1) evidently converges faster than its pairs.

6 Conclusion

In this paper, we introduced a novel inertial algorithm for approximating solutions of Hammerstein inclusions $0 \in u + KFu$ in Hilbert spaces. We also proved the strong convergence of the proposed scheme. Furthermore, we made no assumption of continuity in the main theorem. Moreover, the following observations were made:

1. The Inertial algorithm seems to display its efficiency when compared to the Noninertial algorithms, which perform poorly as the dimension and number of iterations increase.

2. Convergence of the Inertial algorithm seems to be independent of the choice of u_2 and v_2.
3. In sequence and function spaces the Inertial algorithm appears to be more effective and accurate.
4. On $\mathbb{R}^p, p \geq 1$, the optimal choice of c_n appears to be zero.

From the results obtained, the inertial algorithm would, perhaps, be preferred to the non-inertial algorithms in any possible application.

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Authors' contributions
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