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On a nonlinear elasticity problem with friction and Sobolev spaces with variable exponents

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Abstract

We consider a nonlinear elasticity problem in a bounded domain, its boundary is decomposed in three parts: lower, upper, and lateral. The displacement of the substance, which is the unknown of the problem, is assumed to satisfy the homogeneous Dirichlet boundary conditions on the upper part, and not homogeneous one on the lateral part, while on the lower part, friction conditions are considered. In addition, the problem is governed by a particular constitutive law of elasticity system with a strongly nonlinear strain tensor. The functional framework leads to using Sobolev spaces with variable exponents. The formulation of the problem leads to a variational inequality, for which we prove the existence and uniqueness of the solution of the associated variational problem.

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1 Introduction

The study of partial differential equation problems with variable exponents comes from the theory of nonlinear elasticity, elastic mechanics, fluid dynamics, electrorheological fluids, image processing, etc. (see [2, 15, 19]).

First, we introduce the notations needed in this article. Let Ω be a connected open bounded domain of $\mathbb{R}^{\mathbb{N}}$ ($\mathbb{N} = 3$) with Lipschitz boundary Γ. To a given field of displacement u, we associate a nonlinear deformation tensor E defined by

$$E\big(\nabla u(x)\big) = \frac{1}{2}\big({}^{T}\nabla u + \nabla u + {}^{T}\nabla u\,\nabla u\big),$$

whose components are

$$E_{ij}\big(\nabla u(x)\big) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{m=1}^{3}\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right), \quad 1 \le i,j \le 3. \tag{1.1}$$

The corresponding nonlinear constraints tensor $\sigma(u) = (\sigma_{ij}(u(x)))_{1\leq i,j\leq 3}$ is then given by

$$\sigma_{ij}\big(u(x)\big) = \sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}\big(\nabla u(x)\big), \quad 1 \leq i,j \leq 3, \tag{1.2}$$

which describes a nonlinear relation between the stress tensor $(\sigma_{ij})_{i,j=1,2,3}$ and the deformation tensor $(E_{ij})_{i,j=1,2,3}$. The coefficients of elasticity a_{ijkh} (see [3]) satisfy the following symmetry properties:

$$a_{ijkh} = a_{jikh} = a_{ijhk} \quad \text{for all } 1 \leq i,j,k,h \leq 3. \tag{1.3}$$

The aim of this paper is to prove the existence and uniqueness of weak solution for the following nonlinear problem, encountered in the theory of nonlinear elasticity [3]: Let w be a bounded domain in $\mathbb{R}^2$ situated in the plane of equation $x_3 = 0$. We suppose that w represents the lower surface of the domain occupied by the substance. The upper surface Γ_1 is defined by

$$\big\{\big(x',x_3\big) \in \mathbb{R}^3, x_3 = h\big(x'\big) \text{ and } x' \in w\big\},$$

where h is a function defined and bounded on w, that is to say, there exist h_* and h^* in $\mathbb{R}$ such that

$$0 < h_* \leq h\big(x'\big) \leq h^*, \quad \forall\big(x',0\big) = (x_1,x_2,0) \in w.$$

We study the displacement of a substance in

$$\Omega = \big\{\big(x',x_3\big) \in \mathbb{R}^3 : \big(x',0\big) \in w \text{ and } 0 < x_3 < h\big(x'\big)\big\}$$

the boundary $\partial\Omega = \Gamma = \overline{w} \cup \overline{\Gamma_1} \cup \overline{\Gamma_L}$, where Γ_L is the lateral surface of Ω.

The outer normal vector unitary on Γ is denoted by $n = (n_1, n_2, n_3)$. The outer normal vector unitary on w is the vector $(0, 0, -1)$.

Einstein's convention, which consists of making the sum on the repeated indices, will be used unless otherwise stated.

We define the normal and the tangential components u_n and $u_t = (u_{t_1}, u_{t_2}, u_{t_3})$, of the displacement variable u by

$$u_n = u.n = u_i n_i, \qquad u_{t_i} = u_i - u_n n_i, \quad i \text{ and } j = 1,2,3. \tag{1.4}$$

For normal and tangential components σ_n and $\sigma_t = (\sigma_{t_1}, \sigma_{t_2}, \sigma_{t_3})$ of the strain tensor, the definition is as follows:

$$\sigma_n = (\sigma.n).n = \sigma_{ij}n_i n_j, \qquad \sigma_{t_i} = \sigma_{ij}n_j - \sigma_n n_i \quad i \text{ and } j = 1,2,3. \tag{1.5}$$

In this section, we are interested in the following equation:

$$-\frac{\partial}{\partial x_j}\sigma_{ij}\big(u(x)\big) = f_i(x), \quad i \text{ and } j = 1,2,3, \tag{1.6}$$

where $f = (f_1,f_2,f_3)$ represents a mass density of the external forces.

For boundary conditions, it is assumed that

$$u = 0 \quad \text{on } \Gamma_1, \tag{1.7}$$

$$u = g \quad \text{on } \Gamma_L, \tag{1.8}$$

$$u.n = 0 \quad \text{on } w. \tag{1.9}$$

Condition (1.9) means that there is a tangential force exerted by the surface w on the substance. This tangential effort cannot exceed a certain threshold. The Tresca law assumes that this threshold is fixed and known

$$|\sigma_t| \leq K \quad \text{on } w, \tag{1.10}$$

where K is a given positive function called coefficient of friction and $|\sigma_t|$ is the modulus of the tangential constraint defined on w by (1.5).

As long as the tangential constraint σ_t has not reached the threshold

K, the substance moves with a given displacement s, which is the displacement of the lower surface w (adhesion). When the threshold is reached, the substance and the surface move tangentially relative to each other and there is proportional sliding. What can be summarized as follows [8]:

$$\left.\begin{array}{lll} |\sigma_t| < K & \Rightarrow & u_t = s, \\ |\sigma_t| = K & \Rightarrow & \exists\lambda \geq 0 \quad \text{such as } u_t = s - \lambda\sigma_t, \end{array}\right\} \quad \text{on } w, \tag{1.11}$$

where the positive real λ is unknown.

This problem models the behavior of a heterogeneous material with the above Tresca friction free boundary condition. The consideration of this general material is in no way restrictive. Indeed, we can apply this study to the most particular elastic materials, but this particular case makes it easy to describe the different stages of this work. The tensor of the constraints considered here is nonlinear and grouped, as special cases, some models used in Ciarlet [3], Lions [12], and Dautray and Lions [4]. Let us cite by way of example (see [3, 12]).

The complete problem (P_0) is therefore to find the displacement field u, satisfying the following equation and boundary conditions:

$$\left.\begin{array}{ll} -\frac{\partial}{\partial x_j}\sigma_{ij}(u(x)) = f_i(x) & \text{in } \Omega, 1 \leq i \leq 3, \\ \sigma_{ij}(u(x)) = \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u(x)) & \text{in } \Omega, 1 \leq i,j \leq 3, \\ E_{ij}(\nabla u(x)) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{m=1}^{3} \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right) & \text{in } \Omega, 1 \leq i,j \leq 3, \\ u = 0 \quad \text{on } \Gamma_1, & \\ u = g \quad \text{on } \Gamma_L, & \\ \left.\begin{array}{lll} u.n = 0 & & \\ |\sigma_t| < K & \Rightarrow & u_t = s, \\ |\sigma_t| = K & \Rightarrow & \exists\lambda \geq 0 \quad \text{such as } u_t = s - \lambda\sigma_t, \end{array}\right\} \text{ on } w. & \end{array}\right\} \tag{P_0}$$

We consider the functional framework of the considered problem (P_0) using Lebesgue and Sobolev spaces with variable exponents, see for example [6]. However, it is not necessary to use this notion of Lebesgue and Sobolev spaces with variable exponents to study this problem. But we see it as a good generalization to the same study with Lebesgue and Sobolev spaces with fixed exponents.

Several authors studied the system of elasticity with laws of particular behavior and using various techniques in Sobolev spaces with constant exponents. For example, in [3] Ciarlet used the implicit function theorem to show the existence and uniqueness of a solution; in [4] Dautray and Lions studied the linear problem in a regular boundary domain; in [21] Zoubai and Merouani studied the existence and uniqueness of the solutions of the nonlinear elasticity system by topological degree; and in [13, 20] Zoubai and Merouani studied the existence and uniqueness of the solution of Dirichlet's and Neumann's problems in Sobolev spaces with variable exponents.

In Sect. 2, we recall some definitions and properties of Lebesgue and Sobolev spaces with variable exponents (see for example [5–7, 10, 11] for the proofs and more details). This notion of Sobolev spaces with variable exponents is also used in many works (see for example [1, 9, 14]).

The need to work with the concept of Sobolev spaces with variable exponents is motivated by the appearance of these spaces when modeling electrorheological and thermorheological fluids (see [16]) and in image restoration (see [2]).

In Sect. 3, using this notion of Sobolev spaces with variable exponents, we give the convenient functional framework for the considered problem (P_0) to lead to variational problem 3.1. Then we prove in Theorem 3.2 the existence part by checking all hypotheses of Theorem 8.1 page 251 in [12]. And finally, in Sect. 4, we obtain also the uniqueness of the solution to variational problem (3.1).

2 Generalized Lebesgue and Sobolev spaces

Let $\Omega \subset \mathbb{R}^N$, let $p(\cdot) : \Omega \longrightarrow [1, +\infty]$ be a measurable function, called the variable exponent. In the following, we adopt the following notations:

$$C_+(\overline{\Omega}) = \left\{p(\cdot) \in C(\overline{\Omega}), p(x) \geq 1 \text{ for all } x \in \overline{\Omega}\right\},$$

and

$$p^- = \operatorname*{ess\,inf}_{x\in\Omega} p(x), \qquad p^+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

We define the generalized Lebesgue space $L^{p(\cdot)}(\Omega)$, also called Lebesgue space with variable exponent, as the set of measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modulus

$$\rho_{p(\cdot)}(u) = \int_\Omega \left|u(x)\right|^{p(x)} dx \tag{2.1}$$

is finished.

For $x \in \Omega$, $p(x) > 1$, the function of $\mathbb{R}_+ \to \mathbb{R}$ given by $Y \mapsto Y^{p(x)}$ is convex, so also the function $u \mapsto \rho_{p(\cdot)}(u)$.

Moreover, for $1 < p^+ < +\infty$, we put the function

$$u \mapsto \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) = \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}. \tag{2.2}$$

Note that $\|u\|_{L^{p(\cdot)}(\Omega)} = 0$ implies that $\lambda = 0$, then we must have $u = 0$, so that this inf bound is finite. For every $\alpha \in \mathbb{R}$ and $u \in L^{p(\cdot)}(\Omega)$, using the convexity of $\rho_{p(\cdot)}$, we have

$$\begin{aligned}\|\alpha u\|_{L^{p(\cdot)}(\Omega)} &= \inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(\frac{\alpha u + (1-\alpha)0}{\lambda}\right) \leq 1\right\} \\ &= \inf\left\{\lambda > 0 : |\alpha|\rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\} \\ &= |\alpha| \inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\} = |\alpha| \|u\|_{L^{p(\cdot)}(\Omega)}.\end{aligned}$$

Also let u and v be in $L^{p(\cdot)}(\Omega)$ such that

$$\rho_{p(\cdot)}\left(\frac{u}{\|u\|_{L^{p(\cdot)}(\Omega)}}\right) \leq 1 \quad \text{and} \quad \rho_{p(\cdot)}\left(\frac{v}{\|v\|_{L^{p(\cdot)}(\Omega)}}\right) \leq 1,$$

so

$$\begin{aligned}\rho_{p(\cdot)}\left(\frac{u+v}{\|u\|_{L^{p(\cdot)}(\Omega)} + \|v\|_{L^{p(\cdot)}(\Omega)}}\right) \leq{}& \frac{\|u\|_{L^{p(\cdot)}(\Omega)}}{\|u\|_{L^{p(\cdot)}(\Omega)} + \|v\|_{L^{p(\cdot)}(\Omega)}} \rho_{p(\cdot)}\left(\frac{u}{\|u\|_{L^{p(\cdot)}(\Omega)}}\right) \\ &+ \frac{\|v\|_{L^{p(\cdot)}(\Omega)}}{\|u\|_{L^{p(\cdot)}(\Omega)} + \|v\|_{L^{p(\cdot)}(\Omega)}} \rho_{p(\cdot)}\left(\frac{v}{\|v\|_{L^{p(\cdot)}(\Omega)}}\right) \leq 1.\end{aligned}$$

So with $\lambda = \|u\|_{L^{p(\cdot)}(\Omega)} + \|v\|_{L^{p(\cdot)}(\Omega)}$ we obtain

$$\|u+v\|_{L^{p(\cdot)}(\Omega)} \leq \|u\|_{L^{p(\cdot)}(\Omega)} + \|v\|_{L^{p(\cdot)}(\Omega)},$$

therefore the given function (2.2) defines a norm of $L^{p(\cdot)}(\Omega)$, called the norm of Luxembourg [5].

The space $(L^{p(\cdot)}(\Omega), \|.\|_{L^{p(\cdot)}(\Omega)})$ is a Banach space and $\mathcal{D}(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$. Moreover, if $p^- > 1$, $L^{p(\cdot)}(\Omega)$ is [6] uniformly convex and therefore reflexive, and its dual is isomorphic to $L^{p'(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for $x \in \Omega$.

We also have the following inequality called Hölder type inequality:

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \leq 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and all $v \in L^{p'(\cdot)}(\Omega)$.

Now we define the generalized Sobolev space also called Sobolev space with variable exponent

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega); \nabla u \in \left(L^{p(\cdot)}(\Omega)\right)^N\right\}$$

which endowed with the norm

$$u \mapsto \|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

is a Banach space.

The space $W_0^{1,p(\cdot)}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

Let $p(\cdot) \in C_+(\overline{\Omega})$, $p^- \geq 1$, and $u \in W_0^{1,p(\cdot)}(\Omega)$, we have the inequality of Poincaré

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C\|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

where C depends on $p(\cdot)$ and therefore on Ω.

In particular, see for example in [6] Theorem 8.1.6 page 249, see also [5, 7, 10, 11] that if $p^- > 1$, the space $W_0^{1,p(\cdot)}(\Omega)$ is a separable, reflexive Banach space, and uniformly convex. Its dual space is denoted by $W^{-1,p'(\cdot)}(\Omega)$.

In the writing of variational formulations, the convex modulus $\rho_{p(\cdot)}$ appears, which leads us to state the following results.

Proposition 2.1 ([11]) *If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p^+ < +\infty$, the following relationships are true*:

$$\text{(i)} \quad \|u\|_{L^{p(\cdot)}(\Omega)} < 1 \quad (\textit{resp.} = 1, > 1) \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u) < 1 \quad (\textit{resp.} = 1, > 1),$$

$$\text{(ii)} \quad \|u\|_{L^{p(\cdot)}(\Omega)} > 1 \quad \Rightarrow \quad \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)},$$

$$\text{(iii)} \quad \|u\|_{L^{p(\cdot)}(\Omega)} < 1 \quad \Rightarrow \quad \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)},$$

$$\text{(iv)} \quad \lim_{n\to\infty} \rho_{p(\cdot)}(u_n) = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \|u_n\|_{L^{p(\cdot)}(\Omega)} = 0.$$

Proposition 2.2 ([11]) *If $q \in C_+(\overline{\Omega})$ and if for all $x \in \overline{\Omega}$, $q(x) < p^*(x)$, then the injection of $W^{1,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$ is continuous and compact, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \textit{if } p(x) < N, \\ \infty & \textit{if } p(x) \geq N. \end{cases}$$

In particular, the injection of $W_0^{1,p(\cdot)}(\Omega)$ into $L^{p(\cdot)}(\Omega)$ is continuous and compact.

Proposition 2.3 ([18]) *We note*

$$p^x(\cdot) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \textit{if } p(x) < N, \\ \infty & \textit{if } p(x) \geq N. \end{cases}$$

Let $q \in C_+(\partial\Omega)$. If for all $x \in \partial\Omega$, $q(x) < p^\partial(x)$, then the following injections of $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^\partial(\cdot)}(\partial\Omega) \hookrightarrow L^{q(\cdot)}(\partial\Omega)$ are continuous and compact.

Definition 2.1 The continuous function $p : \overline{\Omega} \to [1, +\infty)$ satisfies Hölder's continuity condition if there is a constant C such that

$$\left|p(x) - p(y)\right| \leq \frac{C}{-\log|x-y|} \quad \forall x, y \in \overline{\Omega} \text{ with } |x-y| < \frac{1}{2}.$$

Remark 2.1 Although this condition of regularity is not necessary to define Lebesgue and Sobolev spaces with variable exponents, it proves to be very useful for these spaces to introduce some properties, such that $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$.

Moreover, if $1 < p^- \leq p^+ < N$, then the Sobolev injection of $W^{1,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$ remains true for $q(\cdot) = p^*(\cdot)$ (for more details, see [5]).

Remark 2.2 According to the Poincaré inequality, it is obvious that the norms

$$u \mapsto \|\nabla u\|_{L^{p(\cdot)}(\Omega)} \quad \text{and} \quad u \mapsto \|u\|_{W^{1,p(\cdot)}(\Omega)}$$

are equivalent on $W_0^{1,p(\cdot)}(\Omega)$.

Remark 2.3 Let $a \geq 0$, $b \geq 0$ and let $1 \leq p^- \leq p^+ < +\infty$, then for all $x \in \Omega$

$$(a+b)^{p(x)} \leq 2^{p^+-1}\big(a^{p(x)} + b^{p(x)}\big).$$

3 Variational problem and existence result

We introduce now the following functional space:

$$V(\Omega) = \big\{\varphi \in \big(W^{1,p(\cdot)}_{\Gamma_1 \cup \Gamma_L}(\Omega)\big)^3 \cap \big(W^{2,p(\cdot)}(\Omega)\big)^3 : \varphi \cdot n = 0 \text{ on } w\big\}$$

with

$$W^{1,p(\cdot)}_{\Gamma_1 \cup \Gamma_L}(\Omega) = \big\{\varphi \in W^{1,p(\cdot)}(\Omega) : \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_L, \big\},$$

$\|.\|_{V(\Omega)} = \|.\|_{(W^{1,p(\cdot)}_{\Gamma_1 \cup \Gamma_L}(\Omega))^3}$.

$$G \in \big(W^{2,p(\cdot)}(\Omega)\big)^3 \quad \text{such that } G|_{\Gamma_1 \cup \Gamma_L} = g \text{ and } G \cdot n = 0 \text{ on } w. \tag{3.1}$$

The variational formulation of problem (P_0), see for example [8], leads to the following variational problem.

Problem 3.1 *Let $f \in (L^{p'(\cdot)}(\Omega))^3$ and G satisfying (3.1) be given.*

Find u such that $u - G \in V(\Omega)$ and satisfying the following variational inequality hold

$$\begin{aligned} &\sum_{i,j=1}^{3}\sum_{k,h=1}^{3} \int_\Omega a_{ijkh}(x) E_{kh}(\nabla u) \frac{\partial}{\partial x_j}\big(\varphi_i - (u_i - G_i)\big)\, dx'\, dx_3 + J(\varphi) - J(u-G) \\ &\quad \geq \int_\Omega f\big(\varphi - (u-G)\big)\, dx'\, dx_3, \quad \forall \varphi \in V(\Omega), \end{aligned} \tag{3.2}$$

where

$$J(\varphi) = \int_w K|\varphi - s|\, dx'.$$

To prove the existence of a solution to Problem 3.1, let us assume the following assumptions:

(H_1) $\quad 3 < p(x) < +\infty \quad$ for $x \in \Omega$,

(H_2) $\quad \exists \alpha_0 > 0; \quad \exists \beta > 0$ such that $\alpha_0 \leq a_{ijkh}(x) \leq \beta$ a.e. in Ω, $\quad \forall i,j,k,h = 1$ to 3

(H_3) $\quad f = (f_1, f_2, f_3) \in \big(L^{p'(\cdot)}(\Omega)\big)^3, \quad p'(x) = \dfrac{p(x)}{p(x)-1}.$

Taking

$$\mathbf{W}^{p(\cdot)}_{\Gamma_1\cup\Gamma_L}(\Omega) = \left(W^{1,p(\cdot)}_{\Gamma_1\cup\Gamma_L}(\Omega)\right)^3 \cap \left(W^{2,p(\cdot)}(\Omega)\right)^3,$$

we need the three properties of the operator E_{kh} in the following theorem.

Theorem 3.1 *For u such that $u - G \in \mathbf{W}^{p(\cdot)}_{\Gamma_1\cup\Gamma_L}(\Omega)$, with for $x \in \Omega$ $3 < p(x) < +\infty$, the components E_{kh} of the deformation tensor of St. Venant E satisfy the following properties*:

1. (*Continuity*) *E_{kh} is a continuous function, $k, h = 1$ to 3*;

2. *For all $v \in W^{p(\cdot)}_{\Gamma_1\cup\Gamma_L}(\Omega)$, we have $E_{kh}(\nabla u)\frac{\partial v_i}{\partial x_j} \in L^1(\Omega), \forall i, j, k, h = 1$ to 3*;

3. (*Coercivity*) *$\exists \alpha > 0$; such as $E_{kh}(\xi)\xi \geq \alpha|\xi|^{p(\cdot)}, \forall k, h = 1$ to 3.*

Proof First **1**. For $x \in \Omega$, $p(x) > 3$, and $u, v \in W^{1,p(\cdot)}(\Omega)$, we have $uv \in W^{1,p(\cdot)}(\Omega)$.

So, for $v \in \mathbf{W}^{p(\cdot)}_{\Gamma_1\cup\Gamma_L}(\Omega)$, we have

$$\frac{\partial v_h}{\partial x_k}, \frac{\partial v_k}{\partial x_h} \text{ and } \sum_{m=1}^{3} \frac{\partial v_m}{\partial x_k}\frac{\partial v_m}{\partial x_h} \in W^{1,p(\cdot)}(\Omega),$$

thus $E_{hk}(\nabla v) \in W^{1,p(\cdot)}(\Omega)$. Moreover, for $p(x) > 3$, we have the continuous injection of $W^{1,p(\cdot)}(\Omega)$ in $\mathbb{C}(\Omega)$, thus **1**. holds.

Second **2**. For $x \in \Omega$,

$$\left|E_{hk}(\nabla u)\right|^{p(x)} = \left(\frac{1}{2}\right)^{p(x)} \left|\left(\frac{\partial u_h}{\partial x_k} + \frac{\partial u_k}{\partial x_h} + \sum_{m=1}^{3} \frac{\partial u_m}{\partial x_k}\frac{\partial u_m}{\partial x_h}\right)\right|^{p(x)},$$

using Remark 2.3, we have

$$\left|E_{hk}(\nabla u)\right|^{p(x)} \leq \left(\frac{1}{2}\right)^{p(x)} \times 2^{p^+-1}\left(\left|\frac{\partial u_h}{\partial x_k} + \frac{\partial u_k}{\partial x_h}\right|^{p(x)} + \left|\sum_{m=1}^{3} \frac{\partial u_m}{\partial x_k}\frac{\partial u_m}{\partial x_h}\right|^{p(x)}\right).$$

Using again Remark 2.3, we obtain

$$\left|\frac{\partial u_h}{\partial x_k} + \frac{\partial u_k}{\partial x_h}\right|^{p(x)} \leq 2^{p^+-1}\left(\left|\frac{\partial u_h}{\partial x_k}\right|^{p(x)} + \left|\frac{\partial u_k}{\partial x_h}\right|^{p(x)}\right),$$

thus $E_{hk}(\nabla u) \in L^{p(\cdot)}(\Omega)$ for $h, k = 1$ to 3. As $p(x) > p'(x)$, as soon as $p(x) > 3$ and Ω is bounded, we get $E_{hk}(\nabla u) \in L^{p'(\cdot)}(\Omega)$ for $h, k = 1$ to 3.

Thus, for $v \in W^{p(\cdot)}(\Omega)$, we have $\frac{\partial v_i}{\partial x_j} \in L^{p(\cdot)}(\Omega)$ for $i, j = 1$ to 3. Hence, by the Hölder inequality, we obtain that **2**. holds. The third **3**. property comes from [17]. □

Theorem 3.2 *Let given $f \in (L^{p'(\cdot)}(\Omega))^3$, G satisfying (3.1), and the hypotheses (H_1) to (H_3) hold. Then there exists a solution u to Problem 3.1.*

Proof For the existence part, we apply Theorem 8.5 page 251 in [12] and the first three properties of E_{kh} cited in Theorem 3.1. First, we rewrite variational inequality (3.2) in the form of this Theorem 8.5 page 251 in [12].

- As we see for example in [6] Theorem 8.1.6 page 249, $(W^{1,p(\cdot)}_{\Gamma_1\cup\Gamma_L}(\Omega))^3 \cap (W^{2,p(\cdot)}(\Omega))^3$ is a separable and reflexive Banach space, then its closed subspace $V(\Omega)$ is also a separable and reflexive Banach space.
- The application $V(\Omega) \to \mathbb{R}$ defined by

$$\varphi \mapsto \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u)\frac{\partial \varphi_i}{\partial x_j}\,dx'\,dx_3$$

is linear and continuous, so it is an element of $V'(\Omega)$. We note by $T(u-G)$ this application, so we have

$$\bigl\langle T(u-G),\varphi\bigr\rangle_{V'(\Omega),V(\Omega)} = \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u)\frac{\partial \varphi_i}{\partial x_j}\,dx'\,dx_3.$$

- Similarly, we have the application $V(\Omega) \to \mathbb{R}$, which associates

$$\varphi \mapsto \int_\Omega f\varphi\,dx'\,dx_3$$

is linear and continuous, so is an element $V'(\Omega)$. We note by f this application, so we have

$$\langle f,\varphi\rangle_{V'(\Omega),V(\Omega)} = \int_\Omega f\varphi\,dx'\,dx_3,$$

therefore, problem (3.2) becomes

$$\bigl\langle T(u-G)-f,\varphi-(u-G)\bigr\rangle_{V'(\Omega),V(\Omega)} + J(\varphi)-J(u-G) \geq 0, \quad \forall \varphi \in V(\Omega),$$

with

$$J(\varphi) = \int_w K|\varphi - s|\,dx'.$$

- We check now that the operator T is pseudo-monotonic.
  a) Let u be bounded in $V(\Omega)$, we have

$$\begin{aligned}\bigl\|T(u)\bigr\|_{V'(\Omega)} &= \sup_{\substack{\|\varphi\|_{V(\Omega)=1}\\ \varphi\in V(\Omega)}} \bigl|\bigl\langle T(u),\varphi\bigr\rangle\bigr| \\ &= \sup_{\substack{\|\varphi\|_{V(\Omega)=1}\\ \varphi\in V(\Omega)}} \left|\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u)\frac{\partial \varphi_i}{\partial x_j}\,dx'\,dx_3\right| \\ &\leq \sup_{\substack{\|\varphi\|_{V(\Omega)=1}\\ \varphi\in V(\Omega)}} \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\left|\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u)\frac{\partial \varphi_i}{\partial x_j}\,dx'\,dx_3\right|.\end{aligned}$$

  Let $u \in V(\Omega)$, by Remark 2.3 we get $E_{kh}(\nabla u) \in L^{p(\cdot)}(\Omega)$, and as $p(x) > p'(x)$, as soon as $p(x) > 3$ and Ω bounded, we have

$$E_{kh}(\nabla u) \in L^{p'(\cdot)}(\Omega) \quad \forall 1 \leq k,h, \leq 3. \tag{3.3}$$

Using now hypothesis (H_2) and the Hölder inequality, with (3.3) we obtain

$$\begin{aligned}
\|T(u)\|_{V'(\Omega)} &\leq 2\beta \sup_{\substack{\|\varphi\|_{V(\Omega)}=1\\ \varphi\in V(\Omega)}} \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} \|E_{kh}(\nabla u)\|_{L^{p'(\cdot)}(\Omega)} \left\|\frac{\partial\varphi_i}{\partial x_j}\right\|_{L^{p(\cdot)}(\Omega)} \\
&\leq 6\beta \sup_{\substack{\|\varphi\|_{V(\Omega)}=1\\ \varphi\in V(\Omega)}} \sum_{i=1}^{3}\sum_{k,h=1}^{3} \|E_{kh}(\nabla u)\|_{L^{p'(\cdot)}(\Omega)} \|\varphi_i\|_{W^{1,p(\cdot)}(\Omega)} \\
&\leq 18\beta \sup_{\substack{\|\varphi\|_{V(\Omega)}=1\\ \varphi\in V(\Omega)}} \sum_{k,h=1}^{3} \|E_{kh}(\nabla u)\|_{L^{p'(\cdot)}(\Omega)} \|\varphi\|_{(W^{1,p(\cdot)}(\Omega))^3} \\
&\leq 18\beta \sup_{\substack{\|\varphi\|_{V(\Omega)}=1\\ \varphi\in V(\Omega)}} \sum_{k,h=1}^{3} \|E_{kh}(\nabla u)\|_{L^{p'(\cdot)}(\Omega)} \|\varphi\|_{V(\Omega)} \\
&\leq 18\beta \sum_{k,h=1}^{3} \|E_{kh}(\nabla u)\|_{L^{p'(\cdot)}(\Omega)}.
\end{aligned}$$

From (3.3) and (2.1) we get

$$\int_\Omega \left|E_{kh}(\nabla u)\right|^{p'(x)}\,dx = \rho_{p'(\cdot)}\bigl(E_{kh}(\nabla u)\bigr) < \infty, \quad \forall 1\leq h,k\leq 3,$$

and from (ii)–(iii) of Proposition 2.1 we have

$$\min\bigl\{\|E_{kh}(\nabla u)\|^{p'^-}_{L^{p'(\cdot)}(\Omega)}, \|E_{kh}(\nabla u)\|^{p'^+}_{L^{p'(\cdot)}(\Omega)}\bigr\} \leq \rho_{p'(\cdot)}\bigl(E_{kh}(\nabla u)\bigr) < \infty.$$

For all h and all $k\in\{1,2,3\}$, we get that $\|E_{kh}\|_{L^{p'(\cdot)}(\Omega)}$ is bounded for all h and all $k\in\{1,2,3\}$, consequently $\|T(u)\|_{V'(\Omega)}$ is bounded.

b) Let $u,v,w\in V(\Omega)$ and $\lambda\in\mathbb{R}$, we check that the application of $\mathbb{R}$ in $\mathbb{R}$: $\lambda\mapsto\langle T(u+\lambda v),w\rangle$ is continuous. For this, let us consider $\{\lambda_n\}$ to be a sequence of $\mathbb{R}$ that converges to λ. Let us denote

$$\mathcal{F}_n(x) = \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u+\lambda_n\nabla v)\frac{\partial w_i}{\partial x_j}$$

and

$$\mathcal{F}(x) = \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u+\lambda\nabla v)\frac{\partial w_i}{\partial x_j}.$$

The E_{kh} being continuous, we therefore have, for all h and all $k\in\{1,2,3\}$,

$$\mathcal{F}_n(x) = \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u+\lambda_n\nabla v)\frac{\partial w_i}{\partial x_j} \text{ converges to}$$

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u+\lambda\nabla v)\frac{\partial w_i}{\partial x_j} \text{ a.e. in } \Omega,$$

and we have also with (H_2) and the definition of E_{kh}:

$$\begin{aligned}
&\left|\sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)E_{kh}(\nabla u+\lambda_n\nabla v)\frac{\partial w_i}{\partial x_j}\right| \\
&\quad\leq \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\beta\left|E_{kh}(\nabla u+\lambda_n\nabla v)\right|\left|\frac{\partial w_i}{\partial x_j}\right| \\
&\quad\leq \frac{1}{2}\beta\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\left(\left|\frac{\partial u_h}{\partial x_k}+\lambda_n\frac{\partial v_h}{\partial x_k}\right|+\left|\frac{\partial u_k}{\partial x_h}+\lambda_n\frac{\partial v_k}{\partial x_h}\right|\right. \\
&\qquad\left.+\sum_{m=1}^{3}\left|\frac{\partial u_m}{\partial x_k}+\lambda_n\frac{\partial v_m}{\partial x_k}\right|\left|\frac{\partial u_m}{\partial x_h}+\lambda_n\frac{\partial v_m}{\partial x_h}\right|\right)\left|\frac{\partial w_i}{\partial x_j}\right|.
\end{aligned}$$

As the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is convergent in $\mathbb{R}$, then $\exists m\in\mathbb{R}:|\lambda_n|\leq m$, so

$$\begin{aligned}
\left|\mathcal{F}_n(x)\right| \leq \frac{1}{2}\beta\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}&\left(\left(\left|\frac{\partial u_h}{\partial x_k}\right|+m\left|\frac{\partial v_h}{\partial x_k}\right|\right)+\left(\left|\frac{\partial u_k}{\partial x_h}\right|+m\left|\frac{\partial v_k}{\partial x_h}\right|\right)\right. \\
&\left.+\sum_{m=1}^{3}\left(\left|\frac{\partial u_m}{\partial x_k}\right|+m\left|\frac{\partial v_m}{\partial x_k}\right|\right)\left(\left|\frac{\partial u_m}{\partial x_h}\right|+m\left|\frac{\partial v_m}{\partial x_h}\right|\right)\right)\left|\frac{\partial w_i}{\partial x_j}\right|.
\end{aligned}$$

We define now the function L by

$$\begin{aligned}
L(x) = \frac{1}{2}\beta\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}&\left(\left(\left|\frac{\partial u_h}{\partial x_k}\right|+m\left|\frac{\partial v_h}{\partial x_k}\right|\right)+\left(\left|\frac{\partial u_k}{\partial x_h}\right|+m\left|\frac{\partial v_k}{\partial x_h}\right|\right)\right. \\
&\left.+\sum_{m=1}^{3}\left(\left|\frac{\partial u_m}{\partial x_k}\right|+m\left|\frac{\partial v_m}{\partial x_k}\right|\right)\left(\left|\frac{\partial u_m}{\partial x_h}\right|+m\left|\frac{\partial v_m}{\partial x_h}\right|\right)\right)\left|\frac{\partial w_i}{\partial x_j}\right|.
\end{aligned}$$

We obtain that $L\in L^{p(\cdot)}(\Omega)$.

Thus from Lebesgue's dominated convergence theorem, we deduce that

$$\left\langle T(u+\lambda_n v),w\right\rangle \longrightarrow \left\langle T(u+\lambda v),w\right\rangle,$$

which shows that T is hemi-continuous.

c) By hypothesis (H_2) and the monotony of E_{kh}, we have

$$\begin{aligned}
\left\langle T(u)-T(v),u-v\right\rangle = \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_{\Omega} a_{ijkh}(x)\big(E_{kh}(\nabla u) \\
-E_{kh}(\nabla v)\big)\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial v_i}{\partial x_j}\right)dx'\,dx_3\geq 0,
\end{aligned}$$

so T is monotonous. As it is also bounded and hemi-continuous, then T is pseudo-monotonic.

- The functional J is proper, convex, and lower semi-continuous on $V(\Omega)$. Indeed, let u and v be two elements of $V(\Omega)$, and $\lambda\in[0,1]$, we have

d) J **is convex**, indeed

$$\begin{aligned}
&J\bigl(\lambda u + (1-\lambda)v\bigr) \\
&\quad = \int_w K\bigl(\bigl|\lambda u + (1-\lambda)v - s\bigr|\bigr)\,dx' s|)\,dx' = \int_w K(\bigl|\lambda(u-s) + (1-\lambda)(v-s)\bigr|\,dx' \\
&\quad \leq \int_w K\bigl(\bigl|\lambda(u-s)\bigr|\bigr)\,dx' + \int_w K\bigl(\bigl|(1-\lambda)(v-s)\bigr|\bigr)\,dx' \\
&\quad \leq \lambda \int_w K|u-s|\,dx' + (1-\lambda)\int_w K|v-s|\,dx' = \lambda J(u) + (1-\lambda)J(v).
\end{aligned}$$

e) J **is lower semi-continuous**, indeed

$$\begin{aligned}
\bigl|J(u) - J(v)\bigr| &= \left|\int_w K\bigl(|u-s| - |v-s|\bigr)\,dx'\right|| \,dx' \leq \int_w |K||u-v|\,dx' \\
&\leq \|K\|_{L^\infty(w)}|w|^{\frac{1}{p(\cdot)}}\|u-v\|_{(L^{p'(\cdot)}(w))^2} \\
w|^{\frac{1}{p(x)}} C\|u-v\|_{(W^{1,p(x)}(\Omega))^3} &\leq C\|K\|_{L^\infty(w)}|w|^{\frac{1}{p(x)}}\|u-v\|_{V(\Omega)},
\end{aligned}$$

where C is the constant of the continuous injection from $V(\Omega)$ on $(L^{p'(\cdot)}(w))^2$. Thus, J is Lipschitzian, so it is fortiori lower semi-continuous on $V(\Omega)$.

We can now apply Theorem 8.5 page 251 in [12] to obtain the existence of u such that $u - G$ in $V(\Omega)$ satisfying variational inequality (3.2). □

4 On the uniqueness of the result

Theorem 4.1 *Let the functions*

$$\frac{\partial u_i}{\partial x_j} : \Omega \longrightarrow \left]-\infty, \frac{1}{3}\right], \qquad x \longrightarrow \frac{\partial u_i}{\partial x_j}(x) \quad \textit{for } i,j = 1 \textit{ to } 3;$$

then the operators

$$E_{ij}(\cdot) \textit{ of } \mathbf{W}^{p(\cdot)}(\Omega) \textit{ in } \bigl(\mathbf{W}^{p(\cdot)}(\Omega)\bigr)', \quad i,j = 1 \textit{ to } 3,$$

are monotonous.

Proof We have the following result in [13]. We give also the proof for the convenience of the readers. Using the rule $\frac{1}{2}(a^2 + b^2) \geq -ab$, with $a = \frac{\partial u_m}{\partial x_i}$ and $b = \frac{\partial u_m}{\partial x_j}$, we have

$$\begin{aligned}
E_{ij}(u) &\geq \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{4}\sum_{m=1}^{3}\left(\left(\frac{\partial u_m}{\partial x_i}\right)^2 + \left(\frac{\partial u_m}{\partial x_j}\right)^2\right) \\
&= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{4}\left(\sum_{m=1}^{3}\left(\frac{\partial u_m}{\partial x_i}\right)^2 + \sum_{m=1}^{3}\left(\frac{\partial u_m}{\partial x_j}\right)^2\right), \quad i,j = 1 \text{ to } 3,
\end{aligned}$$

and consequently, $\forall i,j = 1$ to 3,

$$\begin{aligned}\langle E_{ij}(u) - E_{ij}(v), u - v\rangle \geq{}& \frac{1}{2}\left\langle \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), u - v\right\rangle \\ & - \frac{1}{4}\left\langle \left(\sum_{m=1}^{3}\left(\frac{\partial u_m}{\partial x_i}\right)^2 + \sum_{m=1}^{3}\left(\frac{\partial u_m}{\partial x_j}\right)^2\right)\right. \\ & \left. - \left(\sum_{m=1}^{3}\left(\frac{\partial v_m}{\partial x_i}\right)^2 + \sum_{m=1}^{3}\left(\frac{\partial v_m}{\partial x_j}\right)^2\right), u - v\right\rangle. \end{aligned} \tag{4.1}$$

To conclude, we must prove that the second member of (4.1) is positive. For that, we separate the second member of (4.1) in linear and nonlinear parts.

Let the linear function $\Omega \xrightarrow{J_x} \mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{A_{ij}} \mathbb{R}$ be defined by

$$(A_{ij} \circ J_x)(x) = A_{ij}\left(\frac{\partial u}{\partial x_i}(x), \frac{\partial u}{\partial x_j}(x)\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}(x) + \frac{\partial u_j}{\partial x_i}(x)\right), \quad i,j = 1 \text{ to } 3,$$

and the nonlinear function $\Omega \xrightarrow{J_x} \mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{B_{ij}} \mathbb{R}$ be defined by

$$(B_{ij} \circ J_x)(x) = B_{ij}\left(\frac{\partial u}{\partial x_i}(x), \frac{\partial u}{\partial x_j}(x)\right) = -\frac{1}{4}\left(\sum_{j=1}^{3}\left(\frac{\partial u_j}{\partial x_i}(x)\right)^2 + \sum_{i=1}^{3}\left(\frac{\partial u_i}{\partial x_j}(x)\right)^2\right).$$

The functions A_{ij} and B_{ij} are continuous for $p(\cdot) > 3$. It remains to show that, $\forall i,j = 1$ to 3, the A_{ij} are increasing on $\mathbb{R}$, the B_{ij} are increasing on $\mathbb{R}^-$, and the $A_{ij} + B_{ij}$ are increasing on $]-\infty, \frac{1}{3}]$.

1. Let us show that the A_{ij} are increasing: let the function

$$\Omega \xrightarrow{J_x} \mathbb{R} \xrightarrow{\frac{\partial u_i}{\partial x_j}} \mathbb{R} \text{ be defined by } \left(\frac{\partial u_i}{\partial x_j} \circ J_x\right)(x) = \frac{\partial u_i}{\partial x_j}(x), \quad i,j = 1 \text{ to } 3.$$

We note

$$\frac{\partial u}{\partial x_j}(x) = t_j \quad \text{and} \quad \frac{\partial u}{\partial x_i}(x) = \tau_i$$

and

$$\frac{\partial u_i}{\partial x_j}(x) = t_{ij} \quad \text{and} \quad \frac{\partial u_j}{\partial x_i}(x) = \tau_{ji}.$$

The function $t \longmapsto \frac{1}{2}t$ of $\mathbb{R} \longrightarrow \mathbb{R}$, being increasing on $\mathbb{R}$, we have

$$\begin{aligned}\frac{1}{2}\left\langle \frac{\partial u_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j}, \frac{\partial u_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\right\rangle &= \frac{1}{2}\int_{\Omega}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\right)\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\right) dx \\ &= \frac{1}{2}\left\| \frac{\partial u_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\right\|^2_{L^2(\Omega)} \geq 0.\end{aligned}$$

Therefore, the A_{ij} are increasing.

2. Let us show that the B_{ij} are increasing: let the function

$$\Omega \xrightarrow{J_x} \mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{B_{ij}} \mathbb{R}$$

be defined by

$$(B_{ij} \circ J_x)(x) = B_{ij}(t_j, \tau_i) = -\frac{1}{4}\left(\sum_{j=1}^{3}\left(\frac{\partial u_j}{\partial x_i}(x)\right)^2 + \sum_{i=1}^{3}\left(\frac{\partial u_i}{\partial x_j}(x)\right)^2\right), \quad i,j = 1 \text{ to } 3.$$

As in point 1., we note

$$t_{ij} = \frac{\partial u_i}{\partial x_j}(x), \qquad \tau_{ji} = \frac{\partial u_j}{\partial x_i}(x), \quad \forall i,j = 1 \text{ to } 3,$$

$$B_{ij}(t_j, \tau_i) = -\frac{1}{4}\left(\sum_{j=1}^{3}\left(\frac{\partial u_j}{\partial x_i}(x)\right)^2 + \sum_{i=1}^{3}\left(\frac{\partial u_i}{\partial x_j}(x)\right)^2\right),$$

so

$$\begin{aligned} B_{ij}(t_j, \tau_i) &= -\frac{1}{4}\left(\sum_{i=1}^{3} t_{ij}^2 + \sum_{j=1}^{3} \tau_{ji}^2\right) \\ &\geq -\frac{1}{4}\left(6 \times \operatorname*{Max}_{1\leq i,j\leq 3}\left(t_{ij}^2, \tau_{ji}^2\right)\right) = -\frac{3}{2}\varkappa^2. \end{aligned}$$

For the function $f(\varkappa) = -\frac{3}{2}\varkappa^2$ being continuous and increasing on $\mathbb{R}^-$, we deduce that the B_{ij} are increasing on $\mathbb{R}^-$.

3. We show that the $A_{ij} + B_{ij}$ are increasing: the proofs of points 1. and 2. imply that the sum $A_{ij} + B_{ij}$, $\forall i,j = 1$ to 3, corresponds to the sum of the two functions $f(\varkappa) + g(\varkappa) = \varkappa - \frac{3}{2}\varkappa^2$, $\mathbb{R} \longrightarrow \mathbb{R}$, obviously continuous and increasing on $]-\infty, \frac{1}{3}]$, as the derivative of the convex function $h(x) = \frac{1}{2}x^2 - \frac{1}{2}x^3$ on $]-\infty, \frac{1}{3}]$. So, (4.1) is verified, and consequently

$$\left\langle E_{ij}(u) - E_{ij}(v), u - v\right\rangle \geq \left\langle (A_{ij} + B_{ij})(u) - (A_{ij} + B_{ij})(v), u - v\right\rangle \geq 0, \quad \forall i,j = 1 \text{ to } 3.$$

In other words, the $E_{ij}(u), i,j = 1$ to 3, are monotonous $\mathbf{W}^{p(\cdot)}(\Omega)$ in $(\mathbf{W}^{p(\cdot)}(\Omega))'$, $i,j = 1$ to 3. □

Theorem 4.2 *Variational inequality* (3.2) *has a unique solution satisfying Theorem* 3.2.

Proof Suppose now that variational inequality (3.2) has two solutions u_1, u_2 satisfying Theorem 3.2, and $(E_{kh}(\xi) - E_{kh}(\eta))(\xi_{ij} - \eta_{ij}) = 0$ if and only if $\xi = \eta$. So we have

$$\begin{aligned} &\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u_1)\frac{\partial}{\partial x_j}\left(\varphi_i - (u_{1i} - G_i)\right) dx'\,dx_3 \\ &\quad + \int_w K\left(|\varphi - s| - \left|(u_1 - G) - s\right|\right) dx' \\ &\geq \int_\Omega f\left(\varphi - (u_1 - G)\right) dx'\,dx_3, \quad \forall \varphi \in V(\Omega) \end{aligned} \tag{4.2}$$

and

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_{\Omega} a_{ijkh}(x)E_{kh}(\nabla u_2)\frac{\partial}{\partial x_j}\big(\varphi_i-(u_{2i}-G_i)\big)\,dx'\,dx_3 + \int_{w} K\big(|\varphi-s|-\big|(u_2-G)-s\big|\big)\,dx' \geq \int_{\Omega} f\big(\varphi-(u_2-G)\big)\,dx'\,dx_3, \quad \forall\varphi\in V(\Omega). \tag{4.3}$$

We take $\varphi = u_2 - G$ in (4.2) and $\varphi = u_1 - G$ in (4.3), so

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_{\Omega} a_{ijkh}(x)\big(E_{kh}(\nabla u_1)-E_{kh}(\nabla u_2)\big)\frac{\partial}{\partial x_j}\big((u_{1i}-G_i)-(u_{2i}-G_i)\big)\leq 0,$$

but according to Theorem 4.1 on the monotony of E_{kh}, we have

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_{\Omega} a_{ijkh}(x)\big(E_{kh}(\nabla u_1)-E_{kh}(\nabla u_2)\big)\frac{\partial}{\partial x_j}(u_{1i}-u_{2i})\geq 0,$$

so

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_{\Omega} a_{ijkh}(x)\big(E_{kh}(\nabla u_1)-E_{kh}(\nabla u_2)\big)\frac{\partial}{\partial x_j}(u_{1i}-u_{2i}) = 0.$$

Thus $\nabla u_1 = \nabla u_2$ in $(L^{p(\cdot)}(\Omega))^9$, and by the inequality of Poincaré, we get $u_1 = u_2$ in $(L^{p(\cdot)}(\Omega))^3$, and so $u_1 = u_2$ dans $(W^{1,p(\cdot)}(\Omega))^3$. □

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No abbreviations used.

Availability of data and materials
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Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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