# On a class of generalized saddle-point problems arising from contact mechanics 

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#### Abstract

In the present paper we consider a class of generalized saddle-point problems described by means of the following variational system: $$
\begin{aligned} & a(u, v-u)+b(v-u, \lambda)+j(v)-j(u)+J(u, v)-J(u, u) \geq(f, v-u)_{x}, \\ & b(u, \mu-\lambda)-\psi(\mu)+\psi(\lambda) \leq 0, \end{aligned}
$$ ( $v \in K \subseteq X, \mu \in \Lambda \subset Y$ ), where $\left(X,(\cdot, \cdot)_{X}\right)$ and $\left(Y,(\cdot, \cdot)_{Y}\right)$ are Hilbert spaces. We use a fixed-point argument and a saddle-point technique in order to prove the existence of at least one solution. Then, we obtain uniqueness and stability results. Subsequently, we pay special attention to the case when our problem can be seen as a perturbed problem by setting $\psi(\cdot)=\epsilon \bar{\psi}(\cdot)(\epsilon>0)$. Then, we deliver a convergence result for $\epsilon \rightarrow 0$, the case $\psi \equiv 0$ appearing like a limit case. The theory is illustrated by means of examples arising from contact mechanics, focusing on models with multicontact zones.


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## 1 Introduction

In the present paper we bring the attention to the following mixed variational problem.

Problem 1 Given $f \in X$, find $(u, \lambda) \in K \times \Lambda$ such that, for all $v \in K$ and $\mu \in \Lambda$,

$$
\begin{aligned}
& a(u, v-u)+b(v-u, \lambda)+j(v)-j(u)+J(u, v)-J(u, u) \geq(f, v-u)_{X} \\
& b(u, \mu-\lambda)-\psi(\mu)+\psi(\lambda) \leq 0
\end{aligned}
$$

We are going to study this problem under the following hypotheses:
$\left(h_{1}\right)\left(X,(\cdot, \cdot)_{X},\|\cdot\|_{X}\right)$ and $\left(Y,(\cdot, \cdot)_{Y},\|\cdot\|_{Y}\right)$ are Hilbert spaces.
$\left(h_{2}\right)$ The form $a: X \times X \rightarrow \mathbb{R}$ is symmetric, bilinear continuous (of rank $M_{a}>0$ ) and $X$-elliptic (of rank $m_{a}>0$ ).
( $h_{3}$ ) (1) The form $b: X \times Y \rightarrow \mathbb{R}$ is bilinear continuous (of rank $M_{b}>0$ ).

[^0](2) There exists $\alpha>0$ such that
\[

$$
\begin{equation*}
\inf _{\mu \in Y, \mu \neq 0_{Y}} \sup _{v \in X, v \neq 0_{X}} \frac{b(v, \mu)}{\|v\|_{X}\|\mu\|_{Y}} \geq \alpha . \tag{1}
\end{equation*}
$$

\]

$\left(h_{4}\right)$ The functional $j: X \rightarrow[0, \infty)$ is convex and Lipschitz continuous (of rank $L_{j}>0$ ).
$\left(h_{5}\right)$ The functional $\psi: Y \rightarrow[0, \infty)$ is convex and lower semicontinuous. When $\psi \not \equiv 0$, we assume that there exist $c_{\psi}>0$ and $q>1$ such that $\psi(\mu) \geq c_{\psi}\|\mu\|_{Y}^{q}$ for all $\mu \in Y$.
$\left(h_{6}\right) \bullet K \subseteq X$ is an unbounded closed convex subset such that $0_{X} \in K$ and $\Lambda \subset Y$ is a bounded closed convex subset such that $0_{Y} \in \Lambda$
or

- $K \subseteq X$ is a linear subspace and $\Lambda \subset Y$ is an unbounded closed convex subset such that $0_{Y} \in \Lambda$.
$\left(h_{7}\right)$ The functional $J: X \times X \rightarrow[0, \infty)$ satisfies:
(1) For each $v \in X, J(v, \cdot)$ is convex and Lipschitz continuous of rank $L_{J}(v)>0$.
(2) There exists $M_{J}>0$ such that, for all $v_{1}, v_{2}, w_{1}, w_{2} \in X, J\left(v_{1}, w_{2}\right)-J\left(v_{1}, w_{1}\right)+J\left(v_{2}, w_{1}\right)-$ $J\left(v_{2}, w_{2}\right) \leq M_{J}\left\|v_{1}-v_{2}\right\|_{X}\left\|w_{1}-w_{2}\right\|_{X}$.
(3) There exists $c_{J}>0$ such that $|J(u, v)| \leq c_{J}\left(\|u\|_{X}+1\right)\|v\|_{X}$ for all $u, v \in X$.

Problem 1 can be seen as a generalization of the following mixed variational problem: given $f \in X$, find $(u, \lambda) \in X \times \Lambda$ such that, for all $v \in X$ and $\mu \in \Lambda \subset Y$,

$$
\begin{align*}
& a(u, v)+b(v, \lambda)=(f, v)_{X},  \tag{2}\\
& b(u, \mu-\lambda) \leq 0 . \tag{3}
\end{align*}
$$

Such a variational system appears in the weak formulations of some contact problems: see, e.g., [10] for unilateral contact problems or [11] for a class of bilateral contact problems. The unique solution of the problem (2) and (3) is the unique saddle point of the functional

$$
X \times \Lambda \ni(\mu, v) \rightarrow \frac{1}{2} a(v, v)+b(v, \mu)-(f, v)_{X} .
$$

By analyzing contact models with two-contact zones, one can arrive at a variational system of the form below: given $f \in X$, find $(u, \lambda) \in X \times \Lambda$ such that, for all $v \in X$ and $\mu \in \Lambda \subset Y$,

$$
\begin{align*}
& a(u, v-u)+b(v-u, \lambda)+j(v)-j(u) \geq(f, v-u)_{X},  \tag{4}\\
& b(u, \mu-\lambda) \leq 0, \tag{5}
\end{align*}
$$

see, e.g., [15]. According to Theorem 2 in [15], the problem (4) and (5) has at least one solution $(u, \lambda)$ that is unique in its first component; each solution is a saddle point of the functional

$$
X \times \Lambda \ni(\mu, v) \rightarrow \frac{1}{2} a(v, v)+j(v)+b(v, \mu)-(f, v)_{X} .
$$

By considering the perturbed problem

$$
a(u, v-u)+b(v-u, \lambda)+j(v)-j(u) \geq(f, v-u)_{X},
$$

$$
b(u, \mu-\lambda)-\epsilon\|\mu\|_{Y}^{2}+\epsilon\|\lambda\|_{Y}^{2} \leq 0
$$

one can associate the functional

$$
K \times \Lambda \ni(\mu, v) \rightarrow \frac{1}{2} a(v, v)+j(v)-\epsilon\|\mu\|_{Y}^{2}+b(v, \mu)-(f, v)_{X} \quad(\epsilon>0) ;
$$

see the recent paper [3].
In the present paper, we focus on models with multicontact zones, arriving at variational formulations that can be cast in the more general form below.

Problem 2 Given $f \in X$, find $(u, \lambda) \in K \times \Lambda$ such that, for all $v \in K$ and $\mu \in \Lambda$,

$$
\begin{aligned}
& a(u, v-u)+b(v-u, \lambda)+j(v)-j(u)+J(u, v)-J(u, u) \geq(f, v-u)_{X}, \\
& b(u, \mu-\lambda) \leq 0 .
\end{aligned}
$$

Problem 2 can be seen as a particular case of Problem 1, by setting $\psi \equiv 0$. Problem 1 is interesting in its own right and, even more complicated at first glance than Problem 2 due to the additional term " $-\psi(\mu)+\psi(\lambda)$ ", actually, it brings us a significant advantage: assuming that $\psi$ is strictly convex then its solution is unique in both components. In contrast, Problem 2 has at least one solution, unique only in its first component.

By setting $\psi(\cdot)=\epsilon \bar{\psi}(\cdot)$ with $\epsilon>0$, Problem 1 can be seen as a perturbed problem. If $\psi$ is strictly convex then the unique solution of the "perturbed" problem is the unique saddle point of a bifunctional that is strictly convex in its first argument and strictly concave in the second one. When $\epsilon$ goes to zero, then the case $\psi \equiv 0$ appears like a limit case. Thus, numerical reasons enhance our interest to study the case $\psi \not \equiv 0$.
In order to prove the well-posedness of Problem 1 under the hypotheses $\left(h_{1}\right)-\left(h_{7}\right)$, we use a fixed-point argument, a saddle-point technique, arguments in the theory of variational-quasivariational inequalities as well as a minimization argument.

The present paper brings a contribution to the literature devoted to the saddle-point problems and their applications in mechanics; references relevant to the matter are, for instance, [ $1,2,8,10,11,17]$.

The reading of the present work requires a background in convex analysis, nonlinear analysis, calculus of variations, continuum mechanics, and contact mechanics; the reader can consult, e.g., $[4,6,7,9,10,13,14,18,20,21]$; for a more complex view, the reader can also consult $[5,12,19,22,23]$.

To increase the clarity of the presentation, we provide below some tools we need.

Definition 1 Let $A$ and $B$ be two nonempty sets. A pair $(u, \lambda) \in A \times B$ is said to be a saddle point of a functional $\mathcal{L}: A \times B \rightarrow \mathbb{R}$ if and only if

$$
\mathcal{L}(u, \mu) \leq \mathcal{L}(u, \lambda) \leq \mathcal{L}(v, \lambda) \quad \text { for all } v \in A, \mu \in B .
$$

Proposition 1 Let $\left(X,(\cdot, \cdot)_{X},\|\cdot\|_{X}\right),\left(Y,(\cdot, \cdot)_{Y},\|\cdot\|_{Y}\right)$ be Hilbert spaces, let
and let $\mathcal{L}: A \times B \rightarrow \mathbb{R}$ be a functional such that:

$$
\begin{align*}
& v \rightarrow \mathcal{L}(v, \mu) \quad \text { is convex and lower semicontinuous for all } \mu \in B,  \tag{7}\\
& \mu \rightarrow \mathcal{L}(v, \mu) \quad \text { is concave and upper semicontinuous for all } v \in A . \tag{8}
\end{align*}
$$

Then:

- the set of the saddle points of $\mathcal{L}, A_{0} \times B_{0}$, is a convex set;
- if $v \rightarrow \mathcal{L}(\nu, \mu)$ is strictly convex for all $\mu \in B$, then $A_{0}$ has at most one point;
- if $\mu \rightarrow \mathcal{L}(v, \mu)$ is strictly concave for all $v \in A$, then $B_{0}$ has at most one point.

For a proof, see Proposition 1.5 on page 169 in [6].

Proposition 2 Let $\left(X,(\cdot, \cdot)_{X},\|\cdot\|_{X}\right),\left(Y,(\cdot, \cdot)_{Y},\|\cdot\|_{Y}\right)$ be two Hilbert spaces. Assume (6), (7), and (8). Moreover, we assume that

$$
\begin{aligned}
& \text { A is bounded or } \lim _{\|v\|_{X} \rightarrow \infty, v \in A} \mathcal{L}\left(v, \mu_{*}\right)=\infty \text { for some } \mu_{*} \in B \text { and } \\
& \text { B is bounded or } \lim _{\|\mu\|_{Y} \rightarrow \infty, \mu \in B} \inf _{v \in A} \mathcal{L}(v, \mu)=-\infty .
\end{aligned}
$$

Then, the functional $\mathcal{L}$ has at least one saddle point.

For the proof, see Proposition 2.4 on page 176 in [6].
The rest of the paper has the following structure. In Sect. 2 we deliver some auxiliary results. In Sect. 3 we study the well-posedness of Problem 1. In Sect. 4 we obtain a convergence result. In Sect. 5 we give examples of $X, Y, a, b, j, J, K, \Lambda$, and $\psi$ such that the working hypotheses are fulfilled; these examples are related to some weak formulations via Lagrange multipliers for a class of contact models with multicontact zones.

## 2 Auxiliary results

In this section we consider the following auxiliary problem.

Problem 3 Let $\varsigma \in K$. Given $f \in X$, find $\left(u_{\varsigma}, \lambda_{\varsigma}\right) \in K \times \Lambda$ such that, for all $v \in K$ and $\mu \in \Lambda$,

$$
\begin{aligned}
& a\left(u_{\varsigma}, v-u_{\varsigma}\right)+b\left(v-u_{\varsigma}, \lambda_{\varsigma}\right)+j(v)-j\left(u_{\varsigma}\right)+J(\varsigma, v)-J\left(\varsigma, u_{\varsigma}\right) \geq\left(f, v-u_{\varsigma}\right)_{X}, \\
& b\left(u_{\varsigma}, \mu-\lambda_{\varsigma}\right)-\psi(\mu)+\psi\left(\lambda_{\varsigma}\right) \leq 0 .
\end{aligned}
$$

Let us associate to Problem 3 the following functional:

$$
\mathcal{L}_{\zeta}: K \times \Lambda \rightarrow \mathbb{R}, \quad \mathcal{L}_{\zeta}(v, \mu)=\frac{1}{2} a(v, v)+b(v, \mu)-(f, v)_{X}-\psi(\mu)+j(v)+J(\varsigma, v) .
$$

Proposition 3 Let $\varsigma \in K$. Assume that $\left(h_{1}\right)-\left(h_{7}\right)$ hold true. Then:
(i) the functional $\mathcal{L}_{\varsigma}$ has at least one saddle point;
(ii) a pair $\left(u_{\zeta}, \lambda_{\varsigma}\right) \in K \times \Lambda$ verifies Problem 3 if and only if it is a saddle point of $\mathcal{L}_{\zeta}$.

Proof (i) We are going to apply Proposition 2. We know that $K \subseteq X$ and $\Lambda \subset Y$ are nonempty closed convex sets and it is obvious that

$$
\begin{aligned}
& v \rightarrow \mathcal{L}_{\zeta}(v, \mu) \quad \text { is convex and lower semicontinuous for all } \mu \in \Lambda, \\
& \mu \rightarrow \mathcal{L}_{\zeta}(v, \mu) \quad \text { is concave and upper semicontinuous for all } v \in X
\end{aligned}
$$

Since $K \subseteq X$ is unbounded, we have to verify that

$$
\begin{equation*}
\lim _{\|v\|_{X} \rightarrow \infty, v \in K} \mathcal{L}_{\zeta}\left(v, \mu_{*}\right)=\infty \quad \text { for some } \mu_{*} \in \Lambda . \tag{9}
\end{equation*}
$$

Indeed, let $\mu_{*}=0_{Y}$. We write,

$$
\mathcal{L}_{\zeta}\left(v, 0_{Y}\right)=\frac{1}{2} a(v, v)-(f, v)_{X}+j(v)-\psi\left(0_{Y}\right)+J(\varsigma, v) \quad \text { for all } v \in K .
$$

As $j(\cdot), J(\cdot, \cdot)$ are nonnegative,

$$
\mathcal{L}_{\zeta}\left(v, 0_{Y}\right) \geq \frac{m_{a}}{2}\|v\|_{X}^{2}-\|f\|_{X}\|v\|_{X}-\psi\left(0_{Y}\right) \quad \text { for all } v \in K
$$

Passing to the limit as $\|v\|_{X} \rightarrow \infty$, we obtain (9).
If $\Lambda$ is unbounded, then we have to justify that

$$
\begin{equation*}
\lim _{\|\mu\|_{Y} \rightarrow \infty, \mu \in \Lambda} \inf _{v \in K} \mathcal{L}_{\zeta}(v, \mu)=-\infty \tag{10}
\end{equation*}
$$

keeping in mind that $K \subseteq X$ is a linear subspace, according to $\left(h_{6}\right)$.
Indeed, let $\mu \in \Lambda$ and let $u_{\varsigma}^{\mu} \in K$ be the unique solution of the variational inequality of the second kind,

$$
\begin{equation*}
a\left(u_{\varsigma}^{\mu}, v-u_{\varsigma}^{\mu}\right)+j(v)-j\left(u_{\varsigma}^{\mu}\right)+J(\varsigma, v)-J\left(\varsigma, u_{\varsigma}^{\mu}\right)+b\left(v-u_{\varsigma}^{\mu}, \mu\right) \geq\left(f, v-u_{\varsigma}^{\mu}\right)_{X} \tag{11}
\end{equation*}
$$

or, equivalently,

$$
a\left(u_{\varsigma}^{\mu}, v-u_{\varsigma}^{\mu}\right)+j(v)-j\left(u_{\varsigma}^{\mu}\right)+J(\varsigma, v)-J\left(\varsigma, u_{\varsigma}^{\mu}\right) \geq\left(f^{\mu}, v-u_{\varsigma}^{\mu}\right)_{X},
$$

for all $v \in K$, where $f^{\mu} \in X$ is defined by Riesz's representation theorem as follows,

$$
\begin{equation*}
\left(f^{\mu}, v\right)_{X}=(f, v)_{X}-b(v, \mu) \quad \text { for all } v \in X \tag{12}
\end{equation*}
$$

As $u_{\varsigma}^{\mu}$ minimizes the functional

$$
K \ni v \rightarrow \frac{1}{2} a(v, v)+j(v)+J(\varsigma, v)-\left(f^{\mu}, v\right)_{X}
$$

then,

$$
\inf _{v \in K} \mathcal{L}_{\varsigma}(v, \mu)=\frac{1}{2} a\left(u_{\varsigma}^{\mu}, u_{\varsigma}^{\mu}\right)+j\left(u_{\varsigma}^{\mu}\right)+J\left(\varsigma, u_{\varsigma}^{\mu}\right)-\left(f, u_{\varsigma}^{\mu}\right)_{X}+b\left(u_{\varsigma}^{\mu}, \mu\right)-\psi(\mu) .
$$

Let us set $v=0_{X}$ in (11). Therefore,

$$
\begin{align*}
& \frac{1}{2} a\left(u_{\varsigma}^{\mu}, u_{\varsigma}^{\mu}\right)-\left(f, u_{\varsigma}^{\mu}\right)_{X}+j\left(u_{\varsigma}^{\mu}\right)+b\left(u_{\varsigma}^{\mu}, \mu\right)+J\left(\varsigma, u_{\varsigma}^{\mu}\right)  \tag{13}\\
& \quad \leq-\frac{1}{2} a\left(u_{\varsigma}^{\mu}, u_{\varsigma}^{\mu}\right)+j\left(0_{X}\right)
\end{align*}
$$

Consequently,

$$
\inf _{v \in K} \mathcal{L}_{\zeta}(\nu, \mu) \leq-\frac{m_{a}}{2}\left\|u_{\varsigma}^{\mu}\right\|_{X}^{2}-\psi(\mu)+j\left(0_{X}\right) \leq-\frac{m_{a}}{2}\left\|u_{\varsigma}^{\mu}\right\|_{X}^{2}+j\left(0_{X}\right) .
$$

Setting now $v=u_{\varsigma}^{\mu}-w$ with $w \in K$, in (11), we obtain

$$
b(w, \mu) \leq(f, w)_{X}-a\left(u_{\varsigma}^{\mu}, w\right)+j\left(u_{\varsigma}^{\mu}-w\right)-j\left(u_{\varsigma}^{\mu}\right)+J\left(\varsigma, u_{\varsigma}^{\mu}-w\right)-J\left(\varsigma, u_{\varsigma}^{\mu}\right) .
$$

Then, we use the inf-sup property of $b(\cdot, \cdot)$ to write

$$
\alpha\|\mu\|_{Y} \leq\|f\|_{X}+M_{a}\left\|u_{\zeta}^{\mu}\right\|_{X}+L_{j}+L_{J}(\varsigma)
$$

Therefore,

$$
\alpha^{2}\|\mu\|_{Y}^{2} \leq 3\left(\|f\|_{X}^{2}+M_{a}^{2}\left\|u_{\varsigma}^{\mu}\right\|_{X}^{2}+\left(L_{j}+L_{J}(\varsigma)\right)^{2}\right)
$$

As a result, there exists $c=c\left(\alpha, m_{a}, M_{a}, L_{j}, L_{J}(\varsigma)\right)>0$ such that

$$
\inf _{v \in K} \mathcal{L}_{\zeta}(v, \mu) \leq-c\left(\|\mu\|_{Y}^{2}-\|f\|_{X}^{2}-\left(L_{j}+L_{J}(\varsigma)\right)^{2}\right)+j\left(0_{X}\right)
$$

We observe that (10) is verified after we pass to the limit in the previous inequality. Thus, we conclude (i) by applying Proposition 2.
(ii) Let $\left(u_{\varsigma}, \lambda_{\varsigma}\right) \in K \times \Lambda$ be a solution of Problem 3. It is easy to observe that the second line of Problem 3 is equivalent with

$$
\begin{equation*}
\mathcal{L}_{\varsigma}\left(u_{\varsigma}, \mu\right) \leq \mathcal{L}_{\varsigma}\left(u_{\varsigma}, \lambda_{\varsigma}\right) \quad \text { for all } \mu \in \Lambda . \tag{14}
\end{equation*}
$$

On the other hand, keeping in mind the symmetry of $a(\cdot, \cdot)$, for all $v \in K$,

$$
\begin{aligned}
& \mathcal{L}_{\varsigma}\left(u_{\varsigma}, \lambda_{\varsigma}\right)-\mathcal{L}_{\varsigma}\left(v, \lambda_{\varsigma}\right) \\
& \leq-\frac{1}{2} a\left(u_{\varsigma}-v, u_{\varsigma}-v\right)-\left[a\left(u_{\varsigma}, v-u_{\varsigma}\right)+b\left(v-u_{\varsigma}, \lambda_{\varsigma}\right)+j(v)-j\left(u_{\varsigma}\right)\right. \\
&\left.+J(\varsigma, v)-J\left(\varsigma, u_{\varsigma}\right)-\left(f, v-u_{\varsigma}\right)_{X}\right] \leq 0 .
\end{aligned}
$$

Therefore, $\left(u_{\varsigma}, \lambda_{\varsigma}\right)$ is a saddle point of $\mathcal{L}_{\zeta}$. Conversely, if $\left(u_{\zeta}, \lambda_{\varsigma}\right) \in K \times \Lambda$ is a saddle point of $\mathcal{L}_{\zeta}$, keeping in mind (14), it is enough to prove that the saddle point $\left(u_{\zeta}, \lambda_{\varsigma}\right)$ verifies the first line of Problem 3. To start, we write

$$
\mathcal{L}_{\zeta}\left(u_{\varsigma}, \lambda_{\varsigma}\right)-\mathcal{L}_{\zeta}\left(w, \lambda_{\varsigma}\right) \leq 0 \quad \text { for all } w \in K
$$

Thus, for all $w \in K$,

$$
\begin{aligned}
& \frac{1}{2} a\left(u_{\varsigma}, u_{\varsigma}\right)-\frac{1}{2} a(w, w)+j\left(u_{\varsigma}\right)-j(w)+J\left(\varsigma, u_{\varsigma}\right)-J(\varsigma, w) \\
& \quad+b\left(u_{\varsigma}-w, \lambda_{\varsigma}\right)+\left(f, w-u_{\varsigma}\right)_{X} \leq 0
\end{aligned}
$$

Setting $w=u_{\varsigma}+t\left(v-u_{\varsigma}\right)$ with $t \in(0,1]$ and $v \in K$, we obtain

$$
\begin{aligned}
& t a\left(u_{\varsigma}, v-u_{\varsigma}\right)+\frac{t^{2}}{2} a\left(v-u_{\varsigma}, v-u_{\varsigma}\right)+t\left(j(v)-j\left(u_{\varsigma}\right)\right) \\
& \quad+t\left(J(\varsigma, v)-J\left(\varsigma, u_{\varsigma}\right)\right)+t b\left(v-u_{\varsigma}, \lambda_{\varsigma}\right) \geq t\left(f, v-u_{\varsigma}\right)_{X}
\end{aligned}
$$

Subsequently, we divide by $t>0$ and then we pass to the limit as $t \rightarrow 0$. As a result, the pair $\left(u_{\varsigma}, \lambda_{\varsigma}\right)$ verifies the first line of Problem 3.

Proposition 4 Assume $\left(h_{1}\right)-\left(h_{7}\right)$. If, in addition, $\psi$ is strictly convex, then the functional $\mathcal{L}_{\zeta}$ has a unique saddle point.

Proof According to Proposition 3, the functional $\mathcal{L}_{\zeta}$ has at least one saddle point. However, $X \ni v \rightarrow a(v, v) \in[0, \infty)$ and $Y \ni \mu \rightarrow \psi(\mu) \in[0, \infty)$ are strictly convex maps. Hence,
$v \rightarrow \mathcal{L}_{\varsigma}(\nu, \mu) \quad$ is strictly convex for all $\mu \in \Lambda$,
$\mu \rightarrow \mathcal{L}_{\varsigma}(v, \mu) \quad$ is strictly concave for all $v \in K$.

We use now Proposition 1 to complete the proof.
Proposition 5 Assume $\left(h_{1}\right)-\left(h_{7}\right)$. Then, Problem 3 has at least one solution that is unique in its first argument. If, in addition, $\psi$ is strictly convex then Problem 3 has a unique solution $\left(u_{\varsigma}, \lambda_{\varsigma}\right) \in K \times \Lambda$.

Proof We apply Proposition 3. Furthermore, as $v \rightarrow \mathcal{L}_{\zeta}(\nu, \mu)$ is strictly convex for all $\mu \in \Lambda$ we deduce that Problem 3 has at least one solution that is unique in its first argument. If, in addition, $\psi$ is strictly convex then we will apply Proposition 4.

## 3 Well-posedness results

To start, we deliver an existence, uniqueness, and stability result by using a Banach fixedpoint argument.

Theorem 1 Assume that $\left(h_{1}\right)-\left(h_{7}\right)$ hold true. If, in addition $M_{J}<m_{a}$, then Problem 1 has at least one solution that is unique and stable in its first component.

Proof Let $\varsigma \in K$ and let $\left(u_{\varsigma}, \lambda_{\varsigma}\right) \in K \times \Lambda$ be a solution of Problem 3. We define an operator

$$
T: K \rightarrow K, \quad T_{\varsigma}=u_{\varsigma} .
$$

According to Proposition 5 the operator $T$ is well defined. Let us prove that $T$ is a contraction. To start, we take $\varsigma_{1}, \varsigma_{2} \in K$ and we denote by $\left(u_{\varsigma_{1}}, \lambda_{\varsigma_{1}}\right)$ and $\left(u_{\varsigma_{2}}, \lambda_{\varsigma_{2}}\right)$ two corresponding solutions of Problem 3. Thus, for every $i \in\{1,2\}$ we can write, for all $v \in K$ and
$\mu \in \Lambda$,

$$
\begin{aligned}
& a\left(u_{\varsigma_{i}}, v-u_{\varsigma_{i}}\right)+b\left(v-u_{\varsigma_{i}}, \lambda_{\varsigma_{i}}\right)+j(v)-j\left(u_{\varsigma_{i}}\right)+J\left(\varsigma_{i}, v\right)-J\left(\varsigma_{i}, u_{\varsigma_{i}}\right) \\
& \quad \geq\left(f, v-u_{\varsigma_{i}}\right)_{X}, \\
& b\left(u_{\varsigma_{i}}, \mu-\lambda_{\varsigma_{i}}\right)-\psi(\mu)+\psi\left(\lambda_{\varsigma_{i}}\right) \leq 0 .
\end{aligned}
$$

By setting $v=u_{\varsigma 2}$ and $\mu=\lambda_{\varsigma 2}$ if $i=1$ and $v=u_{\varsigma 1}$ and $\mu=\lambda_{\varsigma 1}$ if $i=2$, then,

$$
m_{a}\left\|u_{\varsigma_{1}}-u_{\varsigma_{2}}\right\|_{X}^{2} \leq M_{J}\left\|\varsigma_{1}-\varsigma_{2}\right\|_{X}\left\|u_{\varsigma_{1}}-u_{\varsigma_{2}}\right\|_{X}
$$

and from this we deduce that

$$
\left\|u_{\varsigma_{1}}-u_{\varsigma_{2}}\right\|_{X} \leq \frac{M_{J}}{m_{a}}\left\|\varsigma_{1}-\varsigma_{2}\right\|_{X} .
$$

Therefore,

$$
\left\|T \varsigma_{1}-T \varsigma_{2}\right\|_{X} \leq \frac{M_{J}}{m_{a}}\left\|\varsigma_{1}-\varsigma_{2}\right\|_{X}
$$

As $M_{J}<m_{a}$, we conclude that $T$ is a contraction. Thus, we can apply Banach's fixed-point theorem.
Let $\varsigma^{*}$ be the unique fixed point of $T$. It is easy to observe that the pair $\left(u_{\varsigma^{*}}, \lambda_{\varsigma^{*}}\right) \in K \times \Lambda$ is a solution of Problem 1.

Let $\left(u_{i}, \lambda_{i}\right) \in K \times \Lambda(i \in\{1,2\})$ be two solutions of Problem 1. Thus, for all $v \in K$ and $\mu \in \Lambda$,

$$
\begin{aligned}
& a\left(u_{i}, v-u_{i}\right)+b\left(v-u_{i}, \lambda_{i}\right)+j(v)-j\left(u_{i}\right)+J\left(u_{i}, v\right)-J\left(u_{i}, u_{i}\right) \geq\left(f, v-u_{i}\right)_{X}, \\
& b\left(u_{i}, \mu-\lambda_{i}\right)-\psi(\mu)+\psi\left(\lambda_{i}\right) \leq 0 .
\end{aligned}
$$

Setting $v=u_{2}$ and $\mu=\lambda_{2}$ if $i=1$ and $v=u_{1}$ and $\mu=\lambda_{1}$ if $i=2$, then we are led to

$$
\left(m_{a}-M_{J}\right)\left\|u_{1}-u_{2}\right\|_{X}^{2} \leq 0 .
$$

As $M_{J}<m_{a}$, we obtain $u_{1}=u_{2}$.
Let $f_{1}, f_{2} \in X$ be two data and let $\left(u_{1}, \lambda_{1}\right),\left(u_{2}, \lambda_{2}\right)$ be two corresponding solutions. Then, for each $i \in\{1,2\}$ we can write,

$$
\begin{aligned}
& a\left(u_{i}, v-u_{i}\right)+b\left(v-u_{i}, \lambda_{i}\right)+j(v)-j\left(u_{i}\right) \\
& \quad+J\left(u_{i}, v_{i}\right)-J\left(u_{i}, u_{i}\right) \geq\left(f_{i}, v-u_{i}\right)_{X} \quad \text { for all } v \in K, \\
& b\left(u_{i}, \mu-\lambda_{i}\right)-\psi(\mu)+\psi\left(\lambda_{i}\right) \leq 0 \quad \text { for all } \mu \in \Lambda .
\end{aligned}
$$

Setting $v=u_{2}, \mu=\lambda_{2}$ if $i=1$ and $v=u_{1}, \mu=\lambda_{1}$ if $i=2$ then,

$$
a\left(u_{1}-u_{2}, u_{1}-u_{2}\right) \leq\left|\left(f_{1}-f_{2}, u_{1}-u_{2}\right)_{X}\right|+M_{J}\left\|u_{1}-u_{2}\right\|_{X}^{2} .
$$

As $M_{J}<m_{a}$ we deduce that

$$
\left\|u_{1}-u_{2}\right\|_{X} \leq \frac{1}{m_{a}-M_{J}}\left\|f_{1}-f_{2}\right\|_{X}
$$

As a result, the solution is stable in its first component.

Corollary 1 Admit $\left(h_{1}\right)-\left(h_{4}\right),\left(h_{6}\right)$, and $\left(h_{7}\right)$ and the smallness assumption $M_{J}<m_{a}$. Then, Problem 2 has at least one solution $\left(u_{0}, \lambda_{0}\right)$ that is unique and stable in its first component.

Proof We set $\psi \equiv 0$ in Problem 1 and then we apply Theorem 1.

Theorem 2 (A uniqueness result) Assume that $\left(h_{1}\right)-\left(h_{7}\right)$ hold true together with the smallness assumption $M_{J}<m_{a}$. If, in addition, $\psi$ is strictly convex, then Problem 1 has a unique solution.

Proof According to Theorem 1, we can denote by $u$ the unique first component of each pair solution of Problem 1. Assuming that $\left(u, \lambda_{1}\right)$ and $\left(u, \lambda_{2}\right)$ are two solutions of Problem 1, by the second line of Problem 1 we can write

$$
\begin{align*}
& \psi\left(\lambda_{1}\right)-b\left(u, \lambda_{1}\right) \leq \psi(\mu)-b(u, \mu) ;  \tag{15}\\
& \psi\left(\lambda_{2}\right)-b\left(u, \lambda_{2}\right) \leq \psi(\mu)-b(u, \mu) . \tag{16}
\end{align*}
$$

Therefore, $\lambda_{1}$ and $\lambda_{2}$ are minimizers for the following functional

$$
\mathcal{H}_{u}: \Lambda \rightarrow \mathbb{R}, \quad \mathcal{H}_{u}=\psi(\mu)-b(u, \mu) .
$$

Keeping in mind the properties of $\psi(\cdot)$ and $b(\cdot, \cdot)$, by using a standard minimization argument, see, e.g., Theorem 1.36 in [21], it is easy to observe that $\mathcal{H}_{u}$ has a unique minimizer. Note that, when $\Lambda$ is unbounded, the property $\psi(\mu) \geq c_{\psi}\|\mu\|_{Y}^{q}(q>1)$ is crucial in order to obtain the coercivity of $\mathcal{H}_{u}$. In consequence, Problem 1 has a unique solution.

Remark 1 It is worth underlining that, under the hypotheses of Theorem 2, the unique solution of Problem 1 is the unique saddle point of the functional $\mathcal{L}_{\zeta^{*}}$, where $\varsigma^{*}$ is the unique fixed point of the operator $T$.

Remark 2 The smallness assumption $M_{J}<m_{a}$ was introduced for mathematical reasons. Obtaining results without any smallness assumption is left open.

## 4 A convergence result

In this section we pay attention to a special situation when Problem 1 can be seen as a perturbed problem by considering $\psi(\mu)=\epsilon \bar{\psi}(\mu)(\epsilon>0)$.

Problem 4 Let $\epsilon>0$. Given $f \in X$, find $\left(u_{\epsilon}, \lambda_{\epsilon}\right) \in K \times \Lambda$ such that,

$$
\begin{align*}
& a\left(u_{\epsilon}, v-u_{\epsilon}\right)+b\left(v-u_{\epsilon}, \lambda_{\epsilon}\right)+j(v)-j\left(u_{\epsilon}\right)  \tag{17}\\
& \quad+J\left(u_{\epsilon}, v\right)-J\left(u_{\epsilon}, u_{\epsilon}\right) \geq\left(f, v-u_{\epsilon}\right)_{X} \quad \text { for all } v \in K, \\
& b\left(u_{\epsilon}, \mu-\lambda_{\epsilon}\right)-\epsilon \bar{\psi}(\mu)+\epsilon \bar{\psi}\left(\lambda_{\epsilon}\right) \leq 0 \quad \text { for all } \mu \in \Lambda . \tag{18}
\end{align*}
$$

We admit the following hypotheses:
(H1) $\left(h_{1}\right)-\left(h_{4}\right) ;\left(h_{6}\right)-\left(h_{7}\right) ; M_{J}<m_{a}$;
(H2) $\bar{\psi}: Y \rightarrow[0, \infty)$ is strictly convex and lower semicontinuous and, in addition, there exist $c_{\bar{\psi}}, \bar{c}_{\bar{\psi}}>0$ and $\bar{q}>1$ such that, for all $\mu \in Y$,

$$
c_{\bar{\psi}}\|\mu\|_{Y}^{\bar{q}} \leq \bar{\psi}(\mu) \leq \bar{c}_{\bar{\psi}}\|\mu\|_{Y}^{\bar{q}} .
$$

The following boundedness result will be helpful.

Lemma 1 Let $\epsilon>0$ and let $\left(u_{\epsilon}, \lambda_{\epsilon}\right)$ be the unique solution of Problem 4. Then, $\left(u_{\epsilon}\right)_{\epsilon>0}$ and $\left(\lambda_{\epsilon}\right)_{\epsilon>0}$ are bounded sequences in $K$ and $\Lambda$, respectively.

Proof Let $\epsilon>0$ and let $\left(u_{\epsilon}, \lambda_{\epsilon}\right)$ be the unique solution of Problem 4. Let us set $v=0_{X}$ and $\mu=0_{Y}$ in Problem 4. Hence,

$$
\begin{aligned}
& a\left(u_{\epsilon}, u_{\epsilon}\right) \leq\left(f, u_{\epsilon}\right)_{X}-b\left(u_{\epsilon}, \lambda_{\epsilon}\right)+j\left(0_{X}\right)-j\left(u_{\epsilon}\right)+J\left(u_{\epsilon}, 0_{X}\right)-J\left(u_{\epsilon}, u_{\epsilon}\right), \\
& -b\left(u_{\epsilon}, \lambda_{\epsilon}\right)-\epsilon \bar{\psi}\left(0_{Y}\right)+\epsilon \bar{\psi}\left(\lambda_{\epsilon}\right) \leq 0 .
\end{aligned}
$$

Combining these above relations, as $J\left(u_{\epsilon}, 0_{X}\right)=0, \bar{\psi}\left(0_{Y}\right)=0$ and $j(\cdot), \psi(\cdot), J(\cdot, \cdot)$ are nonnegative maps, then

$$
a\left(u_{\epsilon}, u_{\epsilon}\right) \leq\left(f, u_{\epsilon}\right)_{X}+j\left(0_{X}\right) .
$$

Hence,

$$
m_{a}\left\|u_{\epsilon}\right\|_{X}^{2} \leq \frac{1}{2 k}\|f\|_{X}^{2}+\frac{k}{2}\left\|u_{\epsilon}\right\|_{X}^{2}+j\left(0_{X}\right)
$$

where $k>0$. By setting $k=m_{a}$ we are led to

$$
\begin{equation*}
\left\|u_{\epsilon}\right\|_{X} \leq M:=\sqrt{\frac{2}{m_{a}}\left(\frac{\|f\|_{X}^{2}}{2 m_{a}}+j\left(0_{X}\right)\right)} . \tag{19}
\end{equation*}
$$

Hence, $\left(u_{\epsilon}\right)_{\epsilon>0}$ is a bounded sequence in $K$.
If $\Lambda$ is bounded, then there exists $c_{\Lambda}>0$ such that

$$
\left\|\lambda_{\epsilon}\right\|_{Y} \leq c_{\Lambda} \quad \text { for all } \epsilon>0
$$

As a result, in the case $\Lambda$ is bounded, $\left(\lambda_{\epsilon}\right)_{\epsilon}$ is a bounded sequence in $\Lambda$. If $\Lambda$ is unbounded (recall that, according to $\left(h_{6}\right), K \subseteq X$ is a linear subspace), we set $v=u_{\epsilon}-\frac{w}{\|w\|_{X}}$ with $w \in$ $K$, in (17). As $a(\cdot, \cdot)$ is bilinear continuous of rank $M_{a}, j(\cdot)$ is nonnegative and Lipschitz continuous of $\operatorname{rank} L_{j}$, and

$$
J\left(u_{\epsilon}, v\right) \leq c_{J}\left(\left\|u_{\epsilon}\right\|_{X}+1\right)\|v\|_{X} \leq c_{J}\left(\left\|u_{\epsilon}\right\|_{X}+1\right)^{2}
$$

then, keeping in mind (19) we arrive at

$$
\frac{b\left(w, \lambda_{\epsilon}\right)}{\|w\|_{X}} \leq\|f\|_{X}+M_{a} M+c_{J}(M+1)^{2}+L_{j} .
$$

By the inf-sup property of the form $b(\cdot, \cdot)$, we obtain

$$
\begin{equation*}
\left\|\lambda_{\epsilon}\right\|_{Y} \leq \frac{1}{\alpha}\left[\|f\|_{X}+M_{a} M+c_{J}(M+1)^{2}+L_{j}\right] . \tag{20}
\end{equation*}
$$

As a consequence, $\left(\lambda_{\epsilon}\right)_{\epsilon>0}$ is a bounded sequence in the unbounded subset $\Lambda$.

Recall that, according to Corollary 1, Problem 2 has at least one solution $\left(u_{0}, \lambda_{0}\right) \in K \times \Lambda$ that is unique in its first argument.

In order to deliver the main result of this section we need the following additional assumption.
(H3) Let $\left(u_{\epsilon}\right)_{\epsilon>0} \subset K$ be such that $u_{\epsilon} \rightarrow u$ in $X$ as $\epsilon \rightarrow 0$ and let $\left(\lambda_{\epsilon}\right)_{\epsilon>0} \subset \Lambda$ be such that $\lambda_{\epsilon} \rightharpoonup \lambda$ in $Y$ as $\epsilon \rightarrow 0$. Then, $b\left(u_{\epsilon}, \lambda_{\epsilon}\right) \rightarrow b(u, \lambda)$ as $\epsilon \rightarrow 0$.
Let $\left(u_{\epsilon}\right)_{\epsilon>0} \subset K$ be such that $u_{\epsilon} \rightarrow u$ in $X$ and let $v \in K$. Then, $\lim \sup _{\epsilon \rightarrow 0}\left[J\left(u_{\epsilon}, v\right)-\right.$ $\left.J\left(u_{\epsilon}, u_{\epsilon}\right)\right] \leq J(u, v)-J(u, u)$.

Theorem 3 Assume (H1)-(H3). Let $\left(\left(u_{\epsilon}, \lambda_{\epsilon}\right)\right)_{\epsilon>0} \subset K \times \Lambda$ be a sequence such that, for each $\epsilon>0,\left(u_{\epsilon}, \lambda_{\epsilon}\right)$ is the unique solution of Problem 4. Then, there exists a subsequence $\left(\left(u_{\epsilon^{\prime}}, \lambda_{\epsilon^{\prime}}\right)\right)_{\epsilon^{\prime}} \subset K \times \Lambda$ and there exists a solution of Problem $2\left(u_{0}, \bar{\lambda}_{0}\right)$ such that $u_{\epsilon^{\prime}} \rightarrow u_{0}$ and $\lambda_{\epsilon^{\prime}} \rightharpoonup \bar{\lambda}_{0}$ as $\epsilon^{\prime} \rightarrow 0$.

Proof Let $\epsilon>0$, let $\left(u_{\epsilon}, \lambda_{\epsilon}\right) \in K \times \Lambda$ be the unique solution of Problem 4 and let $\left(u_{0}, \lambda_{0}\right) \in$ $K \times \Lambda$ be a solution of Problem 2. Keeping in mind Lemma 1 and passing eventually to a subsequence, we deduce that there exist $\tilde{u} \in K$ and $\tilde{\lambda} \in \Lambda$ such that $u_{\epsilon^{\prime}} \rightharpoonup \tilde{u}$ and $\lambda_{\epsilon^{\prime}} \rightharpoonup \tilde{\lambda}$ as $\epsilon^{\prime} \rightarrow 0$.
Let $\epsilon^{\prime}>0$. According to Problem 4 we can write:

$$
\begin{align*}
& a\left(u_{\epsilon^{\prime}}, v-u_{\epsilon^{\prime}}\right)+b\left(v-u_{\epsilon^{\prime}}, \lambda_{\epsilon^{\prime}}\right)+j(v)-j\left(u_{\epsilon^{\prime}}\right)  \tag{21}\\
& \quad+J\left(u_{\epsilon^{\prime}}, v\right)-J\left(u_{\epsilon^{\prime}}, u_{\epsilon^{\prime}}\right) \geq\left(f, v-u_{\epsilon^{\prime}}\right)_{X} \quad \text { for all } v \in K, \\
& b\left(u_{\epsilon^{\prime}}, \mu-\lambda_{\epsilon^{\prime}}\right)-\epsilon^{\prime} \bar{\psi}(\mu)+\epsilon^{\prime} \bar{\psi}\left(\lambda_{\epsilon^{\prime}}\right) \leq 0 \quad \text { for all } \mu \in \Lambda . \tag{22}
\end{align*}
$$

On the other hand,

$$
\begin{align*}
& a\left(u_{0}, v-u_{0}\right)+b\left(v-u_{0}, \lambda_{0}\right)+j(v)-j\left(u_{0}\right)  \tag{23}\\
& \quad+J\left(u_{0}, v\right)-J\left(u_{0}, u_{0}\right) \geq\left(f, v-u_{0}\right)_{X} \quad \text { for all } v \in K, \\
& b\left(u_{0}, \mu-\lambda_{0}\right) \leq 0 \quad \text { for all } \mu \in \Lambda . \tag{24}
\end{align*}
$$

We take $v=u_{\epsilon^{\prime}}$ in (23) and $v=u_{0}$ in (21) to obtain,

$$
\begin{align*}
& a\left(u_{0}-u_{\epsilon^{\prime}}, u_{\epsilon^{\prime}}-u_{0}\right)+J\left(u_{\epsilon^{\prime}}, u_{0}\right)-J\left(u_{\epsilon^{\prime}}, u_{\epsilon^{\prime}}\right)+J\left(u_{0}, u_{\epsilon^{\prime}}\right)-J\left(u_{0}, u_{0}\right)  \tag{25}\\
& \quad+b\left(u_{\epsilon^{\prime}}-u_{0}, \lambda_{0}-\lambda_{\epsilon^{\prime}}\right) \geq 0 .
\end{align*}
$$

Setting now $\mu=\lambda_{\epsilon^{\prime}}$ in (24) and $\mu=\lambda_{0}$ in (22), we obtain

$$
\begin{equation*}
b\left(u_{\epsilon^{\prime}}-u_{0}, \lambda_{0}-\lambda_{\epsilon^{\prime}}\right) \leq \epsilon^{\prime} \bar{\psi}\left(\lambda_{0}\right)-\epsilon^{\prime} \bar{\psi}\left(\lambda_{\epsilon^{\prime}}\right) \leq \epsilon^{\prime} \bar{\psi}\left(\lambda_{0}\right) . \tag{26}
\end{equation*}
$$

Combining (25) and (26) we are led to

$$
\left\|u_{\epsilon^{\prime}}-u_{0}\right\|_{X}^{2} \leq \frac{\epsilon^{\prime}}{m_{a}-M_{J}} \bar{\psi}\left(\lambda_{0}\right) .
$$

By passing to the limit as $\epsilon^{\prime} \rightarrow 0$ in the relation above, we obtain that $u_{\epsilon^{\prime}} \rightarrow u_{0}$. Due to the uniqueness of the limit, we conclude that $\widetilde{u}=u_{0}$.
Because $\left(\lambda_{\epsilon^{\prime}}\right)_{\epsilon^{\prime}}$ is a weakly convergent sequence and $0 \leq \bar{\psi}\left(\lambda_{\epsilon^{\prime}}\right) \leq \bar{c}_{\bar{\psi}}\left\|\lambda_{\epsilon^{\prime}}\right\|_{Y}^{p}(p>0)$, then $\epsilon^{\prime} \bar{\psi}\left(\lambda_{\epsilon^{\prime}}\right) \rightarrow 0$ as $\epsilon^{\prime} \rightarrow 0$. Therefore, keeping in mind the working hypotheses and passing to the limit as $\epsilon^{\prime} \rightarrow 0$ in Problem 4 we conclude that $\left(u_{0}, \tilde{\lambda}\right) \in K \times \Lambda$ verifies Problem 2. Thus, we can consider $\bar{\lambda}_{0}=\tilde{\lambda}$.

Remark 3 The entire sequence $\left(u_{\epsilon}\right)_{\epsilon>0}$ is strongly convergent to $u_{0}$ when $\epsilon \rightarrow 0$ because the unique limit $u_{0}$ is independent of the subsequences $\left(u_{\epsilon^{\prime}}\right)_{\epsilon^{\prime}>0}$.

## 5 Examples

In this section we are going to illustrate the theory by means of examples of contact models involving multicontact zones.
Let us consider an elastic body that occupies a bounded domain $\Omega \subset \mathbb{R}^{3}$, with a Lipschitz continuous boundary. The boundary, denoted by $\Gamma$, is partitioned in three measurable parts $\Gamma_{D}, \Gamma_{N}$, and $\Gamma_{C}$ with positive measure. The body $\Omega$ is clamped on $\Gamma_{D}$, body forces of density $\mathbf{f}_{0}$ act on $\Omega$ and surface traction of density $\mathbf{f}_{2}$ acts on $\Gamma_{N}$. On the part $\Gamma_{C}$ the body is or can be in contact (frictional or frictionless) with foundations or obstacles. The mathematical model can be described by means of the following boundary value problem.

Problem 5 Find $\mathbf{u}: \bar{\Omega} \rightarrow \mathbb{R}^{3}$ and $\sigma: \bar{\Omega} \rightarrow \mathbb{S}^{3}$ such that

$$
\begin{align*}
& \operatorname{Div} \boldsymbol{\sigma}+\mathbf{f}_{0}=\mathbf{0} \quad \text { in } \Omega, \\
& \boldsymbol{\sigma}=\mathcal{E} \boldsymbol{\varepsilon}(\mathbf{u}) \quad \text { in } \Omega, \\
& \mathbf{u}=\mathbf{0} \quad \text { on } \Gamma_{D},  \tag{27}\\
& \boldsymbol{\sigma} \boldsymbol{v}=\mathbf{f}_{2} \quad \text { on } \Gamma_{N},
\end{align*}
$$

contact conditions and friction laws on $\Gamma_{C}$.

As usual, $\mathbf{u}=\left(u_{i}\right)$ stands for the displacement field, $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(\mathbf{u})=\left(\varepsilon_{i j}(\mathbf{u})\right)$ denotes the infinitesimal strain tensor, $\mathcal{E}$ is the elastic tensor and $\sigma=\left(\sigma_{i j}\right)$ is the Cauchy stress tensor in the linearized theory. By means of $\bar{\Omega}$ we denote $\Omega \cup \partial \Omega$. Herein, $\mathbb{S}^{3}$ denotes the space of second-order symmetric tensors on $\mathbb{R}^{3}$. Every field in $\mathbb{R}^{3}$ or $\mathbb{S}^{3}$ is typeset in boldface. Everywhere below, by • and : we will denote the inner product on $\mathbb{R}^{3}$ and $\mathbb{S}^{3}$, respectively, and by $\|\cdot\|_{\mathbb{R}^{3}},\|\cdot\|_{\mathbb{S}^{3}}$ we will denote the Euclidean norm on $\mathbb{R}^{3}$ and $\mathbb{S}^{3}$, respectively. The unit outward normal vector to the boundary is denoted by $v$ and is defined almost everywhere. The normal and the tangential components of the displacement field will be denoted by $u_{v}=\mathbf{u} \cdot \boldsymbol{v}$ and $\mathbf{u}_{\tau}=\mathbf{u}-u_{v} \boldsymbol{v}$, respectively; the normal and the tangential components of the Cauchy vector $\boldsymbol{\sigma} \boldsymbol{v}$ on the boundary will be given by the formulas $\sigma_{\nu}=(\boldsymbol{\sigma} \boldsymbol{v}) \cdot \boldsymbol{v}$, $\boldsymbol{\sigma}_{\tau}=\boldsymbol{\sigma} \boldsymbol{v}-\sigma_{\nu} \boldsymbol{v}$.

In order to deliver a first example, we consider the Problem 5 in which the line (27) consists of the following relations:

$$
\begin{align*}
& -\sigma_{v}=F, \quad\left\|\boldsymbol{\sigma}_{\tau}\right\|_{\mathbb{R}^{3}} \leq k\left|\sigma_{\nu}\right|, \quad \boldsymbol{\sigma}_{\tau}=-k\left|\sigma_{\nu}\right| \frac{\mathbf{u}_{\tau}}{\left\|\mathbf{u}_{\tau}\right\|_{\mathbb{R}^{3}}} \quad \text { if } \mathbf{u}_{\tau} \neq \mathbf{0} \quad \text { on } \Gamma_{\alpha} ;  \tag{28}\\
& -\sigma_{\nu}=p_{\nu}\left(u_{\nu}\right), \quad\left\|\boldsymbol{\sigma}_{\tau}\right\|_{\mathbb{R}^{3}} \leq p_{\tau}\left(u_{\nu}\right), \quad \boldsymbol{\sigma}_{\tau}=-p_{\tau}\left(u_{\nu}\right) \frac{\mathbf{u}_{\tau}}{\left\|\mathbf{u}_{\tau}\right\|_{\mathbb{R}^{3}}} \quad \text { if } \mathbf{u}_{\tau} \neq \mathbf{0} \text { on } \Gamma_{\beta} ;  \tag{29}\\
& \sigma_{\nu} \leq 0, \quad u_{\nu} \leq 0, \quad \sigma_{\nu} u_{\nu}=0, \quad \boldsymbol{\sigma}_{\tau}=\mathbf{0} \quad \text { on } \Gamma_{\delta} ;  \tag{30}\\
& u_{\nu}=0, \quad\left\|\boldsymbol{\sigma}_{\tau}\right\|_{\mathbb{R}^{3}} \leq \zeta, \quad \boldsymbol{\sigma}_{\tau}=-\zeta \frac{\mathbf{u}_{\tau}}{\left\|\mathbf{u}_{\tau}\right\|_{\mathbb{R}^{3}}} \quad \text { if } \mathbf{u}_{\tau} \neq \mathbf{0} \text { on } \Gamma_{\chi} . \tag{31}
\end{align*}
$$

In this case, the part $\Gamma_{C}$ is partitioned into four zones $\Gamma_{\alpha}, \Gamma_{\beta}, \Gamma_{\delta}$, and $\Gamma_{\chi}$. On $\Gamma_{\alpha}$, the body is in frictional contact with a foundation such that the normal stress is imposed, on $\Gamma_{\beta}$ the body is in frictional contact with normal compliance with an obstacle, on $\Gamma_{\delta}$ the body can be in frictionless unilateral contact with a rigid obstacle and on $\Gamma_{\chi}$ the body is in bilateral frictional (Tresca) contact with a rigid foundation. For details on (28)-(31) the reader can consult, e.g., $[9,21]$ and the references therein. Herein, $k$ is a friction coefficient, $F$ is the imposed normal stress, $p_{\nu}, p_{\tau}$ are given normal compliance functions and $\zeta$ is a friction bound. Using a variational technique governed by Lagrange multipliers ( $\lambda$ can be introduced by means of $\sigma_{\tau}$ ), see, e.g., [15], one can arrive at a variational formulation related to Problem 1 via the following example.

## Example 1

- $X=\left\{\mathbf{v} \in H^{1}(\Omega)^{3}: \gamma \mathbf{v}=\mathbf{0}\right.$ a.e. on $\left.\Gamma_{D}\right\}$ and $Y$ is the dual of $S\left(Y=S^{\prime}\right)$, where $S=\boldsymbol{\gamma}(X)=\{\widetilde{\mathbf{v}}=\boldsymbol{\gamma} \mathbf{v} \mid \mathbf{v} \in X\}$. Herein and everywhere below $\boldsymbol{\gamma}$ stands for the Sobolev trace operator for vectors, $\boldsymbol{\gamma}: H^{1}(\Omega)^{3} \rightarrow L^{2}(\Gamma)^{3}$.
- $a: X \times X \rightarrow \mathbb{R}, a(\mathbf{u}, \mathbf{v})=\int_{\Omega} \mathcal{E}(\boldsymbol{\varepsilon}(\mathbf{u})(\mathbf{x})): \boldsymbol{\varepsilon}(\mathbf{v})(\mathbf{x}) d x$ where $\mathcal{E}: \mathbb{S}^{3} \rightarrow \mathbb{S}^{3}$ is a fourth-order tensor such that $\mathcal{E}_{i j k l}=\mathcal{E}_{k l i j}=\mathcal{E}_{j i k l} \in \mathbb{R}, 1 \leq i, j, k, l \leq 3$ and there exists $m_{\mathcal{E}}>0$, $\mathcal{E} \boldsymbol{\tau}: \boldsymbol{\tau} \geq m_{\mathcal{E}}\|\boldsymbol{\tau}\|_{\mathbb{S}^{3}}^{2}$ for all $\boldsymbol{\tau} \in \mathbb{S}^{3}$.
- $b: X \times Y \rightarrow \mathbb{R}, b(\mathbf{v}, \mu)=\langle\mu, \gamma \mathbf{v}\rangle$ where $\langle\cdot, \cdot\rangle$ denotes the duality pairing between $Y$ and $S$.
- $j: X \rightarrow[0, \infty), j(\mathbf{v})=\int_{\Gamma_{\alpha}} k F\left|v_{\nu}\right| d \Gamma$ where $k, F>0$.
- $J: X \times X \rightarrow[0, \infty), J(\mathbf{u}, \mathbf{v})=\int_{\Gamma_{\beta}}\left[p_{v}\left(u_{v}\right)\left|v_{v}\right|+p_{\tau}\left(u_{v}\right)\left\|\mathbf{v}_{\tau}\right\|_{\mathbb{R}^{3}}\right] d \Gamma$, where
$p_{v}, p_{\tau}: \mathbb{R} \rightarrow[0, \infty)$ are given Lipschitz continuous functions such that $p_{v}(r)=p_{\tau}(r)=0$ for all $r \leq 0$.
- $K=\left\{\mathbf{v} \in X \mid v_{v} \leq 0\right.$ a.e. on $\left.\Gamma_{\delta}\right\}$. The set $K$ is an unbounded closed convex set that contains $0_{X}$.
- $\Lambda=\left\{\mu \in Y:\langle\mu, \gamma \mathbf{v}\rangle \leq \int_{\Gamma_{\chi}} \zeta\left\|\mathbf{v}_{\tau}\right\|_{\mathbb{R}^{3}} d \Gamma\right\}$. The set $\Lambda$ is a bounded closed convex set that contains $0_{Y}$.
- If $\psi \equiv 0$, this example is related to a weak formulation of Problem 5. If $\psi(\mu)=\epsilon\|\mu\|_{Y}^{2}$, this example is related to a "perturbed weak formulation".

In order to deliver a second example, we consider the following relations:

$$
\begin{align*}
& u_{v}=0, \quad\left\|\boldsymbol{\sigma}_{\tau}\right\|_{\mathbb{R}^{3}} \leq g\left(\left\|\mathbf{u}_{\tau}\right\|_{\mathbb{R}^{3}}\right), \quad \sigma_{\tau}=-g\left(\left\|\mathbf{u}_{\tau}\right\|_{\mathbb{R}^{3}}\right) \frac{\mathbf{u}_{\tau}}{\left\|\mathbf{u}_{\tau}\right\|_{\mathbb{R}^{3}}} \quad \text { if } \mathbf{u}_{\tau} \neq \mathbf{0} \text { on } \Gamma_{\partial} ;  \tag{32}\\
& u_{v} \leq 0, \quad \sigma_{v} \leq 0, \quad \sigma_{v} u_{v}=0 \quad \text { on } \Gamma_{\delta} ; \tag{33}
\end{align*}
$$

$$
\begin{align*}
& u_{v}=0, \quad\left\|\boldsymbol{\sigma}_{\tau}\right\|_{\mathbb{R}^{3}} \leq \zeta, \quad \boldsymbol{\sigma}_{\tau}=-\zeta \frac{\mathbf{u}_{\tau}}{\left\|\mathbf{u}_{\tau}\right\|_{\mathbb{R}^{3}}} \quad \text { if } \mathbf{u}_{\tau} \neq \mathbf{0} \text { on } \Gamma_{\chi} ;  \tag{34}\\
& -\sigma_{v}=F, \quad\left\|\boldsymbol{\sigma}_{\tau}\right\|_{\mathbb{R}^{3}} \leq k\left|\sigma_{\nu}\right|, \quad \boldsymbol{\sigma}_{\tau}=-k\left|\sigma_{\nu}\right| \frac{\mathbf{u}_{\tau}}{\left\|\mathbf{u}_{\tau}\right\|_{\mathbb{R}^{3}}} \quad \text { if } \mathbf{u}_{\tau} \neq \mathbf{0} \text { on } \Gamma_{\alpha} . \tag{35}
\end{align*}
$$

In this case, the part $\Gamma_{C}$ is partitioned into four zones $\Gamma_{\partial}, \Gamma_{\delta}, \Gamma_{\chi}$, and $\Gamma_{\alpha}$. On $\Gamma_{\partial}$ the body is in bilateral frictional contact with a rigid foundation; herein we use a slip-dependent friction law, $g$ being a given slip-dependent friction bound. Details on (32) can be found in $[9,20]$ and the references therein. The other contact conditions and friction laws were already encountered in the previous example. By a variational approach with Lagrange multipliers ( $\lambda$ is defined by means of $\sigma_{\nu}$ ), we are led to Problem 1 by setting $X, Y, a, b, j, J, \psi, K$, and $\Lambda$ as follows.

## Example 2

- $X=K=\left\{\mathbf{v} \in H^{1}(\Omega)^{3}: \gamma \mathbf{v}=\mathbf{0}\right.$ a.e. on $\Gamma_{D}, v_{v}=0$ a.e. on $\left.\Gamma_{\partial} \cup \Gamma_{\chi}\right\}$. The correspondence law for $a(\cdot, \cdot)$ as in the previous example.
- $S=\left\{\tilde{v} \mid \widetilde{v}=\boldsymbol{\gamma}(\mathbf{v}) \cdot \boldsymbol{v}_{\left.\right|_{\delta}}, \boldsymbol{v}_{\left.\right|_{\Gamma_{\delta}}}=\right.$ const. $\}$, and $Y=S^{\prime}$. The correspondence law for $b(\cdot, \cdot)$ as in the previous example.
- $j: X \rightarrow[0, \infty), j(\mathbf{v})=\int_{\Gamma_{\chi}} \zeta\left\|\mathbf{v}_{\tau}\right\|_{\mathbb{R}^{3}} d \Gamma$.
- $\Lambda=\left\{\mu \in Y:\left\langle\mu, v_{v}\right\rangle \leq 0\right.$ for all $\left.\mathbf{v} \in \mathcal{K}\right\}$, where $\mathcal{K}=\left\{\mathbf{v} \in X: v_{v} \leq 0\right.$ a.e. on $\left.\Gamma_{\delta}\right\}$.
- $J: X \times X \rightarrow[0, \infty), J(\mathbf{u}, \mathbf{v})=\int_{\Gamma_{\partial}} g\left(\left\|\mathbf{u}_{\tau}\right\|_{\mathbb{R}^{3}}\right)\left\|\mathbf{u}_{\tau}\right\|_{\mathbb{R}^{3}} d \Gamma$ where $g: \mathbb{R} \rightarrow[0, \infty)$ is a Lipschitz continuous function.
- If $\psi(\mu)=\epsilon c(\mu, \mu)$ where $c(\cdot, \cdot)$ is a symmetric, bilinear, continuous, and $Y$-elliptic form, then this second example is related to a "perturbed weak formulation".

We can continue by giving more examples involving other contact conditions and friction laws. For simplicity, we will "play" with the aforementioned contact conditions and contact laws in order to deliver an example of a contact problem with more than four contact zones. For instance, we consider Problem 5 setting on the line (27) the following contact conditions and friction laws: (29), (32), (33), (34), and (35). In this case, $\Gamma_{C}$ is partitioned into five parts $\Gamma_{\beta}, \Gamma_{\partial}, \Gamma_{\delta}, \Gamma_{\chi}$, and $\Gamma_{\alpha}$. A weak formulation of Problem 5 via Lagrange multipliers can be related to Problem 1 by means of the next example.

## Example 3

- $X=\left\{\mathbf{v} \in H^{1}(\Omega)^{3}: \gamma \mathbf{v}=\mathbf{0}\right.$ on $\Gamma_{D}, v_{v}=0$ on $\left.\Gamma_{\chi} \cup \Gamma_{\beta} \cup \Gamma_{\partial}\right\} ; K=X$.
- $S=\left\{\tilde{v} \mid \widetilde{v}=\boldsymbol{\gamma}(\mathbf{v}) \cdot \boldsymbol{v}_{\left.\right|_{\Gamma_{\delta}}}, \boldsymbol{v}_{\left.\right|_{\Gamma_{\delta}}}=\right.$ const. $\}, Y=S^{\prime}$.
- The correspondence laws for $a(\cdot, \cdot), b(\cdot, \cdot), \psi$, as in the previous examples.
- $j: X \rightarrow[0, \infty), j(\mathbf{v})=\int_{\Gamma_{\alpha}} k F\left|v_{v}\right| d \Gamma+\int_{\Gamma_{X}} \zeta\left\|\mathbf{v}_{\tau}\right\|_{\mathbb{R}^{3}} d \Gamma$.
- $J(\mathbf{u}, \mathbf{v})=\int_{\Gamma_{\jmath}} g\left(\left\|\mathbf{u}_{\tau}\right\|_{\mathbb{R}^{3}}\right)\left\|\mathbf{u}_{\tau}\right\|_{\mathbb{R}^{3}} d \Gamma+\int_{\Gamma_{\beta}}\left[p_{v}\left(u_{\nu}\right)\left|v_{\nu}\right|+p_{\tau}\left(u_{\nu}\right)\left\|\mathbf{v}_{\tau}\right\|_{\mathbb{R}^{3}}\right] d \Gamma$.
- $\Lambda$ as in the previous example.

Herein, again, $\lambda$ can be defined by means of $\sigma_{\nu}$.

These three examples fulfil the working hypotheses. Their verification is rather standard. However, for the convenience of the reader we can refer to, e.g., $[16,21]$ for helpful arguments.

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