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Nonlinear analysis by applying best approximation method in p-vector spaces

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This paper is dedicated to Professor Shih-sen Chang on his 90th Birthday

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Abstract

It is known that the class of p-vector spaces ($0 < p \leq 1$) is an important generalization of the usual norm spaces with rich topological and geometrical structure, but most tools and general principles with nature in nonlinearity have not been developed yet. The goal of this paper is to develop some useful tools in nonlinear analysis by applying the best approximation approach for the classes of 1-set contractive set-valued mappings in p-vector spaces. In particular, we first develop general fixed point theorems of compact (single-valued) continuous mappings for closed p-convex subsets, which also provide an answer to Schauder's conjecture of 1930s in the affirmative way under the setting of topological vector spaces for $0 < p \leq 1$. Then one best approximation result for upper semicontinuous and 1-set contractive set-valued mappings is established, which is used as a useful tool to establish fixed points of nonself set-valued mappings with either inward or outward set conditions and related various boundary conditions under the framework of locally p-convex spaces for $0 < p \leq 1$. In addition, based on the framework for the study of nonlinear analysis obtained for set-valued mappings with closed p-convex values in this paper, we conclude that development of nonlinear analysis and related tools for singe-valued mappings in locally p-convex spaces for $0 < p \leq 1$ seems even more important, and can be done by the approach established in this paper.

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Keywords: Nonlinear analysis; Best approximation; Fixed points; Schauder conjecture; p-convex; p-vector space; Locally p-convex space; Measure of noncompactness; 1-set contractive mapping; Leray–Schauder alternative; Nonexpansive mapping; p-inward and p-outward set; Opial condition; Demiclosedness principle

1 Introduction

It is known that the class of p-seminorm spaces ($0 < p \leq 1$) is an important generalization of the usual normed spaces with rich topological and geometrical structures, and related study has received a lot of attention (e.g., see Agarwal et al. [1], Alghamdi et al. [5], Balaj [8], Balachandran [7], Bayoumi [9], Bayoumi et al. [10], Bernués and Pena [12], Ding [31], Ennassik and Taoudi [34], Ennassik et al. [33], Gal and Goldstein [40], Gholizadeh et al. [41],

Jarchow [54], Kalton [55, 56], Kalton et al. [57], Machrafi and Oubbi [73], Park [90], Qiu and Rolewicz [99], Rolewicz [103], Silva et al. [112], Simons [113], Tabor et al. [116], Tan [117], Wang [120], Xiao and Lu [123], Xiao and Zhu [124, 125], Yuan [133], and many others). However, to the best of our knowledge, the corresponding basic tools and associated results in the category of nonlinear functional analysis have not been well developed. Thus the goal of this paper is to develop some important tools for nonlinear analysis for 1-set contractive mappings under the framework of p-vector spaces, and in particular, for locally p-convex spaces with $1 < p \leq 1$.

In particular, we first develop general fixed point theorems for compact continuous mappings, which provide an answer to Schauder's conjecture of 1930s in the affirmative way under the general framework of topological vector spaces (with $p = 1$ for p-vector spaces). Then, the one best approximation result for upper semicontinuous and 1-set contractive mappings is given with the boundary condition, which is used as a tool to establish fixed points for nonself set-valued mappings with either inward or outward set conditions in locally p-convex spaces. Finally, we give existence results for solutions of Birkhoff–Kellogg problems, the general principle of nonlinear alternative by including Leray–Schauder alternative, and related results as special classes in locally p-convex spaces. The results given in this paper do not only include the corresponding results in the existing literature as special cases, but are also expected to be useful tools for the study of nonlinear problems arising from social science, engineering, applied mathematics, and related topics and areas under the framewkro of locally p-convex spaces for $0 < p \leq 1$.

Before discussing the study of best approximation and related nonlinear analysis tools under the framework of locally p-convex spaces, we first would like to share with readers that although most of the results in nonlinear analysis are normally highly associated with convexity hypotheses under local convex topological vector spaces (of course, including normed spaces and Banach spaces, nice metric spaces), it seems to be a surprise that p-vector spaces, which in general do not have the local convex structure comparing with locally convex spaces, provide some nice properties in the nature way with some kinds of nice approximation and better (i.e., the bigger) structures for the so-called convexities of p-convex subset play very important roles for us to describe Birkhoff and Kellogg problems and related nonlinear problems (such as fixed point problem and so on) in topological vector spaces (TVS) or locally convex spaces (LCS) based on p-vector space behaviors for p in $(0, 1]$ (a p-vector space reduces to TVS when $p = 1$), and also see the corresponding results and properties as pointed by Remark 2.1 (1), Lemma 2.1(ii), and Lemma 2.3.

Here, we would also like to recall that since the first Birkhoff–Kellogg problem was introduced and an associated theorem was proved by Birkhoff and Kellogg [13] in 1922 in discussing the existence of solutions for the equation $x = \lambda F(x)$, where λ is a real parameter and F is a general nonlinear nonself mapping defined on an open convex subset U of a topological vector space E, now the general form of the Birkhoff–Kellogg problem is to find the so-called invariant direction for the nonlinear set-valued mappings F, i.e., to find $x_0 \in \overline{U}$ and $\lambda > 0$ such that $\lambda x_0 \in F(x_0)$ (one may also consider if x_0 is from the boundary $\overline{U}$).

On the other hand, after the Birkhoff and Kellogg theorem given by Birkhoff and Kellogg in 1920s, the study on Birkhoff–Kellogg problem has received a lot of scholars' attention since then. For example, in 1934, one of the fundamental results in nonlinear functional analysis, famously called Leray–Schauder alternative first establoished by Leray

and Schauder [67] was also studied via topological degree and some other advanced approaches by Agarwal et al. [1], Alghamdi et al. [5], Balaj [8], and many others. Thereafter, certain other types of Leray–Schauder alternatives were proved using techniques other than topological degree, see the works given by Granas and Dugundji [48], Furi and Pera [39] in the Banach space setting and applications to the boundary value problems for ordinary differential equations in noncompact problems, a general class of mappings for nonlinear alternative of Leray–Schauder type in normal topological spaces, and some Birkhoff–Kellogg type theorems for general class mappings in TVS or LCS by Agarwal et al. [2], Agarwal and O'Regan [3, 4], Park [88], O'Regan [81] by using the Leray–Schauder type coincidence theory applying to establish Birkhoff–Kellogg problem, the Furi–Pera type result for a general class of set-valued mappings. In this paper, based on the application of our best approximation as a tool for general 1-set contractive set-valued mappings, we develop the general principle for the existence of solutions for Birkhoff–Kellogg problems, and related nonlinear alternatives, which then also allows us to give general existence of Leray–Schauder type and related fixed point theorems for nonself mappings in general locally p-convex spaces for ($0 < p \leq 1$). These new results given in this paper not only include the corresponding results in the existing literature as special cases, but are also expected to be useful tools for the study of nonlinear problems arising from theory to practice.

Now we give a brief introduction on the best approximation theorem related to the development of the nonlinear analysis as a powerful tool with some background.

We all know that the best approximation in nature is related to fixed points for nonself mappings, which tightly link with the classical Leray–Schauder alternative based on the Leray–Schauder continuation theorem by Leray and Schauder [67], which is a remarkable result in nonlinear analysis; and in addition, there exist several continuation theorems, which have many applications to the study of nonlinear functional equations (see Agarwal et al. [1], Alghamdi et al. [5], Balaj [8], O'Regan and Precup [83]). Historically, it seems that the continuation theorem is based on the idea of obtaining a solution of a given equation, starting from one of the solutions of a simpler equation. The essential part of this theorem is the Leray–Schauder boundary condition. It seems that the continuation method was initiated by Poincare [97] and Bernstein [11]. Certainly, Leray and Schauder [67] in 1934 gave the first abstract formulation of the continuation principle using the topological degree (see also Granas and Dugundji [48], Isac [53], Rothe [104, 105], Zeidler [134]). But in this paper, we will see how the best approximation method could be used for the study of fixed point theorems in p-vector space for $0 < p \leq 1$, which as a basic tool will help us to develop the principle of nonlinear alterative, Leray–Schauder alternative, fixed point theorems of Rothe, Petryshyn, Altman type for nonself mappings, and related fixed point, nonlinear alternative with different boundary conditions. Moreover, the new results given in this paper are highly expected to become useful tools for the study on optimization, nonlinear programming, variational inequality, complementarity, game theory, mathematical economics, and other related social science areas.

It is well known that the best approximation is one of very important aspects for the study of nonlinear problems related to the problems on their solvability for partial differential equations, dynamic systems, optimization, mathematical program, operation research; and in particular, the one approach well accepted for the study of nonlinear problems in optimization, complementarity problems, of variational inequality problems, and

so on, is strongly based on what is today called Fan's best approximation theorem given by Fan [37] in 1969, which acts as a very powerful tool in nonlinear analysis (see also the book of Singh et al. [114] for the related discussion and study on the fixed point theory and best approximation with the KKM-map principle). Among them, the related tools are Rothe type and the principle of Leray–Schauder alterative in topological vector spaces (TVS) and local topological vector spaces (LCS), which are comprehensively studied by Agarwal et al. [1], Alghamdi et al. [5], Balaj [8], Chang et al. [24], Chang et al. [25–27], Carbone and Conti [20], Ennassik and Taoudi [34], Ennassik et al. [33], Isac [53], Granas and Dugundji [48], Kirk and Shahzad [60], Liu [72], Park [91], Rothe [104, 105], Shahzad [109–111], Xu [126], Yuan [132, 133], Zeidler [134] (see also the references therein).

On the other hand, the celebrated so-called KKM principle established in 1929 in [62] was based on the celebrated Sperner combinatorial lemma and first applied to a simple proof of the Brouwer fixed point theorem. Later it became clear that these three theorems are mutually equivalent, and they were regarded as a sort of mathematical trinity (Park [91]). Since Fan extended the classical KKM theorem to infinite-dimensional spaces in 1961 [36–38], there have been a number of generalizations and applications in numerous areas of nonlinear analysis and fixed points in TVS and LCS as developed by Browder (see [14–19] and the related references therein). Among them, Schauder's fixed point theorem [107] in normed spaces is one of the powerful tools in dealing with nonlinear problems in analysis. Most notably, it has played a major role in the development of fixed point theory and related nonlinear analysis and mathematical theory of partial and differential equations and others. A generalization of Schauder's theorem from a normed space to general topological vector spaces is an old conjecture in fixed point theory, which is explained by Problem 54 of the book "The Scottish Book" by Mauldin [75] and stated as Schauder's conjecture: "Every nonempty compact convex set in a topological vector space has the fixed point property, or in its analytic statement, does a continuous function defined on a compact convex subset of a topological vector space to itself have a fixed point?" Recently, this question has been answered by the work of Ennassik and Taoudi [34] by using the p-seminorm methods under locally p-convex spaces; see also related contribution given by Cauty [22], plus the works by Askoura and Godet-Thobie [6], Cauty [21], Chang [23], Chang et al. [24], Chen [29], Dobrowolski [32], Gholizadeh et al. [41], Isac [53], Li [70], Li et al. [69], Liu [72], Nhu [77], Okon [79], Park [90–92], Reich [100], Smart [115], Weber [121, 122], Xiao and Lu [123], Xiao and Zhu [124, 125], Xu [129], Xu et al. [130], Yuan [132, 133] in both TVS, LCS and related references therein under the general framework of p-vector spaces for nonself set-valued or single-valued mappings ($0 < p \leq 1$).

The goal of this paper is to establish general new tools of nonlinear analysis under the framework of locally p-convex (seminorm) spaces for 1-set contractive mappings (here $0 < p \leq 1$), but we do wish these new results such as best approximation, theorems of Birkhoff–Kellogg type, nonlinear alternative, fixed point theorems for nonself set-valued with boundary conditions, Rothe, Petryshyn type, Altman type, Leray–Schedule type, and other related nonlinear problems would play important roles for the nonlinear analysis of p-seminorm spaces for $0 < p \leq 1$. In addition, our results on the fixed point theorem for compact (single-valued) continuous mappings in TVS also provide solutions for Schauder's conjecture since 1930s in the affirmative way under the general setting of p-vector spaces for p-convex sets (which may not be locally convex when $p \in (0, 1)$, see Kalton [55, 56], Kalton et al. [57], Jarchow [54], Roloewicz [103], and the related refer-

ences for the study on the development of related nonlinear analysis). In addition, based on the framework for some key results in nonlinear analysis obtained for set-valued mappings with closed p-convex values in this paper, we also conclude that the development of nonlinear analysis for singe-valued mappings in locally p-convex spaces for $0 < p \leq 1$ seem very importnat, too, and can be developed by the approach and method established in this paper.

The paper has seven sections. Section 1 is the introduction. Section 2 describes general concepts for the p-convex subsets of topological vector spaces ($0 < p \leq 1$). In Sect. 3, some basic results of the KKM principle related to abstract convex spaces are given by including locally p-convex spaces as a special class. In Sect. 4, as the application of the KKM principle in abstract convex spaces, which include locally p-convex spaces as a special class ($0 < p \leq 1$), and plus by combining the embedding lemma for compact p-convex subsets from topological vector spaces into locally p-convex spaces, we provide general fixed point theorems for compact (single-valued) continuous self-mappings defined on p-convex compact in TVS, and condensing upper semicontinuous set-valued mappings defined on noncompact p-convex subsets in locally p-convex vector spaces. In Sect. 5, the general best approximation result for 1-set contractive upper semicontinuous mappings is first given under the framework of locally p-convex spaces, and then it is used as a tool to establish general existence theorems for solutions of Birkhoff–Kellogg (problem) alternative, general principle of nonlinear alterative, including Leray–Schauder alternative, Rothe type, Altman type associated with different boundary conditions in locally p-convex spaces. In Sect. 6, we give a number of new results based on the general principles of Birkhoff–Kellogg theorems and Leray–Schauder alternative established in Sect. 5 for 1-set contractive mappings with different boundary conditions in locally p-convex spaces. In Sect. 7, we focus on the study of fixed point theorems for classes of 1-set contractive set-valued mappings under various boundary conditions by including nonexpansive set-valued mappings under p-norm spaces, uniformly convex Banach spaces, or with Opial condition.

For the convenience of our discussion, throughout this paper, all p-convex topological vector spaces and the compact p-convex sets are always assumed to be Hausdorff, and p satisfies the condition for $0 < p \leq 1$ unless specified otherwise, and also we denote by $\mathbb{N}$ the set of all positive integers, i.e., $\mathbb{N} := \{1, 2, \ldots, \}$.

2　The basic results of *p*-vector spaces

We now recall some notions and definitions for p-convex topological vector spaces which will be used in what follows (see Balachandran [7], Bayoumi [9], Jarchow [54], Kalton [55], Rolewicz [103], Gholizadeh et al. [41], Ennassik et al. [33], Ennassik and Taoudi [34], Xiao and Lu [123], Xiao and Zhu [124], and the references therein).

Definition 2.1 A set A in a vector space X is said to be p-convex for $0 < p \leq 1$ if, for any $x, y \in A$, $0 \leq s, t \leq 1$ with $s^p + t^p = 1$, we have $s^{1/p}x + t^{1/p}y \in A$; and if A is 1-convex, it is simply called convex (for $p = 1$) in general vector spaces; the set A is said to be absolutely p-convex if $s^{1/p}x + t^{1/p}y \in A$ for $0 \leq |s|, |t| \leq 1$ with $|s|^p + |t|^p \leq 1$.

Definition 2.2 If A is a subset of a topological vector space X, the closure of A is denoted by $\overline{A}$, then the p-convex hull of A and its closed p-convex hull are denoted by $C_p(A)$ and

$\overline{C}_p(A)$, respectively, which are the smallest p-convex set containing A and the smallest closed p-convex set containing A, respectively.

Definition 2.3 Let A be p-convex and $x_1, \ldots, x_n \in A$, and $t_i \geq 0$, $\sum_1^n t_i^{\mathrm{p}} = 1$. Then $\sum_1^n t_i x_i$ is called a p-convex combination of $\{x_i\}$ for $i = 1, 2, \ldots, n$. If $\sum_1^n |t_i|^{\mathrm{p}} \leq 1$, then $\sum_1^n t_i x_i$ is called an absolutely p-convex combination. It is easy to see that $\sum_1^n t_i x_i \in A$ for a p-convex set A.

Definition 2.4 A subset A of a vector space X is called circled (or balanced) if $\lambda A \subset A$ holds for all scalars λ satisfying $|\lambda| \leq 1$. We say that A is absorbing if, for each $x \in X$, there is a real number $\rho_x > 0$ such that $\lambda x \in A$ for all $\lambda > 0$ with $|\lambda| \leq \rho_x$.

By Definition 2.4, it is easy to see that the system of all circled subsets of X is easily seen to be closed under the formation of linear combinations, arbitrary unions, and arbitrary intersections. In particular, every set $A \subset X$ determines the smallest circled subset $\hat{A}$ of X in which it is contained: $\hat{A}$ is called the circled hull of A. It is clear that $\hat{A} = \cup_{|\lambda| \leq 1} \lambda A$ holds, so that A is circled if and only if (in short, iff) $\hat{A} = A$. We use $\overline{\hat{A}}$ to denote the closed circled hull of $A \subset X$.

In addition, if X is a topological vector space, we use $\mathrm{int}(A)$ to denote the interior of set $A \subset X$, and if $0 \in \mathrm{int}(A)$, then $\mathrm{int}(A)$ is also circled, and we use ∂A to denote the boundary of A in X unless specified otherwise.

Definition 2.5 A topological vector space is said to be locally p-convex if the origin has a fundamental set of absolutely p-convex 0-neighborhoods. This topology can be determined by p-seminorms which are defined in the obvious way (see p. 52 of Bayoumi [9], Jarchow [54], or Rolewicz [103]).

Definition 2.6 Let X be a vector space and $\mathbb{R}^+$ be a nonnegative part of a real line $\mathbb{R}$. Then a mapping $P : X \longrightarrow \mathbb{R}^+$ is said to be a p-seminorm if it satisfies the requirements for $(0 < p \leq 1)$:
  (i) $P(x) \geq 0$ for all $x \in X$;
  (ii) $P(\lambda x) = |\lambda|^p P(x)$ for all $x \in X$ and $\lambda \in R$;
  (iii) $P(x + y) \leq P(x) + P(y)$ for all $x, y \in X$.

A p-seminorm P is called a p-norm if $x = 0$ whenever $P(x) = 0$, so a vector space with a specific p-norm is called a p-normed space, and of course if $p = 1$, X is a normed space as discussed before (e.g., see Jarchow [54]).

By Lemma 3.2.5 of Balachandran [7], the following proposition gives a necessary and sufficient condition for a p-seminorm to be continuous.

Proposition 2.1 *Let X be a topological vector space, P be a p-seminorm on X, and $V :=$ $\{x \in X : P(x) < 1\}$. Then P is continuous if and only if $0 \in \mathrm{int}(V)$, where $\mathrm{int}(V)$ is the interior of V.*

Now, given a p-seminorm P, the p-seminorm topology determined by P (in short, the p-topology) is the class of unions of open balls $B(x, \epsilon) := \{y \in X : P(y - x) < \epsilon\}$ for $x \in X$ and $\epsilon > 0$.

Definition 2.7 A topological vector space X is said to be locally p-convex if it has a 0-basis consisting of p-convex neighborhoods for $(0 < p \leq 1)$. If $p = 1$, X is a usual locally convex space.

We also need the following notion for the so-called p-gauge (see Balachandran [7]).

Definition 2.8 Let A be an absorbing subset of a vector space X. For $x \in X$ and $0 < p \leq 1$, set $P_A = \inf\{\alpha > 0 : x \in \alpha^{\frac{1}{p}} A\}$, then the nonnegative real-valued function P_A is called the p-gauge (gauge if $p = 1$). The p-gauge of A is also known as the Minkowski p-functional for set A.

By Proposition 4.1.10 of Balachandran [7], we have the following proposition.

Proposition 2.2 *Let A be an absorbing subset of X. Then p-gauge P_A has the following properties*:

(i) $P_A(0) = 0$;
(ii) $P_A(\lambda x) = |\lambda|^p P_A(x)$ *if* $\lambda \geq 0$;
(iii) $P_A(\lambda x) = |\lambda|^p P_A(x)$ *for all* $\lambda \in R$ *provided A is circled*;
(iv) $P_A(x + y) \leq P_A(x) + P_A(y)$ *for all* $x, y \in A$ *provided A is p-convex*.

In particular, P_A is a p-seminorm if A is absolutely p-convex (and also absorbing).

As mentioned above, a given p-seminorm is said to be a p-norm if $x = 0$ whenever $P(x) = 0$. A vector space with a specific p-norm is called a p-normed space. The p-norm of an element $x \in E$ will usually be denoted by $\|x\|_p$. If $p = 1$, X is a usual normed space. If X is a p-normed space, then (X, d_p) is a metric linear space with a translation invariant metric d_p such that $d_p = d_p(x, y) = \|x - y\|_p$ for $x, y \in X$. We point out that p-normed spaces are very important in the theory of topological vector spaces. Specifically, a Hausdorff topological vector space is locally bounded if and only if it is a p-normed space for some p-norm $\| \cdot \|_p$, where $0 < p \leq 1$ (see p. 114 of Jarchow [54]). We also note that examples of p-normed spaces include $L^p(\mu)$ - spaces and Hardy spaces H_p, $0 < p < 1$, endowed with their usual p-norms.

Remark 2.1 We would like to make the following important two points.

(1) First, by the fact that (e.g., see Kalton et al. [57] or Ding [31]) there is no open convex nonvoid subset in $L^p[0,1]$ (for $0 < p < 1$) except $L^p[0,1]$ itself. This means that p-normed paces with $0 < p < 1$ are not necessarily locally convex. Moreover, we know that every p-normed space is locally p-convex; and incorporating Lemma 2.3, it seems that p-vector spaces (for $0 < p \leq 1$) are nicer spaces as we can use a p-vector space to approximate (Hausdorff) topological vector spaces (TVS) in terms of Lemma 2.1(ii) for the convex subsets in TVS by using bigger p-convex subsets in p-vector spaces for $p \in (0, 1)$ by also considering Lemma 2.3. In this way, it seems that P-vector spaces have better properties in terms of p-convexity than the usually $(1-)$ convex subsets used in TVS with $p = 1$.

(2) Second, it is worthwhile noting that a 0-neighborhood in a topological vector space is always absorbing by Lemma 2.1.16 of Balachandran [7] or Proposition 2.2.3 of Jarchow [54].

Now, by Proposition 4.1.12 of Balachandran [7], we also have the following Proposition 2.3 and Remark 2.2 (which is Remark 2.3 of Ennassik and Taoudi [34]).

Proposition 2.3 *Let A be a subset of a vector space X, which is absolutely p-convex* $(0 < p \leq 1)$ *and absorbing. Then we have that*:

(*i*) *The p-gauge* P_A *is a p-seminorm such that if* $B_1 := \{x \in X : P_A(x) < 1\}$ *and* $\overline{B_1} = \{x \in X : P_A(x) \leq 1\}$, *then* $B_1 \subset A \subset \overline{B_1}$; *in particular,* $\ker P_A \subset A$, *where* $\ker P_A := \{x \in X : P_A(x) = 0\}$.

(*ii*) $A = B_1$ *or* $\overline{B_1}$ *depending on whether A is open or closed in the* P_A-*topology.*

Remark 2.2 Let X be a topological vector space, and let U be an open absolutely p-convex neighborhood of the origin, and let ϵ be given. If $y \in \epsilon^{\frac{1}{p}} U$, then $y = \epsilon^{\frac{1}{p}} u$ for some $u \in U$ and $P_U(y) = P_U(\epsilon^{\frac{1}{p}} u) = \epsilon P_U(u) \leq \epsilon$ (as $u \in U$ implies that $P_U(u) \leq 1$). Thus, P_U is continuous at *zero*, and therefore P_U is continuous everywhere. Moreover, we have $U = \{x \in X : P_U(x) < 1\}$.

Indeed, since U is open and the scalar multiplication is continuous, we have that, for any $x \in U$, there exists $0 < t < 1$ such that $x \in t^{\frac{1}{p}} U$, and so $P_U(x) \leq t < 1$. This shows that $U \subset \{x \in X : P_U(x) < 1\}$. The conclusion follows by Proposition 2.3.

The following result is a very important and useful result which allows us to make the approximation for convex subsets in topological vector spaces by p-convex subsets in p-convex vector spaces. For the reader's convenience, we provide a sketch of proof below (see also Lemma 2.1 of Ennassik and Taoudi [33], Remark 2.1 of Qiu and Rolewicz [99]).

Lemma 2.1 *Let A be a subset of a vector space X, then we have*:

(*i*) *If A is p-convex with* $0 < p < 1$, *then* $\alpha x \in A$ *for any* $x \in A$ *and any* $0 < \alpha \leq 1$.

(*ii*) *If A is convex and* $0 \in A$, *then A is p-convex for any* $p \in (0, 1]$.

(*iii*) *If A is p-convex for some* $p \in (0, 1)$, *then A is s-convex for any* $s \in (0, p]$.

Proof (i) As $r \leq 1$, by the fact that "for all $x \in A$ and all $\alpha \in [2^{(n+1)(1-\frac{1}{p})}, 2^{n(1-\frac{1}{p})}]$, we have $\alpha x \in A$" is true for all integer $n \geq 0$, taking into account the fact that $(0, 1] = \cup_{n \geq 0}[2^{(n+1)(1-\frac{1}{p})}, 2^{n(1-\frac{1}{p})}]$, the result is obtained.

(ii) Assume that A is a convex subset of X with $0 \in A$ and take a real number $s \in (0, 1]$. We show that A is s-convex. Indeed, let $x, y \in A$ and $\alpha, \beta > 0$ with $\alpha^p + \beta^p = 1$. Since A is convex, then $\frac{\alpha}{\alpha + \beta} x + \frac{\beta}{\alpha + \beta} y \in A$. Keeping in mind that $0 < \alpha + \beta < \alpha^p + \beta^p = 1$, it follows that $\alpha x + \beta y = (\alpha + \beta)(\frac{\alpha}{\alpha + \beta} x + \frac{\beta}{\alpha + \beta} y) + (1 - \alpha - \beta) 0 \in A$.

(iii) Now, assume that A is r-convex for some $p \in (0, 1)$ and pick up any real $s \in (0, p]$. We show that A is s-convex. To see this, let $x, y \in A$ and $\alpha, \beta > 0$ such that $\alpha^s + \beta^s = 1$. First notice that $0 < \alpha^{\frac{p-s}{p}} \leq 1$ and $0 < \beta^{\frac{p-s}{p}} \leq 1$, which imply that $\alpha^{\frac{p-s}{p}} x \in A$ and $\beta^{\frac{p-s}{p}} y \in A$. By the p-convexity of A and the equality $(\alpha^{\frac{s}{p}})^p + (\beta^{\frac{s}{p}})^p = 1$, it follows that $\alpha x + \beta y = \alpha^{\frac{s}{p}} (\alpha^{\frac{p-s}{p}} x) + \beta^{\frac{s}{p}} (\beta^{\frac{p-s}{p}} y) \in A$. This competes the sketch of the proof. □

Remark 2.3 We would like to point out that results (i) and (iii) of Lemma 2.1 do not hold for $p = 1$. Indeed, any singleton $\{x\} \subset X$ is convex in topological vector spaces; but if $x \neq 0$, then it is not p-convex for any $p \in (0, 1)$ (see also Lemma 2.3 below).

We also need the following proposition, which is Proposition 6.7.2 of Jarchow [54].

Proposition 2.4 *Let K be compact in a topological vector X and* $(1 < p \leq 1)$. *Then the closure* $\overline{C}_p(K)$ *of the p-convex hull and the closure* $\overline{AC}_p(K)$ *of absolutely p-convex hull of K are compact if and only if* $\overline{C}_p(K)$ *and* $\overline{AC}_p(K)$ *are complete, respectively.*

We also need the following fact, which is a special case of Lemma 2.4 of Xiao and Zhu [124].

Lemma 2.2 *Let C be a bounded closed p-convex subset of a p-seminorm X with $0 \in \text{int } C$, where $(0 < p \leq 1)$. For every $x \in X$ define an operator by $r(x) := \dfrac{x}{\max\{1, (P_C(x))^{\frac{1}{p}}\}}$, where P_C is the Minkowski p-functional of C. Then C is a retract of X and $r : X \to C$ is continuous such that*

(1) *if $x \in C$, then $r(x) = x$;*
(2) *if $x \notin C$, then $r(x) \in \partial C$;*
(3) *if $x \notin C$, then the Minkowski p-functional $P_C(x) > 1$.*

Proof Taking $s = p$ in Lemma 2.4 of Xiao and Zhu [124], Proposition 2.3, and Remark 2.2, the proof is compete. □

Remark 2.4 As discussed by Remark 2.2, Lemma 2.2 still holds if "the bounded closed p-convex subset C of the p-normed space $(X, \| \cdot \|_p)$" is replaced by "X is a p-seminorm vector space and C is a bounded closed absorbing p-convex subset with $0 \in \text{int } C$ of X".

Before we close this section, we would like to point out that the structure of p-convexity when $p \in (0, 1)$ is really different from what we normally have for the concept of "convexity" used in topological vector spaces (TVS). In particular, maybe the following fact is one of the reasons for us to use better (p-convex) structures in p-vector spaces to approximate the corresponding structure of the convexity used in TVS (i.e., the p-vector space when $p = 1$). Based on the discussion in p. 1740 of Xiao and Zhu [124](see also Bernués and Pena [12] and Sezer et al. [108]), we have the following fact, which indicates that each p-convex subset is "bigger" than the convex subset in topological vector spaces for $0 < p < 1$.

Lemma 2.3 *Let x be a point of a p-vector space E, where assume $0 < p < 1$, then the p-convex hull and the closure of $\{x\}$ are given by*

$$C_p(\{x\}) = \begin{cases} \{tx : t \in (0, 1]\}, & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0; \end{cases} \tag{1}$$

and

$$\overline{C_p(\{x\})} = \begin{cases} \{tx : t \in [0, 1]\}, & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0. \end{cases} \tag{2}$$

But note that if x is a given one point in p-vector space E, when $p = 1$, we have that $\overline{C_1(\{x\})} = C_1(\{x\}) = \{x\}$. This shows to be significantly different for the structure of p-convexity between $p = 1$ and $p \neq 1$!

As an application of Lemma 2.3, we have the following fact for (set-valued) mappings with nonempty closed p-convex values in p-vector spaces for $p \in (0, 1)$, which are truly different from any (set-valued) mappings defined in topological vector spaces (i.e., for a p-vector space with $p = 1$).

Lemma 2.4 *Let U be a nonempty subset of a p-vector space E (where $0 < p < 1$) with zero $0 \in U$, and assume that a (set-valued) mapping $T : U \to 2^E$ is with nonempty closed p-convex values. Then T has at least one fixed point in U, which is the element zero, i.e., $0 \in \cap_{x \in U} T(x) \neq \emptyset$.*

Proof For each $x \in U$, as $T(x)$ is nonempty closed p-convex, by Lemma 2.3, we have at least $0 \in T(x)$. It implies that $0 \in \cap_{x \in U} T(x)$, and thus zero of E is a fixed point of T. This completes the proof.   □

Remark 2.5 By following Definitions 2.5 and 2.6, the discussion given by Proposition 2.3, and remarks thereafter, each given (open) p-convex subset U in a p-vector space E with the zero $0 \in \text{int}(U)$ always corresponds to a p-seminorm P_U, which is indeed the Minkowski p-functional of U in E, and P_U is continuous in E. In particular, a topological vector space is said to be locally p-convex if the origin 0 of E has a fundamental set (denoted by) $\mathfrak{U}$, which is a family of absolutely p-convex 0-neighborhoods (each denoted by U). This topology can be determined by p-seminorm P_U, which is indeed the family $\{P_U\}_{U \in \mathfrak{U}}$, where P_U is just the Minkowski p-functional for each $U \in \mathfrak{U}$ in E (see also p. 52 of Bayoumi [7], Jarchow [49], or Rolewicz [99]).

Throughout this paper, by following Remark 2.5, without loss of generality unless specified otherwise, for a given p-vector space E, where $p \in (0, 1]$, we always denote by $\mathfrak{U}$ the base of the p-vector space E's topology structure, which is the family of its 0-neighborhoods. For each $U \in \mathfrak{U}$, its corresponding P-seminorm P_U is the Minkowski p-functional of U in E. For a given point $x \in E$ and a subset $C \subset E$, we denote by $d_{P_U}(x, C) := \inf\{P_U(x - y) : y \in C\}$ (in short, denoted by $d_P(x, C)$ if no confusion below) the distance of x and C by the seminorm P_U, where P_U is the Minkowski p-functional for each $U \in \mathfrak{U}$ in E.

3 The KKM principle in abstract convex spaces

As mentioned in the introduction, Knaster, Kuratowski, and Mazurkiewicz (in short, KKM) [62] in 1929 obtained the so-called KKM principle (theorem) to give a new proof for the Brouwer fixed point theorem in finite dimensional spaces; and later, in 1961, Fan [36] (see also Fan [38]) extended the KKM principle (theorem) to any topological vector spaces and applied it to various results including the Schauder fixed point theorem. Since then there have appeared a large number of works devoted to applications of the KKM principle (theorem). In 1992, such a research field was called the KKM theory for the first time by Park [85]. Then the KKM theory was extended to general abstract convex spaces by Park [89] (see also Park [90] and [91]), which actually include locally p-convex spaces $(0 < p \leq 1)$ as a special class.

Here we first give some notion and a brief introduction to the abstract convex spaces, which play an important role in the development of the KKM principle and related applications. Once again, for the corresponding comprehensive discussion on the KKM theory and its various applications to nonlinear analysis and related topics, we refer to Agarwal et al. [1], Alghamdi et al. [5], Balaj [8], Mauldin [75], Granas and Dugundji [48], Park [91] and [92], Yuan [133], and related comprehensive references therein.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a given nonempty set D, and let 2^D denote the family of all subsets of D. We have the following definition for abstract convex spaces essentially by Park [89].

Definition 3.1 An abstract convex space $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a set-valued mapping $\Gamma : \langle D \rangle \to 2^E$ with nonempty values $\Gamma_A := \Gamma(A)$ for each $A \in \langle D \rangle$ such that the Γ-convex hull of any $D' \subset D$ is denoted and defined by $\mathrm{co}_\Gamma D' := \cup \{ \Gamma_A | A \in \langle D' \rangle \} \subset E$.

A subset X of E is said to be a Γ-convex subset of $(E, D; \Gamma)$ relative to D' if, for any $N \in \langle D' \rangle$, we have $\Gamma_N \subseteq X$, that is, $\mathrm{co}_\Gamma D' \subset X$. For the convenience of our discussion, in the case $E = D$, the space $(E, E; \Gamma)$ is simply denoted by $(E; \Gamma)$ unless specified otherwise.

Definition 3.2 Let $(E, D; \Gamma)$ be an abstract convex space and Z be a topological space. For a set-valued mapping (or, say, multimap) $F : E \to 2^Z$ with nonempty values, if a set-value mapping $G : D \to 2^Z$ satisfies $F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y)$ for all $A \in \langle D \rangle$, then G is called a KKM mapping with respect to F. A KKM mapping $G : D \to 2^E$ is a KKM mapping with respect to the identity map 1_E.

Definition 3.3 The partial KKM principle for an abstract convex space $(E, D; \Gamma)$ is that, for any closed-valued KKM mapping $G : D \to 2^E$, the family $\{ G(y) \}_{y \in D}$ has the finite intersection property. The KKM principle is that the same property also holds for any open-valued KKM mapping.

An abstract convex space is called a (partial) KKM space if it satisfies the (partial) KKM principle (resp.). We now give some known examples of (partial) KKM spaces (see Park [89] and also [90]) as follows.

Definition 3.4 A ϕ_A-space $(X, D; \{ \phi_A \}_{A \in \langle D \rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to 2^X$ (that is, singular n-simplices) for $A \in \{ D \}$ with $|A| = n + 1$. By putting $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D \rangle$, the triple $(X, D; \Gamma)$ becomes an abstract convex space.

Remark 3.1 For a ϕ_A-space $(X, D; \{ \phi_A \})$, we see that any set-valued mapping $G : D \to 2^X$ satisfying $\phi_A(\Delta_J) \subset G(J)$ for each $A \in \langle D \rangle$ and $J \in \langle A \rangle$ is a KKM mapping.

By the definition, it is clear that every ϕ_A-space is a KKM space, thus we have the following fact (see Lemma 1 of Park [90]).

Lemma 3.1 *Let $(X, D; \Gamma)$ be a ϕ_A-space and $G : D \to 2^X$ be a set-valued (multimap) with nonempty closed [resp. open] values. Suppose that G is a KKM mapping, then $\{ G(a) \}_{a \in D}$ has the finite intersection property.*

By Definition 2.7, we recall that a topological vector space is said to be locally p-convex if the origin has a fundamental set of absolutely p-convex 0-neighborhoods. This topology can be determined by p-seminorms which are defined in the obvious way (see Jarchow [54] or p. 52 of Bayoumi [9]).

Now we have a new KKM space as follows inducted by the concept of p-convexity (see Lemma 2 of Park [90]).

Lemma 3.2 *Suppose that X is a subset of the topological vector space E and $p \in (0,1]$, and D is a nonempty subset of X such that $C_p(D) \subset X$. Let $\Gamma_N := C_p(N)$ for each $N \in \langle D \rangle$. Then $(X, D; \Gamma)$ is a ϕ_A-space.*

Proof Since $C_p(D) \subset X$, Γ_N is well defined. For each $N = \{x_0, x_1, \ldots, x_n\} \subset D$, we define $\phi_N : \Delta_n \to \Gamma_N$ by $\sum_{i=0}^{n} t_i e_i \mapsto \sum_{i=0}^{n} (t_i)^{\frac{1}{p}} x_i$. Then, clearly, $(X, D; \Gamma)$ is a ϕ_A-space. This completes the proof. □

4 Fixed point theorems for condensing set-valued mappings in *p*-vector spaces

In this section, we establish fixed point theorems for upper semicontinuous set-valued mappings in locally p-convex spaces, compact (single-valued) continuous mappings for p-convex subsest in TVS, and condensing mappings for p-convex subsets under the general framework of locally p-convex spaces, which will be a tool used in Sect. 5, Sect. 6 and Sect. 7 to establish the best approximation, fixed points, the principle of nonlinear alternative, Birkhoff–Kellogg problems, Leray–Schauder alternative, which would be useful tools in nonlinear analysis for the study of nonlinear problems arising from theory to the practice. Here, we first gather together necessary definitions, notations, and known facts needed in this section.

Definition 4.1 Let X and Y be two topological spaces. A set-valued mapping (also called multifunction) $T : X \longrightarrow 2^Y$ is a point to set function such that for each $x \in X$, $T(x)$ is a subset of Y. The mapping T is said to be upper semicontinuous (USC) if the subset $T^{-1}(B) := \{x \in X : T(x) \cap B \neq \emptyset\}$ (resp., the set $\{x \in X : T(x) \subset B\}$) is closed (resp., open) for any closed (resp., open) subset B in Y. The function $T : X \to 2^Y$ is said to be lower semicontinuous (LSC) if the set $T^{-1}(A)$ is open for any open subset A in Y.

As an application of the KKM principle for general abstract convex spaces, we have the following general existence result for the "approximation" of fixed points for upper and lower semicontinuous set-valued mappings in locally p-convex spaces for $0 < p \le 1$ (see also the corresponding related results given by Theorem 2.7 of Gholizadeh et al. [41], Theorem 5 of Park [90], and related discussion therein).

Theorem 4.1 *Let A be a p-convex compact subset of a locally p-convex space X, where $0 < p \le 1$. Suppose that $T : A \to 2^A$ is lower (resp. upper) semicontinuous with nonempty p-convex values. Then, for any given U, which is a p-convex neighborhood of zero in X, there exists $x_U \in A$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.*

Proof Suppose that U is any given p-convex element of $\mathfrak{U}$, there is a symmetric open p-convex neighborhood V of zero for which $\overline{V} + \overline{V} \subset U$ in p-convex neighborhood of zero, we prove the results by two cases for T is lower semicontinuous (LSC) and upper semicontinuous (USC).

Case 1, by assuming T is lower semicontinuous: As X is a locally p-convex vector space, suppose that $\mathfrak{U}$ is the family of neighborhoods of 0 in X. For any element U of $\mathfrak{U}$, there is a symmetric open p-convex neighborhood V of zero for which $\overline{V} + \overline{V} \subset U$. Since A is

compact, there exist $x_0, x_1, \ldots, x_n$ in A such that $A \subset \cup_{i=0}^{n}(x_i + V)$. By using the fact that A is p-convex, we find $D := \{b_0, b_2, \ldots, b_n\} \subset A$ for which $b_i - x_i \in V$ for all $i \in \{0, 1, \ldots, n\}$, and we define C by $C := C_p(D) \subset A$. By the fact that T is LSC, it follows that the subset $F(b_i) := \{c \in C : T(c) \cap (x_i + V) = \emptyset\}$ is closed in C (as the set $x_i + V$ is open) for each $i \in \{0, 1, \ldots, n\}$. For any $c \in C$, we have $\emptyset \neq T(c) \cap A \subset T(c) \cap \cup_{i=0}^{n}(x_i + V)$, it follows that $\cap_{i=0}^{n} F(b_i) = \emptyset$. Now, we apply Lemma 3.1 and Lemma 3.2, which implies that there is $N := \{b_{i_0}, b_{i_1}, \ldots, b_{i_k}\} \in \langle D \rangle$ and $x_U \in C_p(N) \subset A$ for which $x_U \notin F(N)$, and so $T(x_u) \cap (x_{i_j} + V) \neq \emptyset$ for all $j \in \{0, 1, \ldots, k\}$. As $b_i - x_i \in V$ and $\overline{V} + \overline{V} \subset U$, which imply that $x_{i_j} + \overline{V} \subset b_{i_j} + U$, which means that $T(x_U) \cap ((b_{i_j} + U) \neq \emptyset$, it follows that $N \subset \{c \in C : T(x_U) \cap (c + U) \neq \emptyset\}$. By the fact that the subsets C, $T(x_U)$ and U are p-convex, we have that $x_U \in \{c \in C : T(x_U) \cap (c + U) \neq \emptyset\}$, which means that $T(x_U) \cap (x_U + U) \neq \emptyset$.

Case 2, by assuming T is upper semicontinuous: We define $F(b_i) := \{c \in C : T(c) \cap (x_i + \overline{V}) = \emptyset\}$, which is then open in C (as the subset $x_i + \overline{V}$ is closed) for each $i = 0, 1, \ldots, n$. Then the argument is similar to the proof for the case T is USC, and by applying Lemma 3.1 and Lemma 3.2 again, it follows that there exists $x_U \in A$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$. This completes the proof. $\qquad \square$

By Theorem 4.1, we have the following Fan–Glicksberg fixed point theorems (Fan [35]) in locally p-convex vector spaces for $(0 < p \leq 1)$, which also improve or generalize the corresponding results given by Yuan [133], Xiao and Lu [123], Xiao and Zhu [124, 125] into locally p-convex vector spaces.

Theorem 4.2 *Let A be a p-convex compact subset of a locally p-convex space X, where $0 < p \leq 1$. Suppose that $T : A \to 2^A$ is upper semicontinuous with nonempty p-convex closed values. Then T has at least one fixed point.*

Proof Assume that $\mathfrak{U}$ is the family of neighborhoods of 0 in X and $U \in \mathfrak{U}$. By Theorem 4.1, there exists $x_U \in A$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$. Then there exist $a_U, b_U \in A$ for which $b_U \in T(a_U)$ and $b_U \in a_U + U$. Now, regarding two nets $\{a_U\}$ and $\{b_U\}$ in Graph(T), which is a compact graph of mapping T as A is compact and T is semicontinuous, we may assume that a_U has a subnet converging to a and $\{b_U\}$ has a subnet converging to b. As $\mathfrak{U}$ is the family of neighborhoods for 0, we should have $a = b$ (e.g., by the Hausdorff separation property) and $a = b \in T(b)$ due to the fact that Graph(T) is close (e.g., see also Lemma 1.1 of Yuan [132]), thus the proof is complete. $\qquad \square$

For a given set A in a vector space X, we denote by "lin(A)" the "linear hull" of A in X.

Definition 4.2 Let A be a subset of a topological vector space X, and let Y be another topological vector space. We shall say that A can be linearly embedded in Y if there is a linear map $L : \text{lin}(A) \to Y$ (not necessarily continuous) whose restriction to A is a homeomorphism.

The following embedded Lemma 4.1 is a significant result due to Theorem 1 of Kalton [55], which says that although not every compact convex set can be linearly embedded in a locally convex space (e.g., see Roberts [101] and Kalton et al. [57]), but for p-convex sets when $0 < p < 1$, every compact p-convex set in topological vector spaces can be considered

as a subset of a locally p-convex vector space, hence every such set has sufficiently many p-extreme points.

Secondly, by property (ii) of Lemma 2.1, each convex subset of a topological vector space containing zero is always p-convex for $0 < p \leq 1$. Thus it is possible for us to transfer the problem involving p-convex subsets from topological vector spaces into the locally p-convex vector spaces, which indeed allows us to establish the existence of fixed points for comppact (single-valued) continuous mappings for compact p-convex subsets in topological vector spaces $(0 < p \leq 1)$ to cover the case when the underlying is just a topological vector space, which provides the answer for Schauder's conjecture in the affirmative in p-vector spaces.

Lemma 4.1 *Let K be a compact p-convex subset $(0 < p < 1)$ of a topological vector space X. Then K can be linearly embedded in a locally p-convex topological vector space.*

Proof It is Theorem 1 of Kalton [55] which completes the proof. □

Remark 4.1 At this point, it is important to note that Lemma 4.1 does not hold for $p = 1$. By Theorem 9.6 of Kalton et al. [57], it was shown that the spaces $L_p = L_p(0, 1)$, where $0 < p < 1$, contain compact convex sets with no extreme points, which thus cannot be linearly embedded in a locally convex space, see also Roberts [101].

Now we give the the following fixed point theorem for (upper semicontinuous continuity, which actually is not needed by applying Lemma 2.4 directly) set-valued mappings with non-empty p-convex closed values defined on closed p-convex subsets of locally p-convex spaces for $0 < p < 1$; and also for the single-valued version of a continuous mapping which is Theorem 3.3 first given by Ennassik and Taoudi [34] by using the p-seminorm argument in p-vector spaces for $0 < p \leq 1$. Here we like to point out that though the conclusion for the existence of fixed points for upper semicontinuous set-valued mappings below defined on compact p-convex subsets for $0 < p < 1$ is the direct conlusion by Lemma 2.4 (even without assumption for the upper semicontinuity), the following argument shares with readers the way by applying KKM principle how to establish the set-valued versions of fixed points in p-vector spaces with combining some special embedded features for p-convex structures for $0 < p < 1$.

Theorem 4.3 *If K is a nonempty compact p-convex subset of a topological vector space X for $0 < p < 1$, then any upper semicontinuous set-valued mapping $T : K \to 2^K$ with nonempty p-convex closed value has at least a fixed point; and secondly, for $0 < p \leq 1$, any single-valued continuous mapping $T : K \to K$ has at least a fixed point.*

Proof We complete the argument by the following two cases.

First case, for the set-valued mappings by assuming that K is p-convex with $0 < p < 1$. By Lemma 4.1, it follows that K can be linearly embedded in a locally p-convex space E, which means that there exists a linear map $L : \mathrm{lin}(K) \to E$ whose restriction to K is a homeomorphism. Define the mapping $S : L(K) \to L(K)$ by $(Sx) := L(Tx)$ for $x \in X$. This mapping is easily checked to be well defined. The mapping S is upper semicontinuous since L is a (continuous) homeomorphism and T is upper semicontinuous on K. Furthermore, the set $L(K)$ is compact, being the image of a compact set under a continuous mapping L.

It is also p-convex since it is the image of a p-convex set under a linear mapping. Then, by Theorem 4.2, there exists $x \in K$ such that $Lx \in S(Lx) = L(Tx)$, thus there exists $y \in Tx$ such that $Lx = Ly$, which implies that $x = y \in T(x)$ since L is a homeomorphism, which is the fixed point of T.

Second case, considering when T is a single-valued continuous mapping for $0 < p \leq 1$, this is Theorem 3.3 given by Ennassik and Taoudi [34]. Thus the proof is complete. □

Remark 4.2 As mentoined above, by Lemma 2.4, the conclusion in Theorem 4.3 holds for set-valued mappings with non-empty p-convex closed values without upper semicontinuous assumptions when $0 < p < 1$. Secondly, the the single-valued version of Theorem 4.3 which was first given by Ennassik and Taoudi (Theorem 3.3 of [34]) for $0 < p \leq 1$ indeed provides an answer to Schauder's conjecture under the TVS. Here we also mention a number of related works and discussion by authors in this drection, see Mauldin [75], Granas and Dugundji [48], Park [91, 92] and the references therein.

We recall that for two given topological spaces X and Y, a set-valued mapping $T : X \to 2^Y$ is said to be compact if there is a compact subset set C in Y such that $F(X)(= \{y \in F(X), x \in X\})$ is contained in C, i.e., $F(X) \subset C$. Now, we have the following noncompact version of fixed point theorems for compact (single-valued) continuous mappings defined on a general p-convex subset in topologiocal vector spaces for $0 < p \leq 1$.

Theorem 4.4 (Schauder's fixed point theorem for compact mappings) *If C is a nonempty closed p-convex subset of a topological vector space E with $(0 < p \leq 1)$ and $T : C \to C$ is (single-valued) continuous and compact (i.e., the set $T(C)$ is contained in a compact subset of C), then T has at least one fixed point.*

Proof As T is compact, there exists a compact subset A in C such that $T(C) \subset A$. Let $K := \overline{C}_p(A)$ be the closure of the p-convex hull of set A in C. Then K is compact p-convex by Proposition 2.4, and the mapping $T : K \to K$ is continuous. Now, by Theorem 4.3, it follows that T has a fixed point $x \in K \subset C$ such that $x \in T(x)$. This completes the proof. □

As an immediate consequence of Theorem 4.4, we have the following result, which gives an affirmative answer to Schauder's conjecture in in topological vector spaces (TVS).

Corollary 4.1 *If K is a nonempty closed convex subset of a topological vector space X, then any (single-valued) continuous and compact mapping $T : K \to K$ has at least a fixed point.*

Proof Apply Theorem 4.4 with $p = 1$, this completes the proof.   □

Theorem 4.4 improves or unifies corresponding results given by Askoura and Godet-Thobie [6], Cauty [21], Cauty [22], Chen [29], Isac [53], Li [70], Nhu [77], Okon [79], Park [92], Reich [100], Smart [115], Yuan [133], Theorem 3.3 of Ennassik and Taoudi [33], Theorem 3.14 of Gholizadeh et al. [41], Xiao and Lu [123], Xiao and Zhu [124, 125] under the framework of topological vector spaces.

In order to establish fixed point theorems for the classes of 1-set contractive and condensing mappings in p-vector spaces by using the concept of the measure of noncompactness (or the noncompactness measures), which were introduced and widely accepted in

mathematical community by Kuratowski [65], Darbo [30] (see related references therein), we first need to have a brief introduction for the concept of noncompactness measures for the so-called Kuratowski or Hausdorff measures of noncompactness in normed spaces (see Alghamdi et al. [5], Machrafi and Oubbi [73], Nussbaum [78], Sadovskii [106], Silva et al. [112], Xiao and Lu [123] for the general concepts under the framework of p-seminorm or just for locally convex p-convex settings for $0 < p \le 1$, which will be discussed below, too).

For a given metric space (X, d) (or a p-normed space $(X, \|c \cdot \|_p)$), we recall the notions of completeness, boundedness, relative compactness, and compactness as follows. Let (X, d) and (Y, d) be two metric spaces and $T : X \to Y$ be a mapping (or an operator). Then: 1) T is said to be bounded if for each bounded set $A \subset X$, $T(A)$ is bounded set of Y; 2) T is said to be continuous if for every $x \in X$, the $\lim_{n \to \infty} x_n = x$ implies that $\lim_{n \to \infty} T(x_n) = T$; and 3) T is said to be completely continuous if T is continuous and $T(A)$ is relatively compact for each bounded subset A of X.

Let $A_1, A_2 \subset X$ be bounded of a metric space (X, d), we also recall that the Hausdorff metric $d_H(A_1, A_2)$ between A_1 and A_2 is defined by

$$d_H(A_1, A_2) := \max \left\{ \sup_{x \in A_1} \inf_{y \in A_2} d(x, y), \sup_{y \in A_2} \inf_{x \in A_1} d(x, y) \right\}.$$

The Hausdorff and Kuratowski measures of noncompactness (denoted by β_H and β_K, respectively) for a nonempty bounded subset D in X are the nonnegative real numbers $\beta_H(D)$ and $\beta_K(D)$ defined by

$$\beta_H(D) := \inf\{\epsilon > 0 : D \text{ has a finite } \epsilon\text{-net}\}$$

and

$$\beta_K(D) := \inf\big\{\epsilon > 0 :$$
$$D \subset \cup_{i=1}^n D_i, \text{ where } D_i \text{ is bounded and } \operatorname{diam} D_i \le \epsilon, n \text{ is an integer } \big\},$$

here $\operatorname{diam} D_i$ means the diameter of the set D_i, and it is well known that $\beta_H \le \beta_K \le 2\beta_H$. We also point out that the notions above can be well defined under the framework of p-seminorm spaces $(E, \| \cdot \|_p)_{p \in \mathfrak{P}}$ by following a similar idea and method used by Chen and Singh [28], Ko and Tasi [63], and Kozlov et al. [64] (see the references therein for more details).

Let T be a mapping from $D \subset X$ to X. Then we have that: 1) T is said to be a k-set contraction with respect to β_K (or β_H) if there is a number $k \in [0, 1)$ such that $\beta_K(T(A)) \le k\beta_K(A)$ (or $\beta_H(T(A)) \le k\beta_H(A)$) for all bounded sets A in D; and 2) T is said to be β_K-condensing (or β_H-condensing) if $(\beta_K(T(A)) < \beta_K(A))$ (or $\beta_H(T(A)) < \beta_H(A))$ for all bounded sets A in D with $\beta_K(A) > 0$ (or $\beta_H(A) > 0$).

For the convenience of our discussion, throughout the rest of this paper, if a mapping is β_K-condensing (or β_H-condensing), we simply say it is "a condensing mapping" unless specified otherwise.

Moreover, it is easy to see that: 1) if T is a compact operator, then T is a k-set contraction; and 2) if T is a k-set contraction for $k \in (0, 1)$, then T is condensing.

In order to establish the fixed points of set-valued condensing mappings in p-vector spaces for $0 < p \le 1$, we need to recall some notions introduced by Machrafi and Oubbi [73] for the measure of noncompactness in locally p-convex vector spaces, which also satisfies some necessary (common) properties of the classical measures of noncompactness such as β_K and β_H mentioned above introduced by Kuratowski [65], Sadovskii [106](see also the related discussion by Alghamdi et al. [5], Nussbaum [78], Silva et al. [112], Xiao and Lu [123] and the references therein). In particular, the measures of noncompactness in locally p-convex spaces (for $0 < p \le 1$) should have the stable property, which means the measure of noncompactness A is the same by transition to the (closure) for the p-convex hull of subset A.

For the convenience of discussion, we follow up to use α and β to denote the Kuratowski and the Hausdorff measures of noncompactness in topological vector spaces, respectively (see the same way used by Machrafi and Oubbi [73]) unless stated otherwise. The E is used to denote a Hausdorff topological vector space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{Q}\}$, where $\mathbb{R}$ denotes all real numbers and $\mathbb{Q}$ all complex numbers, and $p \in (0, 1]$. Here, the base set of a family of all balanced zero neighborhoods in E is denoted by $\mathfrak{V}_0$.

We recall that $U \in \mathfrak{V}_0$ is said to be shrinkable if it is absorbing, balanced, and $rU \subset U$ for all $r \in (0, 1)$, and we know that any topological vector space admits a local base at zero consisting of shrinkable sets (see Klee [61] or Jarchow [54] for details).

Recall that a topological vector space E is said to be a locally p-convex space if E has a local base at zero consisting of p-convex sets. The topology of a locally p-convex space is always given by an upward directed family P of p-seminorms, where a p-seminorm on E is any nonnegative real-valued and subadditive functional $\| \cdot \|_p$ on E such that $\|\lambda x\|_p = |\lambda|^p \|x\|_p$ for each $x \in E$ and $\lambda \in \mathbb{R}$ (i.e., the real number line). When E is Hausdorff, then for every $x \ne 0$, there is some $p \in P$ such that $P(x) \ne 0$. Whenever the family P is reduced to a singleton, one says that $(E, \| \cdot \|)$ is a p-seminormed space. A p-normed space is a Hausdorff p-seminormed space when $p = 1$, which is the usual locally convex case. Furthermore, a p-normed space is a metric vector space with the translation invariant metric $d_p(x, y) := \|x - y\|_p$ for all $x, y \in E$, which is the same notation as above.

By Remark 2.5, if P is a continuous p-seminorm on E, then the ball $B_p(0, s) := \{x \in E : P(x) < s\}$ is shrinkable for each $s > 0$. Indeed, if $r \in (0, 1)$ and $x \in \overline{rB_p(0, s)}$, then there exists a net $(x_i)_{i \in I} \subset B_p(0, s)$ such that rx_i converges to x. By the continuity of P, we get $P(x) \le r^p s < s$, which means that $r\overline{B_p(0, s)} \subset B_P(0, s)$. In general, it can be shown that every p-convex $U \in \mathfrak{V}_0$ is shrinkable.

We recall that given such a neighborhood U, a subset $A \subset E$ is said to be U-small if $A - A \subset U$ (or small of order U by Robertson [102]). Now, we follow the idea of Kaniok [58] in the setting of a topological vector space E to use zero neighborhoods in E instead of seminorms to define the measure of noncompactness in (local convex) p-vector spaces $(0 < p \le 1)$ as follows: For each $A \subset E$, the U-measures of noncompactness $\alpha_U(A)$ and $\beta_U(A)$ for A are defined by:

$$\alpha_U(A) := \inf\{: r > 0 :$$

$$A \text{ is covered by a finite number of } rU\text{-small sets } A_i \text{ for } i = 1, 2, \ldots, n\}$$

and

$$\beta_U(A) := \inf\left\{r > 0 : \text{there exists } x_1, \ldots, x_n \in E \text{ such that } A \subset \cup_{i=1}^n (x_i + rU)\right\},$$

where we set $\inf \emptyset := \infty$.

By the definition above, it is clear that when E is a normed space and U is the closed unit ball of E, α_U and β_U are nothing else but the Kuratowski measure β_K and Hausdorff measure β_H of noncompactness, respectively. Thus, if $\mathfrak{U}$ denotes a fundamental system of balanced and closed zero neighborhoods in E and $\mathfrak{F}_{\mathfrak{U}}$ is the space of all functions $\phi : \mathfrak{U} \to R$ endowed with the pointwise ordering, then the α_U (resp., β_U) measures for noncompactness introduced by Kaniok [58] can be expressed by the Kuratowski (resp., the Hausdorff) measure of noncompact $\alpha(A)$ (resp., $\beta(A)$) for a subset A of E as the function defined from $\mathfrak{U}$ into $[0, \infty)$:

$$\alpha(A)(U) := \alpha_U(A) \quad (\text{resp.,} \ \beta(A)(U) := \beta_U(A)).$$

By following Machrafi and Oubbi [73], in order to define the measure of noncompactness in (locally convex) p-vector space E, we need the following notions of basic and sufficient collections for zero neighborhoods in a topological vector space. To do this, let us introduce an equivalence relation on V_0 by saying that U is related to V, written $U\mathfrak{R}V$, if and only if there exist $r, s > 0$ such that $rU \subset V \subset sU$. We now have the following definition.

Definition 4.3 (BCZN) We say that $\mathfrak{B} \subset \mathfrak{V}_0$ is a basic collection of zero neighborhoods (in short, BCZN) if it contains at most one representative member from each equivalence class with respect to $\mathfrak{R}$. It will be said to be sufficient (in short, SCZN) if it is basic and, for every $V \in \mathfrak{V}_0$, there exist some $U \in \mathfrak{B}$ and some $r > 0$ such that $rU \subset V$.

Remark 4.3 By Remark 2.5 again, it follows that, for a locally p-convex space E, its base set $\mathfrak{U}$, the family of all open p-convex subsets for 0 is BCZB. We note that: 1) In the case E is a normed space, if f is a continuous functional on E, $U := \{x \in E : |f(x)| < 1\}$ and V is the open unit ball of E, then $\{U\}$ is basic but not sufficient, but $\{V\}$ is sufficient; 2) Secondly, if (E, τ) is a locally convex space, whose topology is given by an upward directed family P of seminorms so that no two of them are equivalent, the collection $(B_p)_{p \in \mathbb{P}}$ is an SCZN, where B_p is the open unit ball of p. Further, if $\mathfrak{W}$ is a fundamental system of zero neighborhoods in a topological vector space E, then there exists an SCZN consisting of $\mathfrak{W}$ members; and 3) By following Oubbi [84], we recall that a subset A of E is called uniformly bounded with respect to a sufficient collection $\mathfrak{B}$ of zero neighborhoods if there exists $r > 0$ such that $A \subset rV$ for all $V \in \mathfrak{B}$. Note that in the locally convex space $C_c(X) := C_c(X, \mathbb{K})$, the set $B_\infty := \{f \in C(X) : \|f\|_\infty \leq 1\}$ is uniformly bounded with respect to the SCZN $\{B_k, k \in \mathbb{K}\}$, where B_k is the (closed or) open unit ball of the seminorm P_k, where $k \in \mathbb{K}$.

Now we are ready to give the definition for the measure of noncompactness in (locally p-convex) topological vector space E as follows.

Definition 4.4 Let $\mathfrak{B}$ be an SCZN in E. For each $A \subset E$, we define the measure of noncompactness of A with respect to $\mathfrak{B}$ by $\alpha_{\mathfrak{B}}(A) := \sup_{U \in \mathfrak{B}} \alpha_U(A)$.

By the definition above, it is clear that: 1) The measure of noncompactness α_B holds the semiadditivity, i.e., $\alpha_B(A \cup B) = \max\{\alpha_B(A), \alpha_B(B)\}$; and 2) $\alpha_B(A) = 0$ if and only if A is a precompact subset of E (for more properties in detail, see Proposition 1 and the related discussion by Machraf and Oubbi [84]).

As we know, under the normed spaces (and even seminormed spaces), Kuratowski [65], Darbo [30], and Sadovskii [106] introduced the notions of k-set-contractions for $k \in (0, 1)$ and the condensing mappings to establish fixed point theorems in the setting of Banach spaces, normed or seminorm spaces. By following the same idea, if E is a Hausdorff locally p-convex space, we have the following definition for general (nonlinear) mappings.

Definition 4.5 A mapping $T : C \to 2^C$ is said to be a k-set contraction (resp., condensing), if there is some SCZN $\mathfrak{B}$ in E consisting of p-convex sets such that (resp., condensing) for any $U \in \mathfrak{B}$, there exists $k \in (0, 1)$ (resp., condensing) such that $\alpha_U(T(A)) \leq k\alpha_U(A)$ for $A \subset C$ (resp., $\alpha_U(T(A)) < \alpha_U(A)$ for each $A \subset C$ with $\alpha_U(A) > 0$).

It is clear that a contraction mapping on C is a k-set contraction mapping (where we always mean $k \in (0, 1)$), and a k-set contraction mapping on C is condensing; and they all reduce to the usual cases by the definitions for the β_K and β_H, which are the Kuratowski measure and Hausdorff measure of noncompactness, respectively, in normed spaces (see Kuratowski [65]).

From now on, we denote by $\mathfrak{V}_0$ the set of all shrinkable zero neighborhoods in E. We have the following result which is Theorem 1 of Machrafi and Oubbi [73], saying that in the general setting of locally p-convex spaces, the measure of noncompactness α for U given by Definition 4.4 is stable from U to its p-convex hull $C_p(A)$ of the subset A in E, which is key for us to establish the fixed points for condensing mappings in locally p-convex spaces for $0 < p \leq 1$. This also means that the key property for the Kuratowski and Hausdorff measures of noncompactness in normed (or p-seminorm) spaces also holds for the measure of noncompactness by Definition 4.4 in the setting of locally p-convex spaces with $(0 < p \leq 1)$ (see more similar and related discussion in detail by Alghamdi et al. [5] and Silva et al. [112]).

Lemma 4.2 *If $U \in \mathfrak{V}_0$ is p-convex for some $0 < p \leq 1$, then $\alpha(C_p(A)) = \alpha(A)$ for every $A \subset E$.*

Proof It is Theorem 1 of Machrafi and Oubbi [73]. The proof is complete.   □

Now, based on the definition for the measure of noncompactness given by Definition 4.4 (originally from Machrafi and Oubbi [73]), we have the following general extended version of Schauder, Darbo, and Sadovskii type fixed point theorems in the context of locally p-convex vector spaces for condensing mappings.

Theorem 4.5 (Schauder's fixed point theorem for condensing mappings) *Let $C \subset E$ be a complete p-convex subset of a Hausdorff locally p-convex space E with $0 < p \leq 1$. If $T : C \to 2^C$ is an upper semicontinuous and (α) condensing set-valued mapping with nonempty p-convex closed values, then T has a fixed point in C and the set of fixed points of T is compact.*

Proof Let $\mathfrak{B}$ be a sufficient collection of p-convex zero neighborhoods in E with respect to which T is condensing for any given $U \in \mathfrak{B}$. We choose some $x_0 \in C$ and let $\mathfrak{F}$ be the

family of all closed p-convex subsets A of C with $x_0 \in A$ and $T(A) \subset A$. Note that $\mathfrak{F}$ is not empty since $C \in \mathfrak{F}$. Let $A_0 = \cap_{A \in \mathfrak{F}} A$. Then A_0 is a nonempty closed p-convex subset of C such that $T(A_0) \subset A_0$. We shall show that A_0 is compact. Let $A_1 = \overline{C_p(T(A_0) \cup \{x_0\})}$. Since $T(A_0) \subset A_0$ and A_0 is closed and p-convex, $A_1 \subset A_0$. Hence, $T(A_1) \subset T(A_0) \subset A_1$. It follows that $A_1 \in \mathfrak{F}$ and therefore $A_1 = A_0$. Now, by Proposition 1 of Machrafi and Oubbi [73] and Lemma 4.2 (i.e., Theorem 1 and Theorem 2 in [73]), we get $\alpha_U(T(A_0)) = \alpha_U(A)$. Our assumption on T shows that $\alpha_U(A_0) = 0$ since T is condensing. As U is arbitrary from the family $\mathfrak{B}$, thus A_0 is p-convex and compact (see Proposition 4 in [73]). Now, the conclusion follows by Theorem 4.2. Secondly, let C_0 be the set of fixed points of T in C. Then it follows that $C_0 \subset T(C_0)$, and the upper semicontinuity of T implies that its graph is closed, so is the set C_0. As T is condensing, we have $\alpha_U(T(C_0)) \leq \alpha_U(C_0)$, which implies that $\alpha_U(C_0) = 0$. As U is arbitrary from the family $\mathfrak{B}$, it implies that C_0 is compact (by Proposition 4 in [73] again). The proof is complete. □

As applications of Theorem 4.5, we have the following few of fixed points for condensing mappings in locally p-convex spaces for $0 < p \leq 1$.

Corollary 4.2 (Darbo type fixed point theorem) *Let C be a complete p-convex subset of a Hausdorff locally p-convex space E with $0 < p \leq 1$. If $T : C \to 2^C$ is a (k)-set-contraction (where $k \in (0,1)$) with closed and p-convex values, then T has a fixed point.*

Corollary 4.3 (Sadovskii type fixed point theorem) *Let $(E, \|\cdot\|)$ be a complete p-normed space and C be a bounded, closed, and p-convex subset of E, where $0 < p \leq 1$. Then every continuous and condensing mapping $T : C \to 2^C$ with closed and p-convex values has a fixed point.*

Proof In Theorem 4.5, let $\mathfrak{B} := \{B_p(0,1)\}$, where $B_p(0,1)$ stands for the closed unit ball of E. By the fact that it is clear that $\alpha(A) = (\alpha_{\mathfrak{B}}(A))^p$ for each $A \subset E$, T satisfies all the conditions of Theorem 4.5. This completes the proof. □

Corollary 4.4 (Darbo type) *Let $(E, \|\cdot\|)$ be a complete p-normed space and C be a bounded, closed p-convex subset of E, where $0 < p \leq 1$. Then each mapping $T : C \to C$ which is a continuous set-contraction with closed p-convex values has a fixed point.*

Theorem 4.5 improves Theorem 5 of Machrafi and Oubbi [73] for general condensing mappings, which are general upper semicontinuous mappings with closed p-convex values, and also unifies the corresponding results in the existing literature, e.g., see Alghamdi et al. [5], Górniewicz [46], Górniewicz et al. [47], Nussbaum [78], Silva et al. [112], Xiao and Lu [123], Xiao and Zhu [124, 125], and the references therein.

Secondly, as an application of the KKM principle for abstract convex spaces with Kalton's remarkable embedded lemma [55] for compact p-convex sets in topological vector spaces, we also establish general fixed point theorems for compact (single-valued) continuous mappings, which allows us to answer Schauder's conjecture in the affirmative way under the general framework of closed p-convex subsets in TVS for $0 < p \leq 1$.

Before ending this section, we would also like to remark that compared with the topological method or related arguments used by Askoura et al. [6], Cauty [21, 22], Nhu [77], Reich [100], the fixed points given in this section improve or unify the corresponding ones

given by Alghamdi et al. [5], Darbo [30], Liu [72], Machrafi and Oubbi [73], Sadovskii [106], Silva et al. [112], Xiao and Lu [123], and those from the references therein.

5 Best approximation for the classes of 1-set contractive mappings in locally *p*-convex spaces

The goal of this section is first to establish one general best approximation result for 1-set upper semicontinuous and hemicompact (see its definition below) nonself set-valued mappings, which in turn is used as a tool to derive the general principle for the existence of solutions for Birkhoff–Kellogg problems (see Birkhoff and Kellogg [13]) and fixed points for nonself 1-set contractive set-valued mappings in locally *p*-convex spaces for $0 < p \leq 1$.

Here, we recall that since the Birkhoff–Kellogg theorem was first introduced and proved by Birkhoff and Kellogg [13] in 1922 in discussing the existence of solutions for the equation $x = \lambda F(x)$, where λ is a real parameter, and F is a general nonlinear nonself mapping defined on an open convex subset U of a topological vector space E, now the general form of the Birkhoff–Kellogg problem is to find the so-called invariant direction for the nonlinear set-valued mappings F, i.e., to find $x_0 \in \overline{U}$ (or $x_0 \in \partial \overline{U}$) and $\lambda > 0$ such that $\lambda x_0 \in F(x_0)$.

Since Birkhoff and Kellogg theorem was given by Birkhoff and Kellogg in 1920s, the study on the Birkhoff–Kellogg problem has received a lot of scholars' attention. For example, one of the fundamental results in nonlinear functional analysis, called the Leray–Schauder alternative by Leray and Schauder [67] in 1934, was established via topological degree. Thereafter, certain other types of Leray–Schauder alternatives were proved using techniques other than topological degree, see the work given by Granas and Dugundji [48], Furi and Pera [39] in the Banach space setting and applications to the boundary value problems for ordinary differential equations, and a general class of mappings for nonlinear alternative of Leray–Schauder type in normal topological spaces, and also Birkhoff–Kellogg type theorems for a general class of mappings in TVS or LCS by Agarwal et al. [2], Agarwal and O'Regan [3, 4], Park [88]; in particular, recently O'Regan [81] used the Leray–Schauder type coincidence theory to establish some Birkhoff–Kellogg problem, Furi–Pera type results for a general class of set-valued mappings.

In this section, one best approximation result for 1-set contractive mappings in *p*-seminorm spaces is first established and is then used to the general principle for solutions of Birkhoff–Kellogg problems and related nonlinear alternatives, which allows us to give general existence results for the Leray–Schauder type and related fixed point theorems of nonself mappings in *p*-seminorm spaces for $(0 < p \leq 1)$. The new results given in this part not only include the corresponding results in the existing literature as special cases, but are also expected to be useful tools for the study of nonlinear problems arising from theory to practice for 1-set contractive mappings.

We also note that the general nonlinear alternative related to the Leray–Schauder alternative under the framework of *p*-seminorm spaces for $(0 < p \leq 1)$ given in this section would be a useful tool for the study of nonlinear problems. In addition, we also note that the corresponding results in the existing literature for Birkhoff–Kellogg problems and the Leray–Schauder alternative have been studied comprehensively by Granas and Dugundji [48], Isac [53], Park [89–91], Carbone and Conti [20], Chang and Yen [27], Chang et al. [25, 26], Kim et al. [59], Shahzad [110]–[111], Singh [114]; and in particular, many general forms have been recently obtained by O'Regan [82] (see also the references therein).

In order to study the existence of fixed points for nonself mappings in p-vector spaces for $0 < p \leq 1$, we need the following definitions.

Definition 5.1 (Inward and outward sets in p-vector spaces) Let C be a subset of a p-vector space E and $x \in E$ for $0 < p \leq 1$. Then the p-inward set $I_C^p(x)$ and p-outward set $O_C^p(x)$ are defined by

$I_C^p(x) := \{x + r(y - x) : y \in C$ for any $r \geq 0$ (1) if $0 \leq r \leq 1$ with $(1 - r)^p + r^p = 1$; or (2) if $r \geq 1$ with $(\frac{1}{r})^p + (1 - \frac{1}{r})^p = 1\}$; and

$O_C^p(x) := \{x + r(y - x) : y \in C$ for any $r \leq 0$ (1) if $0 \leq |r| \leq 1$ with $(1 - |r|)^p + |r|^p = 1$; or (2) if $|r| \geq 1$ with $(\frac{1}{|r|})^p + (1 - \frac{1}{|r|})^p = 1\}$.

From the definition, it is obvious that when $p = 1$, both the inward and outward sets $I_C^p(x)$, $O_C^p(x)$ are reduced to the definitions of the inward set $I_C(x)$ and the outward set $O_C(x)$, respectively, in topological vector spaces introduced by Halpern and Bergman [49] and used for the study of nonself mappings related to nonlinear functional analysis in the literature. In this paper, we mainly focus on the study of the p-inward set $I_C^p(x)$ for the best approximation related to the boundary condition for the existence of fixed points in p-vector spaces. By the special property of p-convex concept when $p \in (0, 1)$ and $p = 1$, we have the following fact.

Lemma 5.1 *Let C be a subset of a p-vector space E and $x \in E$, where for $0 < p \leq 1$. Then, for both p-inward and outward sets $I_C^p(x)$ and $O_C^p(x)$ defined above, we have*

(I) when $p \in (0, 1)$, $I_C^p(x) = [\{x\} \cup C]$ and $O_C^p(x) = [\{x\} \cup \{2x\} \cup -C]$,

(II) when $p = 1$, in general $[\{x\} \cup C] \subset I_C^p(x)$ and $[\{x\} \cup \{2x\} \cup -C] \subset O_C^p(x)$.

Proof First, when $p \in (0, 1)$, by the definitions of $I_C^p(x)$, the only real number $r \geq 0$ satisfying the equation $(1 - r)^p + r^p = 1$ for $r \in [0, 1]$ is $r = 0$ or $r = 1$, and when $r \geq 1$, the equation $(\frac{1}{r})^p + (1 - \frac{1}{r})^p = 1$ implies that $r = 1$. The same reason for $O_C^p(x)$, it follows that $r = 0$ and $r = -1$.

Secondly, when $p = 1$, all $r \geq 0$ and all $r \leq 0$ satisfy the requirement of definition for $I_C^p(x)$ and $O_C^p(x)$, respectively, thus the proof is compete. $\square$

By following the original idea by Tan and Yuan [118] for hemicompact mappings in metric spaces, we introduce the following definition for a mapping being hemicompact in p-seminorm spaces for $p \in (0, 1]$, which is indeed the "**(H) condition**" used in Theorem 5.1 below to prove the existence of best approximation results for 1-set contractive set-valued mappings in p-seminorm vector spaces for $p \in (0, 1]$.

Definition 5.2 (Hemicompact mapping) Let E be a p-vector space with p-seminorm for $1 < p \leq 1$. For a given bonded (closed) subset D in E, a mapping $F : D \to 2^E$ is said to be hemicompact if each sequence $\{x_n\}_{n \in N}$ in D has a convergent subsequence with limit x_0 such that $x_0 \in F(x_0)$, whenever $\lim_{n \to \infty} d_{P_U} P(x_n, F(x_n)) = 0$ for each $U \in \mathfrak{U}$, where $d_{P_U} P(x, C) := \inf\{P_U(x - y) : y \in C\}$ is the distance of a single point x with the subset C in E based on P_U, P_U is the Minkowski p-functional in E for $U \in \mathfrak{U}$, which is the base of the family consisting of all subsets of 0-neighborhoods in E.

Remark 5.1 We would like to point out that Definition 5.2 is indeed an extension for a "hemicompact mapping" defined from a metric space to a p-vector space with the p-seminorm, where $p \in (0, 1]$ (see Tan and Yuan [118]). By the monotonicity of Minkowski p-functionals, i.e., the bigger 0-neighborhoods, the smaller Minkowski p-functionals' values (see also p. 178 of Balachandran [7]), Definition 5.2 describes the converge for the distance between x_n and $F(x_n)$ by using the language of seminorms in terms of Minkowski p-functionals for each 0-neighborhood in $\mathfrak{U}$ (the base), which is the family consisting of its 0-neighborhoods in a locally p-convex space E for $0 < p \leq 1$.

Now we have the following Schauder fixed point theorem for 1-set contractive mappings in locally p-convex spaces for $p \in (0, 1]$.

Theorem 5.1 (Schauder's fixed point theorem for 1-set contractive mappings) *Let U be a nonempty bounded open subset of a (Hausdorff) locally p-convex space E and its zero $0 \in U$, and let $C \subset E$ be a closed p-convex subset of E such that $0 \in C$ with $0 < p \leq 1$. If $F : C \cap \overline{U} \to 2^{C \cap \overline{U}}$ is an upper semicontinuous and 1-set contractive set-valued mapping with nonempty p-convex closed values and satisfies one of the following conditions*:

(H) Condition: *The sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\overline{U}$ has a convergent subsequence with limit $x_0 \in \overline{U}$ such that $x_0 \in F(x_0)$, whenever $\lim_{n \to \infty} d_{P_U}(x_n, F(x_n)) = 0$, where $d_{P_U}(x_n, F(x_n)) := \inf\{P_U(x_n - z) : z \in F(x_n)\}$, where P_U is the Minkowski p-functional for any $U \in \mathfrak{U}$, which is the family of all nonempty open p-convex subsets containing the zero in E.*

(H1) Condition: *There exists x_0 in $\overline{U}$ with $x_0 \in F(x_0)$ if there exists $\{x_n\}_{n \in \mathbb{N}}$ in $\overline{U}$ such that $\lim_{n \to \infty} d_{P_U}(x_n, F(x_n)) = 0$, where P_U is the Minkowski p-functional for any $U \in \mathfrak{U}$, which is the family of all nonempty open p-convex subsets containing the zero in E.*

Then F has at least one fixed point in $C \cap \overline{U}$.

Proof Let $\mathfrak{U}$ be a family of all nonempty open p-convex subsets containing the zero in E and U be any element in $\mathfrak{U}$. As the mapping T is 1-set contractive, take an increasing sequence $\{\lambda_n\}$ such that $0 < \lambda_n < 1$ and $\lim_{n \to \infty} \lambda_n = 1$, where $n \in \mathbb{N}$. Now we define a mapping $F_n : C \to 2^C$ by $F_n(x) := \lambda_n F(x)$ for each $x \in C$ and $n \in \mathbb{N}$. Then it follows that F_n is a λ_n-set-contractive mapping with $0 < \lambda_n < 1$. By Theorem 4.5 on the condensing mapping F_n in p-vector space with p-seminorm P_U for each $n \in \mathbb{N}$, there exists $x_n \in C$ such that $x_n \in F_n(x_n) = \lambda_n F(x_n)$. Thus there exists $y_n \in F(x_n)$ such that $x_n = \lambda_n y_n$. As P_U is the Minkowski p-functional of U in E, it follows that P_U is continuous as $0 \in \text{int}(U) = U$. Note that, for each $n \in \mathbb{N}$, $\lambda_n x_n \in \overline{U} \cap C$, which implies that $x_n = r(\lambda_n y_n) = \lambda_n y_n$, thus $P_U(\lambda_n y_n) \leq 1$ by Lemma 2.2. Note that

$$P_U(y_n - x_n) = P_U(y_n - x_n) = P_U(y_n - \lambda_n y_n) = P_U\left(\frac{(1 - \lambda_n)\lambda_n y_n}{\lambda_n}\right)$$

$$\leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p P_U(\lambda_n y_n) \leq \left(\frac{1 - \lambda_n}{\lambda_n}\right)^p,$$

which implies that $\lim_{n \to \infty} P_U(y_n - x_n) = 0$ for all $U \in \mathfrak{U}$.

Now (1) if F satisfies the (H) condition, it implies that the consequence $\{x_n\}_{n \in \mathbb{N}}$ has a convergent subsequence which converges to x_0 such that $x_0 \in F(x_0)$. Without loss of generality, we assume that $\lim_{n \to \infty} x_n = x_0$, here $y_n \in F(x_n)$ is with $x_n = \lambda_n y_n$, and $\lim_{n \to \infty} \lambda_n = 1$, it

implies that $x_0 = \lim_{n\to\infty}(\lambda_n y_n)$, which means $y_0 := \lim_{n\to\infty} y_n = x_0$. There exists $y_0(= x_0) \in F(x_0)$.

(ii) if F satisfies the (H1) condition, then by the (H1) condition, it follows that there exists x_0 in $\overline{U}$ such that $x_0 \in F(x_0)$, which is a fixed point of F. We complete the proof. $\qquad\square$

Theorem 5.2 (Best approximation for 1-set-contractive mappings) *Let U be a bounded open p-convex subset of a locally p-convex space E ($0 \le p \le 1$) the zero $0 \in U$, and C be a (bounded) closed p-convex subset of E with also zero $0 \in C$. Assume that $F : \overline{U} \cap C \to 2^C$ is a 1-set contractive and upper semicontinuous mapping with nonempty closed p-convex values, and for each $x \in \partial_C U$ with $y \in F(x) \cap (C \setminus \overline{U})$, $(P_U^{\frac{1}{p}}(y) - 1)^p \le P_U(y - x)$ for $0 < p \le 1$ (this is trivial when $p = 1$). In addition, if F satisfies one of the following conditions:*

(H) Condition: *The sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\overline{U}$ has a convergent subsequence with limit $x_0 \in \overline{U}$ such that $x_0 \in F(x_0)$, whenever $\lim_{n\to\infty} d_{P_U}(x_n, F(x_n)) = 0$, where $d_{P_U}(x_n, F(x_n)) := \inf\{P_U(x_n - z) : z \in F(x_n)\}$, where P_U is the Minkowski p-functional for any $U \in \mathfrak{U}$, which is the family of all nonempty open p-convex subsets containing the zero in E.*

(H1) Condition: *There exists x_0 in $\overline{U}$ with $x_0 \in F(x_0)$ if there exists $\{x_n\}_{n\in\mathbb{N}}$ in $\overline{U}$ such that $\lim_{n\to\infty} d_{P_U}(x_n, F(x_n)) = 0$, where P_U is the Minkowski p-functional for any $U \in \mathfrak{U}$, which is the family of all nonempty open p-convex subsets containing the zero in E.*

Then we have that there exist $x_0 \in C \cap \overline{U}$ and $y_0 \in F(x_0)$ such that

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\big(y_0, \overline{I_U^p(x_0)} \cap C\big),$$

where P_U is the Minkowski p-functional of U. More precisely, we have that either (I) or (II) holds:

(I) F has a fixed point $x_0 \in \overline{U} \cap C$ such that $x_0 \in F(x_0)$ (i.e., $0 = P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_U^p(x_0)} \cap C)$,

(II) There exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \overline{U}$ with

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\big(y_0, \overline{I_U^p(x_0)} \cap C\big) = \big(P_U^{\frac{1}{p}}(y_0) - 1\big)^p > 0.$$

Proof As E is a p-convex space and U is a bounded open p-convex subset of E, it suffices to prove that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\overline{U}$ and $y_n \in F(x_n)$ such that $\lim_{n\to\infty} P_U(y_n - x_n) = 0$, and the conclusion follows by applying the (H) condition.

Let $r : E \to U$ be a retraction mapping defined by $r(x) := \dfrac{x}{\max\{1, (P_U(x))^{\frac{1}{p}}\}}$ for each $x \in E$, where P_U is the Minkowski p-functional of U. Since the space E's zero $0 \in U$ ($=$ int U as U is open), it follows that r is continuous by Lemma 2.2. As the mapping F is 1-set contractive, taking an increasing sequence $\{\lambda_n\}$ such that $0 < \lambda_n < 1$ and $\lim_{n\to\infty} \lambda_n = 1$, where $n \in \mathbb{N}$. Now, for each $n \in \mathbb{N}$, we define a mapping $F_n : C \cap \overline{U} \to 2^C$ by $F_n(x) := \lambda_n F \circ r(x)$ for each $x \in C \cap \overline{U}$. By the fact that C and $\overline{U}$ are p-convex, it follows that $r(C) \subset C$ and $r(\overline{U}) \subset \overline{U}$, thus $r(C \cap \overline{U}) \subset C \cap \overline{U}$. Therefore F_n is a mapping from $\overline{U} \cap C$ to itself. For each $n \in \mathbb{N}$, by the fact that F_n is a λ_n-set-contractive mapping with $0 < \lambda_n < 1$, it follows by Theorem 4.5 for the condensing mapping that there exists $z_n \in C \cap \overline{U}$ such that $z_n \in F_n(z_n) = \lambda_n F \circ r(z_n)$. As $r(C \cap \overline{U}) \subset C \cap \overline{U}$, let $x_n = r(z_n)$. Then we have that $x_n \in C \cap \overline{U}$, and there exists $y_n \in F(x_n)$ with $x_n = r(\lambda_n y_n)$ such that one of the following (1) or (2) holds for each $n \in \mathbb{N}$: (1) $\lambda_n y_n \in C \cap \overline{U}$; or (2) $\lambda_n y_n \in C \setminus \overline{U}$.

Now we prove the conclusion by considering the following two cases under the (H) condition and (H1) condition.

Case (I) For each $n \in N$, $\lambda_n y_n \in C \cap \overline{U}$; or

Case (II) There exists a positive integer n such that $\lambda_n y_n \in C \setminus \overline{U}$.

First, by case (I), for each $n \in \mathbb{N}$, $\lambda_n y_n \in \overline{U} \cap C$, which implies that $x_n = r(\lambda_n y_n) = \lambda_n y_n$, thus $P_U(\lambda_n y_n) \leq 1$ by Lemma 2.2. Note that

$$P_U(y_n - x_n) = P_U(y_n - x_n) = P_U(y_n - \lambda_n y_n) = P_U\left(\frac{(1-\lambda_n)\lambda_n y_n}{\lambda_n}\right)$$
$$\leq \left(\frac{1-\lambda_n}{\lambda_n}\right)^p P_U(\lambda_n y_n) \leq \left(\frac{1-\lambda_n}{\lambda_n}\right)^p,$$

which implies that $\lim_{n\to\infty} P_U(y_n - x_n) = 0$. Now, for any $V \in \mathbb{U}$, without loss of generality, let $U_0 = V \cap U$. Then we have the following conclusion:

$$P_{U_0}(y_n - x_n) = P_{U_0}(y_n - x_n) = P_{U_0}(y_n - \lambda_n y_n) = P_{U_0}\left(\frac{(1-\lambda_n)\lambda_n y_n}{\lambda_n}\right)$$
$$\leq \left(\frac{1-\lambda_n}{\lambda_n}\right)^p P_{U_0}(\lambda_n y_n) \leq \left(\frac{1-\lambda_n}{\lambda_n}\right)^p,$$

which implies that $\lim_{n\to\infty} P_{U_0}(y_n - x_n) = 0$, where P_{U_0} is the Minkowski p-functional of U_0 in E.

Now, if F satisfies the (H) condition, if follows that the consequence $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence which converges to x_0 such that $x_0 \in F(x_0)$. Without loss of generality, we assume that $\lim_{n\to\infty} x_n = x_0$, where $y_n \in F(x_n)$ is with $x_n = \lambda_n y_n$, and $\lim_{n\to\infty} \lambda_n = 1$, and as $x_0 = \lim_{n\to\infty}(\lambda_n y_n)$, which implies that $y_0 = \lim_{n\to\infty} y_n = x_0$. Thus there exists $y_0(= x_0) \in F(x_0)$, we have $0 = d_p(x_0, F(x_0)) = d(y_0, \overline{U} \cap C) = d_p(y_0, I_{\overline{U}}^p(x_0) \cap C))$ as indeed $x_0 = y_0 \in F(x_0) \in \overline{U} \cap C \subset I_{\overline{U}}^p(x_0) \cap C)$.

If F satisfies the (H1) condition, if follows that there exists $x_0 \in \overline{U} \cap C$ with $x_0 \in F(x_0)$. Then we have $0 = P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, I_{\overline{U}}^p(x_0) \cap C)$.

Second, by case (II) there exists a positive integer n such that $\lambda_n y_n \in C \setminus \overline{U}$. Then we have that $P_U(\lambda_n y_n) > 1$, and also $P_U(y_n) > 1$ as $\lambda_n < 1$. As $x_n = r(\lambda_n y_n) = \frac{\lambda_n y_n}{(P_U(\lambda_n y_n))^{\frac{1}{p}}}$, which implies that $P_U(x_n) = 1$, thus $x_n \in \partial_C(U)$. Note that

$$P_U(y_n - x_n) = P_U\left(\frac{(P_U(y_n)^{\frac{1}{p}} - 1)y_n}{P_U(y_n)^{\frac{1}{p}}}\right) = \left(P_U^{\frac{1}{p}}(y_n) - 1\right)^p.$$

By the assumption, we have $(P_U^{\frac{1}{p}}(y_n) - 1)^p \leq P_U(y_n - x)$ for $x \in C \cap \partial\overline{U}$, it follows that

$$P_U(y_n) - 1 \leq P_U(y_n) - \sup\{P_U(z) : z \in C \cap \overline{U}\}$$
$$\leq \inf\{P_U(y_n - z) : z \in C \cap \overline{U}\} = d_p(y_n, C \cap \overline{U}).$$

Thus we have the best approximation: $P_U(y_n - x_n) = d_P(y_n, \overline{U} \cap C) = (P_U^{\frac{1}{p}}(y_n) - 1)^p > 0$.

Now we want to show that $P_U(y_n - x_n) = d_P(y_n, \overline{U} \cap C) = d_p(y_n, I_{\overline{U}}^p(x_0) \cap C) > 0$.

By the fact that $(\overline{U} \cap C) \subset I_{\overline{U}}^p(x_n) \cap C$, let $z \in I_{\overline{U}}^p(x_n) \cap C \setminus (\overline{U} \cap C)$, we first claim that $P_U(y_n - x_n) \leq P_U(y_n - z)$. If not, we have $P_U(y_n - x_n) > P_U(y_n - z)$. As $z \in I_{\overline{U}}^p(x_n) \cap C \setminus (\overline{U} \cap C)$,

there exists $y \in \overline{U}$ and a nonnegative number c (actually $c \geq 1$ as shown soon below) with $z = x_n + c(y - x_n)$. Since $z \in C$, but $z \notin \overline{U} \cap C$, it implies that $z \notin \overline{U}$. By the fact that $x_n \in \overline{U}$ and $y \in \overline{U}$, we must have the constant $c \geq 1$; otherwise, it implies that $z(= (1 - c)x_n + cy) \in \overline{U}$, this is impossible by our assumption, i.e., $z \notin \overline{U}$. Thus we have that $c \geq 1$, which implies that $y = \frac{1}{c}z + (1 - \frac{1}{c})x_n \in C$ (as both $x_n \in C$ and $z \in C$). On the other hand, as $z \in I_{\overline{U}}^p(x_n) \cap C \setminus (\overline{U} \cap C)$ and $c \geq 1$ with $(\frac{1}{c})^p + (1 - \frac{1}{c})^p = 1$, combining with our assumption that, for each $x \in \partial_C \overline{U}$ and $y \in F(x_n) \setminus \overline{U}$, $P_{\overline{U}}^{\frac{1}{p}}(y) - 1 \leq P_{\overline{U}}^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$, it then follows that

$$P_U(y_n - y) = P_U \left[\frac{1}{c}(y_n - z) + \left(1 - \frac{1}{c} \right)(y_n - x_n) \right]$$
$$\leq \left[\left(\frac{1}{c} \right)^p P_U(y_n - z) + \left(1 - \frac{1}{c} \right)^p P_U(y_n - x_n) \right] < P_U(y_n - x_n),$$

which contradicts that $P_U(y_n - x_n) = d_P(y_n, \overline{U} \cap C)$ as shown above, we know that $y \in \overline{U} \cap C$, we should have $P_U(y_n - x_n) \leq P_U(y_n - y)$! This helps us to complete the claim: $P_U(y_n - x_n) \leq P_U(y_n - z)$ for any $z \in I_{\overline{U}}^p(x_n) \cap C \setminus (\overline{U} \cap C)$, which means that the following best approximation of Fan type (see [37, 38]) holds:

$$0 < d_P(y_n, \overline{U} \cap C) = P_U(y_n - x_n) = d_p \left(y_n, I_{\overline{U}}^p(x_n) \cap C \right).$$

Now, by the continuity of P_U, it follows that the following best approximation of Fan type is also true:

$$0 < P_U(y_n - x_n) = d_P(y_n, \overline{U} \cap C) = d_p \left(y_n, I_{\overline{U}}^p(x_n) \cap C \right) = d_p \left(y_n, \overline{I_{\overline{U}}^p(x_n)} \cap C \right).$$

The proof is complete.   □

Remark 5.2 Based on the proof of Theorem 5.2, we have that 1): For the condition "$x \in \partial_C U$ with $y \in F(x)$, $P_{\overline{U}}^{\frac{1}{p}}(y) - 1 \leq P_{\overline{U}}^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$", indeed we only need that for "$x \in \partial_C U$ with $y \in F(x) \cap (C \setminus \overline{U})$), $P_{\overline{U}}^{\frac{1}{p}}(y) - 1 \leq P_{\overline{U}}^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$"; 2): Theorem 5.2 also improves the corresponding best approximation for 1-set contractive mappings given by Li et al. [69], Liu [72], Xu [129], Xu et al. [130], and the results from the references therein; and 3): When $p = 1$, we have a similar best approximation result for the mapping F in the locally convex spaces with outward set boundary condition below (see Theorem 3 of Park [87] and the related discussion by references therein).

For the p-vector space with $p = 1$ being a topological vector space E, we have the following best approximation for the outward set $\overline{O_{\overline{U}}(x_0)}$ based on the point $\{x_0\}$ with respect to the convex subset U in E.

Theorem 5.3 (Best approximation for outward sets) *Let U be a bounded open convex subset of a locally convex space E (i.e., $p = 1$) with zero $0 \in \text{int } U = U$ (the interior $\text{int } U = U$ as U is open), and let C be a closed p-convex subset of E with also zero $0 \in C$. Assume that $F : \overline{U} \cap C \rightarrow C$ is a 1-set-contractive continuous mapping with closed p-convex values satisfying condition (H) or $(H1)$ above. Then there exist $x_0 \in \overline{U} \cap X$ and $y_0 \in F(x_0)$ such that*

$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{O_{\overline{U}}(x_0)} \cap C)$, where P_U is the Minkowski *p*-functional of U. *More precisely, we have that either* (I) *or* (II) *holds*:

(I) F *has a fixed point* $x_0 \in \overline{U} \cap C$, *i.e.*, $P_U(y_0 - x_0) = P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{O_{\overline{U}}(x_0)} \cap C)) = 0$;

(II) *There exist* $x_0 \in \partial_C(U)$ *and* $y_0 \in F(x_0) \backslash \overline{U}$ *with*

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, O_{\overline{U}}(x_0) \cap C) = d_p(y_0, \overline{O_{\overline{U}}(x_0)} \cap C) > 0.$$

Proof We define a new mapping $F_1 : \overline{U} \cap C \to 2^C$ by $F_1(x) := \{2x\} - F(x)$ for each $x \in \overline{U} \cap C$, then F_1 is also a compact and upper semicontinuous mapping with nonempty closed convex values, and F_1 satisfies all hypotheses of Theorem 5.2 with $p = 1$. If follows by Theorem 5.2 that there exist $x_0 \in \overline{U} \cap X$ and $y_1 \in F_1(x_0)$ such that $P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p(y_1, \overline{I_{\overline{U}}(x_0)} \cap C)$. More precisely, we have that either (I) or (II) holds:

(I) F_1 has a fixed point $x_0 \in \overline{U} \cap C$ (so $0 = P_U(y_1 - x_0) = P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p(y_1, \overline{I_{\overline{U}}(x_0)} \cap C)$);

(II) There exist $x_0 \in \partial_C(U)$ and $y_1 \in F_1(x_0) \backslash \overline{U}$ with

$$P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p(y_1, \overline{O_{\overline{U}}(x_0)} \cap C) > 0.$$

Now, for any $x \in O_{\overline{U}}(x_0)$, there exist $r < 0$, $u \in \overline{U}$ such that $x = x_0 + r(u - x_0)$. Let $x_1 = 2x_0 - x$, then $x_1 = 2x_0 - x_0 - r(u - x_0) = x_0 + (-r)(u - x_0) \in I_{\overline{U}}(x_0)$. Let $y_1 = 2x_0 - y_0$ for some $y_0 \in F(x_0)$. As we have $P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p(y_1, \overline{I_{\overline{U}}(x_0)} \cap C)$, it follows that $P_U(y_1 - x_0) \leq P_U(y_1 - x_1)$, which implies that

$$P_U(x_0 - y_0) = P_U(y_1 - x_0) \leq P_U(y_1 - x_1) = P_U\big(2x_0 - y_0 - (2x_0 - x)\big) = P_U(y_0 - x)$$

for all $x \in O_{\overline{U}}(x_0)$. Thus we have $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, O_{\overline{U}}(x_0) \cap C)$, and by the continuity of P_U, it follows that

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\big(y_0, \overline{O_{\overline{U}}(x_0)} \cap C\big)\big(P_U(y_0) - 1\big) > 0.$$

This completes the proof.   □

Now, by the application of Theorem 5.2 and Theorem 5.3, Remark 5.2, and the argument used in Theorem 5.2, we have the following general principle for the existence of solutions for Birkhoff–Kellogg problems in *p*-seminorm spaces, where $(0 < p \leq 1)$.

Theorem 5.4 (Principle of Birkhoff–Kellogg alternative) *Let U be a bounded open p-convex subset of a locally p-convex space E $(0 \leq p \leq 1)$ with zero $0 \in \operatorname{int} U = (U)$ (the interior $\operatorname{int} U$ as U is open), and C be a closed p-convex subset of E with also zero $0 \in C$. Assume that $F : \overline{U} \cap C \to C$ is a 1-set-contractive continuous mapping with closed p-convex values satisfying the* (H) *or* (H1) *condition above. Then F has at least one of the following two properties*:

(I) *F has a fixed point $x_0 \in \overline{U} \cap C$ such that $x_0 \in F(x_0)$,*

(II) *There exist $x_0 \in \partial_C(U)$, $y_0 \in F(x_0) \backslash \overline{U}$, and $\lambda = \dfrac{1}{(P_U(y_0))^{\frac{1}{p}}} \in (0, 1)$ such that $x_0 = \lambda y_0 \in \lambda F(x_0)$. In addition, if for each $x \in \partial_C U$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ (this is trivial*

when $p = 1$), then the best approximation between x_0 and y_0 is given by

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C) = \left(P_U^{\frac{1}{p}}(y_0) - 1\right)^p > 0.$$

Proof If (I) is not the case, then (II) is proved by Remark 5.2 and by following the proof in Theorem 5.2 for case (ii): $y_0 \in C \setminus \overline{U}$ with $y_0 = f(x_0) \in f(x_0)$. Indeed, as $y_0 \notin \overline{U}$, it follows that $P_U(y_0) > 1$, and $x_0 = f(y_0) = y_0 \frac{1}{(P_U(y_0))^{\frac{1}{p}}}$. Now let $\lambda = \frac{1}{(P_U(y_0))^{\frac{1}{p}}}$, we have $\lambda < 1$ and $x_0 = \lambda y_0$ with $y_0 \in F(x_0)$. Finally, the additional assumption in (II) allows us to have the best approximation between x_0 and y_0 obtained by following the proof of Theorem 5.2 as $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C) > 0$. This completes the proof. □

As an application of Theorem 5.2 for the nonself set-valued mappings discussed in Theorem 5.3 with outward set condition, we can also have the following general principle of Birkhoff–Kellogg alternative in locally convex spaces by applying Theorem 5.4 for $p = 1$.

Theorem 5.5 (Principle of Birkhoff–Kellogg alternative in TVS) *Let U be a bounded open convex subset of the locally convex space E with the zero $0 \in U$, and let C be a closed convex subset of E with also zero $0 \in C$. Assume that $F : \overline{U} \cap C \to 2^C$ is a 1-set contractive and upper semicontinuous mapping with nonempty closed convex values satisfying the (H) or (H1) condition above. Then it has at least one of the following two properties*:

(I) F has a fixed point $x_0 \in \overline{U} \cap C$ such that $x_0 \in F(x_0)$;

(II) There exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \overline{U}$ and $\lambda \in (0, 1)$ such that $x_0 = \lambda y_0$, and the best approximation between x_0 and y_0 is given by $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C) > 0$.

On the other hand, by the proof of Theorem 5.2, we note that for case (II) of Theorem 5.2, the assumption "each $x \in \partial_C U$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$" is only used to guarantee the best approximation "$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C) > 0$", thus we have the following Leray–Schauder alternative in locally p-convex spaces, which, of course, includes the corresponding results in locally convex spaces as special cases.

Theorem 5.6 (The Leray–Schauder nonlinear alternative) *Let C be a closed p-convex subset of a p-seminorm space E with $0 \leq p \leq 1$ and the zero $0 \in C$. Assume that $F : C \to 2^C$ is a 1-set contractive and upper semicontinuous mapping with nonempty closed p-convex values satisfying the (H) or (H1) condition above. Let $\varepsilon(F) := \{x \in C : x \in \lambda F(x) \text{ for some } 0 < \lambda < 1\}$. Then either F has a fixed point in C or the set $\varepsilon(F)$ is unbounded.*

Proof We prove the conclusion by assuming that F has no fixed point, then we claim that the set $\varepsilon(F)$ is unbounded. Otherwise, assume that the set $\varepsilon(F)$ is bounded and assume that P is the continuous p-seminorm for E, then there exists $r > 0$ such that the set $B(0, r) := \{x \in E : P(x) < r\}$, which contains the set $\varepsilon(F)$, i.e., $\varepsilon(F) \subset B(0, r)$, which means, for any $x \in \varepsilon(F)$, $P(x) < r$. Then $B(0.r)$ is an open p-convex subset of E and the zero $0 \in B(0, r)$ by Lemma 2.2 and Remark 2.4. Now, let $U := B(0, r)$ in Theorem 5.4, it follows that the mapping $F : B(0, r) \cap C \to 2^C$ satisfies all the general conditions of Theorem 5.4, and we have that for any $x_0 \in \partial_C B(0, r)$, no any $\lambda \in (0, 1)$ such that $x_0 = \lambda y_0$, where $y_0 \in F(x_0)$. Indeed, for any $x \in \varepsilon(F)$, it follows that $P(x) < r$ as $\varepsilon(F) \subset B(0, r)$, but for any $x_0 \in \partial_C B(0, r)$,

we have $P(x_0) = r$, thus conclusion (II) of Theorem 5.4 does not hold. By Theorem 5.4 again, F must have a fixed point, but this contradicts our assumption that F is fixed point free. This completes the proof. $\qquad\square$

Now assume a given locally p-convex space E equipped with the P-seminorm (by assuming it is continuous at zero) for $0 < p \leq 1$, then we know that $P : E \to \mathbb{R}^+$, $P^{-1}(0) = 0$, $P(\lambda x) = |\lambda|^p P(x)$ for any $x \in E$ and $\lambda \in \mathbb{R}$. Then we have the following useful result for fixed points due to Rothe and Altman types in p-vector spaces, which plays important roles for optimization problem, variational inequality, complementarity problems (see Isac [53] or Yuan [133] and the references therein for related study in detail).

Corollary 5.1 *Let U be a bounded open p-convex subset of a locally p-convex space E and zero $0 \in U$, plus C is a closed p-convex subset of E with $U \subset C$, where $0 < p \leq 1$. Assume that $F : \overline{U} \to 2^C$ is a 1-set contractive upper semicontinuous mapping with nonempty closed p-convex values satisfying the (H) or $(H1)$ condition above. If one of the following is satisfied,*
  1. *(Rothe type condition): $P_U(y) \leq P_U(x)$ for $y \in F(x)$, where $x \in \partial U$;*
  2. *(Petryshyn type condition): $P_U(y) \leq P_U(y - x)$ for $y \in F(x)$, where $x \in \partial U$;*
  3. *(Altman type condition): $|P_U(y)|^{\frac{2}{p}} \leq [P_U(y) - x)]^{\frac{2}{p}} + [P_U(x)]^{\frac{2}{p}}$ for $y \in F(x)$, where $x \in \partial U$,*

then F has at least one fixed point.

Proof By conditions (1), (2), and (3), it follows that the conclusion of (II) in Theorem 5.4 "there exist $x_0 \in \partial_C(U)$ and $\lambda \in (0, 1)$ such that $x_0 \notin \lambda F(x_0)$" does not hold, thus by the alternative of Theorem 5.4, F has a fixed point. This completes the proof. $\qquad\square$

By the fact that when $p = 1$, each locally p-convex space is a locally convex spaces, we have the following classical Fan's best approximation (see [37]) as a powerful tool for the study in the optimization, mathematical programming, games theory, mathematical economics, and other related topics in applied mathematics.

Corollary 5.2 (Fan's best approximation) *Let U be a bounded open convex subset of a locally convex space E with the zero $0 \in U$ and C be a closed convex subset of E with also zero $0 \in C$, and assume $F : \overline{U} \cap C \to 2^C$ is a 1-set contractive and upper semicontinuous mapping with nonempty closed convex values satisfying the (H) or $(H1)$ condition above. Assume P_U to be the Minkowski p-functional of U in E. Then there exist $x_0 \in \overline{U} \cap X$ and $y_0 \in T(x_0)$ such that $P_U(y_0 - x_0) = d_p(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C)$. More precisely, we have the following either (I) or (II) holding, where $W_{\overline{U}}(x_0)$ is either an inward set $I_{\overline{U}}(x_0)$ or an outward set $O_{\overline{U}}(x_0)$:*

(I) *F has a fixed point $x_0 \in \overline{U} \cap C$, $0 = P_U(y_0 - x_0) = P_U(y_0 - x_0) = d_p(y_0, \overline{U} \cap C) = d_p(y_0, \overline{W_{\overline{U}}(x_0)} \cap C))$,*

(II) *There exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \backslash \overline{U}$ with*

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\big(y_0, \overline{W_{\overline{U}}(x_0)} \cap C\big) = P_U(y_0) - 1 > 0.$$

Proof When $p = 1$, it automatically satisfies the inequality $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$, and indeed we have that for $x_0 \in \partial_C(U)$, with $y_0 \in F(x_0)$, we have $P_U(y_0 - x_0) = d_p(y_0, \overline{U} \cap C) =$

$d_p(y_0, \overline{W_{\overline{U}}(x_0)} \cap C) = P_U(y_0) - 1$. The conclusions are given by Theorem 5.2 (or Theorem 5.3). The proof is complete. $\qquad\square$

We would like to point out similar results on Rothe and Leray–Schauder alternative developed by Isac [53], Park [86], Potter [98], Shahzad [109–111], Xiao and Zhu [124] (see also related references therein) as tools of nonlinear analysis in topological vector spaces. As mentioned above, when $p = 1$ and take F as a continuous mapping, then we obtain a version of Leray–Schauder in locally convex spaces, and thus we omit its statement in detail.

6 Principle of nonlinear alternatives for classes of 1-set contractive mappings in locally *p*-convex spaces

As applications of results in Sect. 5, we now establish general results for the existence of solutions for Birkhoff–Kellogg problem and the principle of Leray–Schauder alternatives in locally *p*-convex spaces for $0 < p \leq 1$.

Theorem 6.1 (Birkhoff–Kellogg alternative in locally *p*-convex spaces) *Let U be a bounded open p-convex subset of a locally p-convex space E (where $0 \leq p \leq 1$) with the zero $0 \in U$, and let C be a closed p-convex subset of E with also zero $0 \in C$, and assume $F : \overline{U} \cap C \to 2^C$ is a 1-set contractive and upper semicontinuous mapping with nonempty closed p-convex values satisfying condition (H) or (H1) above. In addition, for each $x \in \partial_C(U)$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$), where P_U is the Minkowski p-functional of U. Then we have that either (I) or (II) holds:*

(I) There exists $x_0 \in \overline{U} \cap C$ such that $x_0 \in F(x_0)$;

(II) There exists $x_0 \in \partial_C(U)$ with $y_0 \in F(x_0) \setminus \overline{U}$ and $\lambda > 1$ such that $\lambda x_0 = y_0 \in F(x_0)$, i.e., $F(x_0) \cap \{\lambda x_0 : \lambda > 1\} \neq \emptyset$.

Proof By following the argument and symbols used in the proof of Theorem 5.2, we have that either

(1) F has a fixed point $x_0 \in \overline{U} \cap C$; or

(2) There exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with $x_0 = f(y_0)$ such that

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C) = P_U(y_0) - 1 > 0,$$

where $\partial_C(U)$ denotes the boundary of U relative to C in E, and f is the restriction of the continuous retraction r with respect to the set U in E.

If F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. As given by the proof of Theorem 5.2, we have that $y_0 \in F(x_0)$ and $y_0 \notin \overline{U}$, thus $P_U(y_0) > 1$ and $x_0 = f(y_0) = \frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}$, which means $y_0 = (P_U(y_0))^{\frac{1}{p}} x_0$. Let $\lambda = (P_U(y_0))^{\frac{1}{p}}$, then $\lambda > 1$, and we have $\lambda x_0 = y_o \in F(x_0)$. This completes the proof. $\qquad\square$

Theorem 6.2 (Birkhoff–Kellogg alternative in TVS) *Let U be a bounded open convex subset of a locally convex space E with the zero $0 \in U$, and let C be a closed convex subset of E with also zero $0 \in C$, and assume $F : \overline{U} \cap C \to 2^C$ is a 1-set contractive and upper semicontinuous mapping with nonempty closed convex values satisfying condition (H) or (H1)*

above. Then we have either (I) *or* (II) *of the following holds, where* $W_{\overline{U}}(x_0)$ *is either an inward set* $I_{\overline{U}}(x_0)$ *or an outward set* $O_{\overline{U}}(x_0)$:

(I) *There exists* $x_0 \in \overline{U} \cap C$ *such that* $x_0 \in F(x_0)$;

(II) *There exists* $x_0 \in \partial_C(U)$ *with* $y_0 \in F(x_0) \backslash \overline{U}$ *and* $\lambda > 1$ *such that* $\lambda x_0 = y_0 \in F(x_0)$, *i.e.,* $F(x_0) \cap \{\lambda x_0 : \lambda > 1\} \neq \emptyset$.

Proof When $p = 1$, then it automatically satisfies the inequality $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$, and indeed, for $x_0 \in \partial_C(U)$, with $y_0 \in F(x_0)$, we have $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{W_{\overline{U}}(x_0)} \cap C) = P_U(y_0) - 1$. The conclusions are given by Theorems 5.3 and 5.4. The proof is complete. $\qquad\square$

Indeed, we have the following fixed points for nonself mappings in p-vector spaces for $0 < p \leq 1$ under different boundary conditions.

Theorem 6.3 (Fixed points of nonself mappings) *Let U be a bounded open p-convex subset of a locally p-convex space E (where $0 \leq p \leq 1$) with the zero $0 \in U$, and let C be a closed p-convex subset of E with also zero $0 \in C$, and assume $F : \overline{U} \cap C \to 2^C$ is a 1-set contractive and upper semicontinuous mapping with nonempty closed p-convex values satisfying condition (H) or (H1) above. In addition, for each $x \in \partial_C(U)$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$), where P_U is the Minkowski p-functional of U. If F satisfies any one of the following conditions for any $x \in \partial_C(U) \backslash F(x)$,*

(i) *For each $y \in F(x)$, $P_U(y - z) < P_U(y - x)$ for some $z \in \overline{I_{\overline{U}}(x)} \cap C$;*

(ii) *For each $y \in F(x)$, there exists λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{I_{\overline{U}}(x)} \cap C$;*

(iii) *$F(x) \subset \overline{I_{\overline{U}}(x)} \cap C$;*

(iv) *$F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;*

(v) *$F(\partial U) \subset \overline{U} \cap C$;*

(vi) *For each $y \in F(x)$, $P_U(y - x) \neq ((P_U(y))^{\frac{1}{p}} - 1)^p$;*

then F must have a fixed point.

Proof By following the argument and symbols used in the proof of Theorem 5.2 (see also Theorem 5.4), we have that either

(1) F has a fixed point $x_0 \in \overline{U} \cap C$; or

(2) There exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with $x_0 = f(y_0)$ such that

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C) = P_U(y_0) - 1 > 0,$$

where $\partial_C(U)$ denotes the boundary of U relative to C in E, and f is the restriction of the continuous retraction r with respect to the set U in E.

First, suppose that F satisfies condition (i), if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. Then, by condition (i), it follows that $P_U(y_0 - z) < P_U(y_0 - x_0)$ for some $z \in \overline{I_{\overline{U}}(x)} \cap C$, this contradicts the best approximation equations given by (2) above, thus F must have a fixed point.

Second, suppose that F satisfies condition (ii), if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. Then, by condition (ii), there exists $\lambda > 1$ such that $\lambda x_0 + (1 - \lambda)y_0 \in$

$\overline{I_{\overline{U}}(x)} \cap C$. It follows that

$$P_U(y_0 - x_0) \leq P_U(y_0 - (\lambda x_0 + (1 - \lambda y_0))) = P_U(\lambda(y_0 - x_0))$$

$$= |\lambda|^p P_U(y_0 - x_0) < P_U(y_0 - x_0),$$

this is impossible, and thus F must have a fixed point in $\overline{U} \cap C$.

Third, suppose that F satisfies condition (iii), i.e., $F(x) \subset \overline{I_{\overline{U}}(x)} \cap C$, then by (2) we have that $P_U(y_0 - x_0)$, and thus $x_0 = y_0 \in F(x_0)$, which means F has a fixed point.

Forth, suppose that F satisfies condition (iv), if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. As given by the proof of Theorem 5.2, we have that $y_0 \notin \overline{U}$, thus $P_U(y_0) > 1$ and $x_0 = f(y_0) = \frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}$, which means $y_0 = (P_U(y_0))^{\frac{1}{p}} x_0$, where $(P_U(y_0))^{\frac{1}{p}} > 1$, this contradicts assumption (iv), thus F must have a fixed point in $\overline{U} \cap C$.

Fifth, suppose that F satisfies condition (v), then $x_0 \notin F(x_0)$. As $x_0 \in \partial_C U$, now by condition (v), we have that $F(\partial U) \subset \overline{U} \cap C$, it follows that, for any $y_0 \in F(x_0)$, we have $y_0 \in \overline{U} \cap C$, thus $y \notin \overline{U} \backslash \cap C$, which implies that $0 < P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = 0$, this is impossible, thus F must have a fixed point. Here, like pointed out by Remark 5.2, we know that based on condition (v), the mapping F has a fixed point by applying $F(\partial U) \subset \overline{U} \cap C$ is enough, we do not need the general hypothesis: "for each $x \in \partial_C(U)$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$".

Finally, suppose that F satisfies condition (vi), if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. Then condition (v) implies that $P_U(y_0 - x_0) \neq ((P_U(y))^{\frac{1}{p}} - 1)^p$, but our proof in the theorem shows that $P_U(y_0 - x_0) = ((P_U(y))^{\frac{1}{p}} - 1)^p$, this is impossible, thus F must have a fixed point. Then the proof is complete. □

Now, by taking the set C in Theorem 6.1 as the whole p-vector space E itself, we have the following general results for nonself upper semicontinuous set-valued mappings which include the results of Rothe, Petryshyn, Altman, and Leray–Schauder type fixed points as special cases.

Taking $p = 1$ and $C = E$ in Theorem 6.3, we have the following fixed points for nonself upper semicontinuous set-valued mappings associated with inward or outward sets in locally convex spaces.

Theorem 6.4 (Fixed points of nonself mappings with boundary conditions) *Let U be a bounded open convex subset of a locally convex space E with the zero $0 \in U$, and assume that $F : \overline{U} \to 2^E$ is a 1-set contractive and upper semicontinuous mapping with nonempty closed convex values satisfying condition (H) or (H1) above. If F satisfies any one of the following conditions for any $x \in \partial(U) \backslash F(x)$,*

(i) *For each $y \in F(x)$, $P_U(y - z) < P_U(y - x)$ for some $z \in \overline{I_{\overline{U}}(x)}$ (or $z \in \overline{O_{\overline{U}}(x)}$);*

(ii) *For each $y \in F(x)$, there exists λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{I_{\overline{U}}(x)}$ (or $\overline{O_{\overline{U}}(x)}$);*

(iii) *$F(x) \subset \overline{I_{\overline{U}}(x)}$ (or $\overline{O_{\overline{U}}(x)}$);*

(iv) *$F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;*

(v) *$F(\partial(U) \subset \overline{U}$;*

(vi) *For each $y \in F(x)$, $P_U(y - x) \neq P_U(y) - 1$;*

then F must have a fixed point.

In what follows, based on the best approximation theorem in a p-seminorm space, we will also give some fixed point theorems for nonself set-valued mappings with various boundary conditions, which are related to the study for the existence of solutions for PDE and differential equations with boundary problems (see Browder [17], Petryshyn [94, 95], Reich [100]), which would play roles in nonlinear analysis for p-seminorm space as shown below.

First, as discussed by Remark 5.2, the proof of Theorem 5.2 with the strongly boundary condition "$F(\partial(U)) \subset \overline{U} \cap C$" only, we can prove that F has a fixed point, thus we have the following fixed point theorem of Rothe type in locally p-convex spaces.

Theorem 6.5 (Rothe type) *Let U be a bounded open p-convex subset of a locally p-convex space E (where $0 \leq p \leq 1$) with the zero $0 \in U$. Assume that $F : \overline{U} \to 2^E$ is a 1-set contractive and upper semicontinuous mapping with nonempty closed p-convex values satisfying condition (H) or $(H1)$ above, and such that $F(\partial(U)) \subset \overline{U}$, then F must have a fixed point.*

Now, as applications of Theorem 6.5, we give the following Leray–Schauder alternative in p-vector spaces for nonself set-valued mappings associated with the boundary condition, which often appears in the applications (see Isac [53] and the references therein for the study of complementary problems and related topics in optimization).

Theorem 6.6 (Leray–Schauder alternative in locally p-convex spaces) *Let E be a locally p-convex space E, where $0 < p \leq 1$, let $B \subset E$ be bounded closed p-convex such that $0 \in \operatorname{int} B$. Let $F : [0,1] \times B \to E$ be 1-set contractive and upper semicontinuous set-valued with nonempty closed p-convex values satisfying condition (H) or $(H1)$ above, and such that the set $F([0,1] \times B)$ is relatively compact in E. If the following assumptions are satisfied:*
  (1) $x \notin F(t,x)$ for all $x \notin \partial B$ and $t \in [0,1]$,
  (2) $F(\{0\} \times \partial B) \subset B$,
then there is an element $x^* \in B$ such that $x^* \in F(1,x^*)$.

Proof For any $n \in N$, we consider the mapping

$$
F_n(x) = \begin{cases} F(\frac{1-P_B(x)}{\epsilon_n}, \frac{x}{P_B(x)}), & \text{if } 1 - \epsilon \leq P_B(x) \leq 1, \\ F(1, \frac{X}{1-\epsilon_n}), & \text{if } P_B(x) < 1 - \epsilon_n, \end{cases} \tag{3}
$$

where P_B is the Minkowski p-functional of B and $\{\epsilon_n\}_{n \in N}$ is a sequence of real numbers such that $\lim_{n \to \infty} \epsilon_n = 0$ and $0 < \epsilon_n < \frac{1}{2}$ for any $n \in N$. We observe that, for each $n \in N$, the mapping F_n is 1-set contractive upper semicontinuous with nonempty closed p-convex values on B. From assumption (2), we have that $F_n(\partial B) \subset B$, and the assumptions of Theorem 6.5 are satisfied, then for each $n \in N$, there exists an element $u_n \in B$ such that $u_n \in F_n(u_n)$.

We first prove the following statement: "It is impossible to have an infinite number of the elements u_n satisfying the following inequality: $1 - \epsilon_n \leq P_B(u_n) \leq 1$".

If not, we assume to have an infinite number of the elements u_n satisfying the following inequality:

$$
1 - \epsilon_n \leq P_B(u_n) \leq 1.
$$

As $F_n(B)$ is relatively compact and by the definition of mappings F_n, we have that $\{u_n\}_{n\in N}$ is contained in a compact set in E. Without loss of generality (indeed, each compact set is also countably compact), we define the sequence $\{t_n\}_{n\in N}$ by $t_n := \frac{1-P_B(u_n)}{\epsilon}$ for each $n \in N$. Then we have that $\{t_n\}_{n\in N} \subset [0,1]$, and we may assume that $\lim_{n\to\infty} t_n = t \in [0,1]$. The corresponding subsequence of $\{u_n\}_{n\in N}$ is denoted again by $\{u_n\}_{n\in N}$ and it also satisfies the inequality $1 - \epsilon_n \le P_B(u_n) \le 1$, which implies that $\lim_{n\to\infty} P_B(u_n) = 1$.

Now, let u^* be an accumulation point of $\{u_n\}_{n\in N}$, thus we have $\lim_{n\to\infty}(t_n, \frac{u_n}{P_B(u_n)}, u_n) = (t, u^*, u^*)$. By the fact that F is compact, we have assume that $u_n \in F(t_n, \frac{u_n}{P_B(u_n)})$ for each $n \in N$. It follows that $u^* \in F(t, u^*)$, this contradicts assumption (1) as we have $\lim_{n\to\infty} P_B(u_n) = 1$ (which means that $u^* \in \partial B$, this is impossible).

Thus it is impossible "to have an infinite number of elements u_n satisfying the inequality: $1-\epsilon_n \le P_B(u_n) \le 1$", which means that there is only a finite number of elements of sequence $\{u_n\}_{n\in N}$ satisfying the inequality $1 - \epsilon_n \le P_B(u_n) \le 1$. Now, without loss of generality, for $n \in N$, we have the following inequality:

$$P_B(u_n) < 1 - \epsilon_n.$$

By the fact that $\lim_{n\to}(1-\epsilon_n) = 1$, $u_n \in F(1, \frac{u_n}{1-\epsilon})$ for all $n \in N$ and assuming that $\lim_{n\to} u_n = u^*$, the upper semicontinuity of F with nonempty closed values implies that the graph of F is closed, and by the fact $u_n \in F(1, \frac{u_n}{1-\epsilon})$, it implies that $u^* \in F(1, u^*)$. This completes the proof. □

As a special case of Theorem 6.6, we have the following principle for the implicit form of Leray–Schauder type alternative for set-valued mappings in locally p-spaces for $0 < p \le 1$.

Corollary 6.1 (The implicit Leray–Schauder alternative) *Let E be a locally p-convex space E, where $0 < p \le 1$, $B \subset E$ be bounded closed p-convex such that $0 \in \text{int}\, B$. Let $F : [0,1] \times B \to E$ be 1-set contractive and continuous with nonempty closed p-convex values satisfying condition (H) or $(H1)$ above, and the set $F([0,1] \times B)$ be relatively compact in E. If the following assumptions are satisfied*:

(1) $F(\{0\} \times \partial B) \subset B$,

(2) $x \notin F(0,x)$ for all $x \in \partial B$,

then at least one of the following properties is satisfied:

(i) *There exists $x^* \in B$ such that $x^* \in F(1, x^*)$; or*

(ii) *There exists $(\lambda^*, x^*) \in (0,1) \times \partial B$ such that $x^* \in F(\lambda^*, x^*)$.*

Proof The result is an immediate consequence of Theorem 6.6, this completes the proof. □

We would like to point out that similar results on Rothe and Leray–Schauder alternative have been developed by Furi and Pera [39], Granas and Dugundji [48], Górniewicz [46], Górniewicz et al. [47], Isac [53], Li et al. [69], Liu [72], Park [86], Potter [98], Shahzad [109–111], Xu [129], Xu et al. [130] (see also related references therein) as tools of nonlinear analysis in the Banach space setting and applications to the boundary value problems for ordinary differential equations in noncompact problems, a general class of mappings for nonlinear alternative of Leray–Schauder type in normal topological spaces, and some Birkhoff–Kellogg type theorems for general class mappings in topological vector spaces have also been established by Agarwal et al. [2], Agarwal and O'Regan [3, 4], Park [88]

(see the references therein for more details); and in particular, recently O'Regan [81] used the Leray–Schauder type coincidence theory to establish some Birkhoff–Kellogg problem, Furi–Pera type results for a general class of mappings.

Before closing this section, we would like to share that as the application of the best approximation result for 1-set contractive mappings, we can establish fixed point theorems and the general principle of Leray–Schauder alternative for nonself mappings, which seem to play important roles for the nonlinear analysis under the framework of p-seminorm spaces, as the achievement of nonlinear analysis for locally convex spaces, normed spaces, or in Banach spaces.

7 Fixed points for classes of 1-set contractive mappings

In this section, based on the best approximation Theorem 5.2 for classes of 1-set contractive mappings developed in Sect. 5, we will show how it can be used as a useful tool to establish fixed point theorems for nonself upper semicontinuous mappings in p-seminorm spaces (for $p \in (0, 1]$, and including norm spaces, uniformly convex Banach spaces as special classes).

By following Browder [17], Li [68], Goebel and Kirk [43], Petryshyn [94, 95], Tan and Yuan [118], Xu [129] (see also the references therein), we recall some definitions as follows for p-seminorm spaces, where $p \in (0, 1]$.

Definition 7.1 Let D be a nonempty (bounded) closed subset of p-vector spaces $(E, \| \cdot \|_p)$ with p-seminorm, where $p \in (0, 1]$. Suppose that $f : D \to X$ is a (single-valued) mapping, then: (1) f is said to be nonexpansive if for each $x, y \in D$, we have $\|f(x) - f(y)\|_p \leq \|x - y\|_p$; (2) f (actually, $(I - f)$) is said to be demiclosed (see Browder [17]) at $y \in X$ if for any sequence $\{x_n\}_{n \in \mathbb{N}}$ in D, the conditions $x_n \to x_0 \in D$ weakly and $(I - f)(x_n) \to y_0$ strongly imply that $(I - f)(x_0) = y_0$, where I is the identity mapping; (3) f is said to be hemicompact (see p. 379 of Tan and Yuan [118]) if each sequence $\{x_n\}_{n \in \mathbb{N}}$ in D has a convergent subsequence with the limit x_0 such that $x_0 = f(x_0)$, whenever $\lim_{n \to \infty} d_p(x_n, f(x_n)) = 0$, here $d_P(x_n, f(x_n)) := \inf\{P_U(x_n - z) : z \in f(x_n)\}$, and P_U is the Minkowski p-functional for any $U \in \mathfrak{U}$, which is the family of all nonempty open p-convex subsets containing the zero in E; (4) f is said to be demicompact (by Petryshyn [94]) if each sequence $\{x_n\}_{n \in \mathbb{N}}$ in D has a convergent subsequence whenever $\{x_n - f(x_n)\}_{n \in \mathbb{N}}$ is a convergent sequence in X; (5) f is said to be a semiclosed 1-set contractive mapping if f is 1-set contractive mapping, and $(I - f)$ is closed, where I is the identity mapping (by Li [68]); and (6) f is said to be semicontractive (see Petryshyn [95] and Browder [17]) if there exists a mapping $V : D \times D \to 2^X$ such that $f(x) = V(x, x)$ for each $x \in D$, with (a) for each fixed $x \in D$, $V(\cdot, x)$ is nonexpansive from D to X; and (b) for each fixed $x \in D$, $V(x, \cdot)$ is completely continuous from D to X, uniformly for u in a bounded subset of D (which means if v_j converges weakly to v in D and u_j is a bounded sequence in D, then $V(u_j, v_j) - V(u_j, v) \to 0$ strongly in D).

From the definition above, we first observe that definitions (1) to (6) for set-valued mappings can be given in a similar way with the Hausdorff metric H (we omit their definitions in detail here to save space); Secondly, if f is a continuous demicompact mapping, then $(I - f)$ is closed, where I is the identity mapping on X. It is also clear from the definitions that every demicompact map is hemicompact in seminorm spaces, but the converse is not true by the example in p. 380 by Tan and Yuan [118]. It is evident that if f is demicompact, then $I - f$ is demiclosed. It is known that, for each condensing mapping f, when D

or $f(D)$ is bounded, then f is hemicompact; and also f is demicompact in metric spaces by Lemma 2.1 and Lemma 2.2 of Tan and Yuan [118], respectively. In addition, it is known that every nonexpansive map is a 1-set-contractive mapping; and also if f is a hemicompact 1-set-contractive mapping, then f is a 1-set-contractive mapping satisfying the following **(H1) condition** (which is the same as condition (H1) used by Theorem 5.1 in Sect. 5, but slightly different from condition (H) used there in Sect. 5):

(H1) condition Let D be a nonempty bounded subset of a space E and assume $F : \overline{D} \to 2^E$ is a set-valued mapping. If $\{x_n\}_{n \in \mathbb{N}}$ is any sequence in D such that, for each x_n, there exists $y_n \in F(x_n)$ with $\lim_{n \to \infty}(x_n - y_n) = 0$, then there exists a point $x \in \overline{D}$ such that $x \in F(x)$.

We first note that the (H1) condition above is actually the same one as condition (C) used by Theorem 1 of Petryshyn [95]. Secondly, it was shown by Browder [17] that indeed the nonexpansive mapping in a uniformly convex Banach X enjoys condition (H1) as shown below.

Lemma 7.1 *Let D be a nonempty bounded convex subset of a uniformly convex Banach space E. Assume that $F : \overline{D} \to E$ is a nonexpansive (single-valued) mapping, then the mapping $P := I - F$ defined by $P(x) := (x - F(x))$ for each $x \in \overline{D}$ is demiclosed, and in particular, the (H1) condition holds.*

Proof By following the argument given in p. 329 (see the proof of Theorem 2.2 and Corollary 2.1) by Petryshyn [95], the mapping F is demiclosed (which actually is called Browder's demiclosedness principle), which says that by the assumption of (H1) condition, if $\{x_n\}_{n \in \mathbb{N}}$ is any sequence in D such that for each x_n there exists $y_n \in F(x_n)$ with $\lim_{n \to \infty}(x_n - y_n) = 0$, then we have $0 \in (I - F)(\overline{D})$, which means that there exists $x_0 \in \overline{D}$ with $0 \in (I - F)(x_0)$, this implies that $x_0 \in F(x_0)$. The proof is complete.   □

Remark 7.1 When a p-vector space E is with a p-norm, the (H) condition satisfies the (H1) condition. The (H1) condition is mainly supported by the so-called demiclosedness principle after the work by Browder [17].

By applying Theorem 5.2, we have the following result for nonself mappings in p-seminorm spaces for $p \in (0, 1]$.

Theorem 7.1 *Let U be a bounded open p-convex subset of a p-(semi)norm space E $(0 < p \le 1)$ the zero $0 \in U$. Assume that $F : \overline{U} \to 2^E$ is a 1-set contractive and upper semicontinuous mapping with nonempty closed p-convex values, satisfying condition (H) or (H1) above. In addition, for any $x \in \partial\overline{U}$ and $y \in F(x)$, we have $\lambda x \ne y$ for any $\lambda > 1$ (i.e., the Leray–Schauder boundary condition). Then F has at least one fixed point.*

Proof By Theorem 5.2 with $C = E$, it follows that we have the following either (I) or (II) holding:

(I) F has a fixed point $x_0 \in \overline{U}$, i.e., $P_U(y_0 - x_0) = 0$;

(II) There exist $x_0 \in \partial(U)$ and $y_0 \in F(x_0)$ with $P_U(y_0 - x_0) = (P_U^{\frac{1}{p}}(y_0) - 1)^p > 0$.

If F has no fixed point, then above (II) holds and $x_0 \notin F(x_0)$. By the proof of Theorem 5.2, we have that $x_0 = f(y_0)$ and $y_0 \notin \overline{U}$. Thus $P_U(y_0) > 1$ and $x_0 = f(y_0) = \dfrac{y_0}{(P_U(y_0))^{\frac{1}{p}}}$, which means

$y_0 = (P_U(y_0))^{\frac{1}{p}} x_0$, where $(P_U(y_0))^{\frac{1}{p}} > 1$, this contradicts the assumption. Thus F must have a fixed point. The proof is complete. $\qquad\square$

By following the idea used and developed by Browder [17], Li [68], Li et al. [69], Goebel and Kirk [43], Petryshyn [94, 95], Tan and Yuan [118], Xu [129], Xu et al. [130] (see also the references therein), we have a number of existence theorems for the principle of Leray–Schauder type alternatives in p-seminorm spaces $(E, \| \cdot \|_p)$ for $p \in (0, 1]$ as follows.

Theorem 7.2 *Let U be a bounded open p-convex subset of a p-(semi)norm space $(E, \| \cdot \|_p)$ $(0 < p \le 1)$ the zero $0 \in U$. Assume that $F : \overline{U} \to 2^E$ is a 1-set contractive and upper semi-continuous mapping with nonempty closed p-convex values satisfying condition (H) or $(H1)$ above. In addition, there exist $\alpha > 1$, $\beta \ge 0$ such that, for each $x \in \partial\overline{U}$, we have that for any $y \in F(x)$, $\|y - x\|_p^{\alpha/p} \ge \|y\|_p^{(\alpha+\beta)/p} \|x\|_p^{-\beta/p} - \|x\|_p^{\alpha/p}$. Then F has at least one fixed point.*

Proof We prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 7.1 does not hold. If we assume that F has no fixed point, by the boundary condition of Theorem 7.1, there exist $x_0 \in \partial\overline{U}$, $y_0 \in F(x_0)$, and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t - 1)^\alpha - t^{\alpha+\beta} + 1$ for $t \ge 1$. We observe that f is a strictly decreasing function for $t \in [1, \infty)$ as the derivative of $f'(t) = \alpha(t-1)^{\alpha-1} - (\alpha+\beta)t^{\alpha+\beta-1} < 0$ by the differentiation, thus we have $t^{\alpha+\beta} - 1 > (t-1)^\alpha$ for $t \in (1, \infty)$. By combining the boundary condition, we have that $\|y_0 - x_0\|_p^{\alpha/p} = \|\lambda_0 x_0 - x_0\|_p^{\alpha/p} = (\lambda_0 - 1)^\alpha \|x_0\|_p^{\alpha/p} < (\lambda_0^{\alpha+\beta} - 1)\|x_0\|_p^{(\alpha+\beta)/p} \|x_0\|_p^{-\beta/p} = \|y_0\|_p^{(\alpha+\beta)/p} \|x_0\|_p^{-\beta/p} - \|x_0\|_p^{\alpha/p}$, which contradicts the boundary condition given by Theorem 7.2. Thus, the conclusion follows and the proof is complete. $\qquad\square$

Theorem 7.3 *Let U be a bounded open p-convex subset of a p-(semi)norm space $(E, \| \cdot \|_p)$ $(0 < p \le 1)$ the zero $0 \in U$. Assume that $F : \overline{U} \to 2^E$ is a 1-set contractive and upper semi-continuous mapping with nonempty closed p-convex values satisfying condition (H) or $(H1)$ above. In addition, there exist $\alpha > 1$, $\beta \ge 0$ such that, for each $x \in \partial\overline{U}$, we have that for any $y \in F(x)$, $\|y + x\|_p^{(\alpha+\beta)/p} \le \|y\|_p^{\alpha/p} \|x\|_p^{\beta/p} + \|x\|_p^{(\alpha+\beta)/p}$. Then F has at least one fixed point.*

Proof We prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 7.1 does not hold. If we assume F has no fixed point, by the boundary condition of Theorem 7.1, there exist $x_0 \in \partial\overline{U}$, $y_0 \in F(x_0)$, and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t + 1)^{\alpha+\beta} - t^\alpha - 1$ for $t \ge 1$. We then can show that f is a strictly increasing function for $t \in [1, \infty)$, thus we have $t^\alpha + 1 < (t+1)^{\alpha+\beta}$ for $t \in (1, \infty)$. By the boundary condition given in Theorem 7.3, we have that

$$\|y_0 + x_0\|_p^{(\alpha+\beta)/p} = (\lambda_0 + 1)^{\alpha+\beta} \|x_0\|_p^{(\alpha+\beta)/p} > \left(\lambda_0^\alpha + 1\right) \|x_0\|_p^{(\alpha+\beta)/p} = \|y_0\|_p^{\alpha/p} \|x_0\|_p^{\beta/p} + \|x_0\|_p^{\alpha/p},$$

which contradicts the boundary condition given by Theorem 7.3. Thus, the conclusion follows and the proof is complete. $\qquad\square$

Theorem 7.4 *Let U be a bounded open p-convex subset of a p-(semi)norm space $(E, \| \cdot \|_p)$ $(0 < p \le 1)$ the zero $0 \in U$. Assume that $F : \overline{U} \to 2^E$ is a 1-set contractive and upper semi-continuous mapping with nonempty closed p-convex values satisfying condition (H) or $(H1)$ above. In addition, there exist $\alpha > 1$, $\beta \ge 0$ (or alternatively $\alpha > 1$, $\beta \ge 0$) such that, for each*

$x \in \partial\overline{U}$, we have that for any $y \in F(x)$, $\|y - x\|_p^{\alpha/p}\|x\|_p^{\beta/p} \geq \|y\|_p^{\alpha/p}\|y + x\|_p^{\beta/p} - \|x\|_p^{(\alpha+\beta)/p}$. Then F has at least one fixed point.

Proof The same as above, we prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 7.1 does not hold. If we assume F has no fixed point, by the boundary condition of Theorem 7.1, there exist $x_0 \in \partial\overline{U}$, $y_0 \in F(x_0)$, and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t - 1)^\alpha - t^\alpha(t - 1)^\beta + 1$ for $t \geq 1$. We then can show that f is a strictly decreasing function for $t \in [1, \infty)$, thus we have $(t - 1)^\alpha < t^\alpha(t + 1)^\beta - 1$ for $t \in (1, \infty)$.

By the boundary condition given in Theorem 7.4, we have that

$$\|y_0 - x_0\|_p^{\alpha/p}\|x_0\|_p^{\beta/p} = (\lambda_0 - 1)^\alpha\|x_0\|_p^{(\alpha+\beta)/p} < \left(\lambda_0^\alpha(\lambda_0 + 1)^\beta - 1\right)\|x_0\|_p^{(\alpha+\beta)/p}$$
$$= \|y_0\|_p^{\alpha/p}\|y_0 + x_0\|_p^{\beta/p} - \|x_0\|_p^{(\alpha+\beta)/p},$$

which contradicts the boundary condition given by Theorem 7.4. Thus, the conclusion follows and the proof is complete. □

Theorem 7.5 *Let U be a bounded open p-convex subset of a p-(semi)norm space $(E, \|\cdot\|_p)$ $(0 < p \leq 1)$ the zero $0 \in U$. Assume that $F : \overline{U} \to 2^E$ is a 1-set contractive and upper semicontinuous mapping with nonempty closed p-convex values satisfying condition (H) or $(H1)$ above. In addition, there exist $\alpha > 1$, $\beta \geq 0$, we have that for any $y \in F(x)$, $\|y + x\|_p^{(\alpha+\beta)/p} \leq \|y - x\|_p^{\alpha/p}\|x\|_p^{\beta/p} + \|y\|_p^{\beta/p}\|x\|^{\alpha/p}$. Then F has at least one fixed point.*

Proof The same as above, we prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 7.1 does not hold. If we assume F has no fixed point, by the boundary condition of Theorem 7.1, there exist $x_0 \in \partial\overline{U}$, $y_0 \in F(x_0)$, and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t + 1)^{\alpha+\beta} - (t - 1)^\alpha - t^\beta$ for $t \geq 1$. We then can show that f is a strictly increasing function for $t \in [1, \infty)$, thus we have $(t + 1)^{\alpha+\beta} > (t - 1)^\alpha + t^\beta$ for $t \in (1, \infty)$.

By the boundary condition given in Theorem 7.5, we have that $\|y_0 + x_0\|_p^{(\alpha+\beta)/p} = (\lambda_0 + 1)^{\alpha+\beta}\|x_0\|_p^{(\alpha+\beta)/p} > ((\lambda_0 - 1)^\alpha + \lambda_0^\beta)\|x_0\|_p^{(\alpha+\beta)/p} = \|\lambda_0 x_0 - x_0\|_p^{\alpha/p}\|x_0\|_p^{\beta/p} + \|\lambda_0 x_0\|_p^{\beta/p}\|x_0\|_p^{\alpha/p} = \|y_0 - x_0\|_p^{\beta/p}\|x_0\|_p^{\alpha/p} + \|y_0\|_p^{\beta/p}\|x_9\|^{\alpha/p}$, which implies that

$$\|y_0 + x_0\|_p^{(\alpha+\beta)/p} > \|y_0 - x_0\|_p^{\beta/p}\|x_0\|_p^{\alpha/p} + \|y_0\|_p^{\beta/p}\|x_9\|^{\alpha/p},$$

this contradicts the boundary condition given by Theorem 7.5. Thus, the conclusion follows and the proof is complete. □

As an application of Theorems 7.1, by testing the Leray–Schauder boundary condition, we have the following conclusion for each special case, and thus we omit their proofs in detail here.

Corollary 7.1 *Let U be a bounded open p-convex subset of a p-(semi)norm space $(E, \|\cdot\|_p)$ $(0 < p \leq 1)$ the zero $0 \in U$. Assume that $F : \overline{U} \to 2^E$ is a 1-set contractive and upper semicontinuous mapping with nonempty closed p-convex values satisfying condition (H) or $(H1)$*

above. Then F has at least one fixed point if one of the following conditions holds for $x \in \partial \overline{U}$ and $y \in F(x)$:

 (i) $\|y\|_p \leq \|x\|_p$,

 (ii) $\|y\|_p \leq \|y - x\|_p$,

 (iii) $\|y + x\|_p \leq \|y\|_p$,

 (iv) $\|y + x\|_p \leq \|x\|_p$,

 (v) $\|y + x\|_p \leq \|y - x\|_p$,

 (vi) $\|y\|_p \cdot \|y + x\|_p \leq \|x\|_p^2$,

 (vii) $\|y\|_p \cdot \|y + x\|_p \leq \|y - x\|_p \cdot \|x\|_p$.

If the p-seminorm space E is a uniformly convex Banach space $(E, \|\cdot\|)$ (for p-norm space with $p = 1$), then we have the following general existence result (which actually is true for nonexpansive set-valued mappings).

Theorem 7.6 *Let U be a bounded open convex subset of a uniformly convex Banach space $(E, \|\cdot\|)$ (with $p = 1$) with zero $0 \in U$. Assume that $F : \overline{U} \to E$ is a semicontractive and continuous single-valued mapping with nonempty values. In addition, for any $x \in \partial \overline{U}$, we have $\lambda x \neq F(x)$ for any $\lambda > 1$ (i.e., the Leray–Schauder boundary condition). Then F has at least one fixed point.*

Proof By the assumption that F is a semicontractive and continuous single-valued mapping with nonempty values, it follows by Lemma 3.2 in p. 338 of Petryshyn [95] that f is a 1-set contractive single-valued mapping. Moreover, by the assumption that E is uniformly convex Banach, indeed $(I - F)$ is closed at zero, i.e., F is semiclosed (see Browder [17] or Goebel and Kirk [43]). Thus all assumptions of Theorem 7.1 are satisfied with the (H1) condition. The conclusion follows by Theorem 7.1, and the proof is complete.   □

Like Lemma 7.1 shows, a single-valued nonexpansive mapping defined in a uniformly convex Banach space (see also Theorem 7.6) satisfies the (H1) condition. Actually, the nonexpansive set-valued mappings defined on a special class of Banach spaces with the so-called Opial condition do not only satisfy condition (H1), but also belong to the classes of semiclosed 1-set contractive mappings as shown below.

The notion of the so-called Opial condition was first given by Opial [80], and it says that a Banach space X is said to satisfy Opial's condition if $\liminf_{n\to\infty} \|w_n - w\| < \liminf_{n\to\infty} \|w_n - p\|$ whenever (w_n) is a sequence in X weakly convergent to w and $p \neq w$. We know that Opial's condition plays an important role in the fixed point theory, e.g., see Lami Dozo [66], Goebel and Kirk [44], Xu [127], and the references therein. The following result shows that nonexpansive set-valued mappings in Banach spaces with Opial's condition (see Lami Dozo [66]) satisfy condition (H1).

Lemma 7.2 *Let C be a convex weakly compact subset of a Banach space X which satisfies Opial's condition. Let $T : C \to K(C)$ be a nonexpansive set-valued mapping with nonempty compact values. Then the graph of $(I - T)$ is closed in $(X, \sigma(X, X^*) \times (X, \|\cdot\|))$, thus T satisfies the (H1) condition, where I denotes the identity on X, $\sigma(X, X^*)$ the weak topology, and $\|\cdot\|$ the norm (or strong) topology.*

Proof By following Theorem 3.1 of Lami Dozo [66], it follows that the mapping T is demiclosed, thus T satisfies the (H1) condition. The proof is complete.   □

As an application of Lemma 7.2, we have the following results for nonexpansive mappings.

Theorem 7.7 *Let C be a nonempty convex weakly compact subset of a Banach space X which satisfies Opial's condition and $0 \in \text{int } C$. Let $T : C \to K(X)$ be a nonexpansive set-valued mapping with nonempty compact convex values. In addition, for any $x \in \partial \overline{C}$, we have $\lambda x \not\in F(x)$ for any $\lambda > 1$ (i.e., the Leray–Schauder boundary condition). Then F has at least one fixed point.*

Proof As T is nonexpansive, it is 1-set contractive. By Lemma 7.1, it is then semicontractive and continuous. Then the (H1) condition of Theorem 7.1 is satisfied. The conclusion follows by Theorem 7.1, and the proof is complete.　　　　　　　　　　　　　　　□

Before the end of this section, by considering the p-seminorm space $(E, \| \cdot \|)$ is a seminorm space with $p = 1$, the following result is a special case of corresponding results from Theorem 7.2 to Theorem 7.5, and thus we omit its proof.

Corollary 7.2 *Let U be a bounded open convex subset of a norm space $(E, \| \cdot \|)$. Assume that $F : \overline{U} \to 2^E$ is a 1-set contractive and upper semicontinuous mapping with nonempty closed p-convex values satisfying condition (H) or $(H1)$ above. Then F has at least one fixed point if there exist $\alpha > 1$, $\beta \geq 0$ such that any one of the following conditions is satisfied:*
  (i) For each $x \in \partial \overline{U}$ and any $y \in F(x)$, $\|y - x\|^{\alpha} \geq \|y\|^{(\alpha+\beta)} \|x\|^{-\beta} - \|x\|^{\alpha}$;
  (ii) For each $x \in \partial \overline{U}$ and any $y \in F(x)$, $\|y + x\|^{(\alpha+\beta)} \leq \|y\|^{\alpha} \|x\|^{\beta} + \|x\|^{(\alpha+\beta)}$;
  (iii) For each $x \in \partial \overline{U}$ and any $y \in F(x)$, $\|y - x\|^{\alpha} \|x\|^{\beta} \geq \|y\|^{\alpha} \|y + x\|^{\beta} - \|x\|^{(\alpha+\beta)}$;
  (iv) For each $x \in \partial \overline{U}$ and any $y \in F(x)$, $\|y + x\|^{(\alpha+\beta)} \leq \|y - x\|^{\alpha} \|x\|^{\beta} + \|y\|^{\beta} \|x\|^{\alpha}$.

Remark 7.2 As discussed in Lemma 7.1 and the proof of Theorem 7.6, when the p-vector space is a uniformly convex Banach space, semicontractive or nonexpansive mappings automatically satisfy condition (H) or (H1). Moreover, our results from Theorem 7.1 to Theorem 7.6, Corollary 7.1, and Corollary 7.2 also improve or unify corresponding results given by Browder [17], Li [68], Li et al. [69], Goebel and Kirk [43], Petryshyn [94, 95], Reich [100], Tan and Yuan [118], Xu [126], Xu [129], Xu et al. [130], and the results from the references therein by extending the nonself mappings to the classes of 1-set contractive set-valued mappings in p-seminorm spaces with $p \in (0.1]$ (including the normed space or Banach space when $p = 1$ and for p-seminorm spaces).

Before the end of this paper, we would like to share with readers that the main goal of this paper is to develop some new results and tools in the nature way for the category of nonlinear analysis for 1-set contractive mappings under the general framework of locally p-convex spaces (where $(0 < p \leq 1)$), and we expect that they become useful tools for the study on optimization, nonlinear programming, variational inequality, complementarity, game theory, mathematical economics, and other related social science areas. In particular, we first establish one best approximation, acting as a tool to establish the principle of nonlinear alternative, which then allows us to give general principle of nonlinear alternative for 1-set contractive mappings.

As mentioned at the beginning of this paper, we do expect that nonlinear results and principles of the best approximation theorem established in this paper would play a very

important role for the nonlinear analysis under the general framework of locally p-convex spaces for $(0 < p \leq 1)$, as shown by those results given in Sects. 4, 5, 6 and 7 above for the fixed point theorems of nonself mappings, the principle of nonlinear alternative, Rothe type, Leray–Schauder alternative which not only include corresponding results in the existing literature as special cases, but also would be important tools for the study of optimization, nonlinear programming, variational inequality, complementarity, game theory, mathematical economics, and related topics and areas forthcoming; and in Sect. 7, by considering p-seminorm spaces for $p \in (0, 1]$, as an application of best approximation, we unified and improved the corresponding results in the existing literature under the general framework of locally p-convex spaces.

But one thing we like to point out that the results mainly are estabished for set-valued mappings with non-empty closed p-convex values for $0 < p \leq 1$, not much attention given to sing-valued mappings. As suggested by the title of this paper, in Sects. 4, 5, 6 and 7 of this paper, we focus on the development of results in nonlinear analysis mainly related to fixed points, the best approximation, and the general principle of nonlinear alternative and related boundary conditions under the framework of locally p-convex spaces for $0 < p \leq 1$, for nonlinear set-valued mappings, which are upper semicontinuous, 1-set contractive with non-empty closed p-convex values. On the other hand, as shown by Lemma 2.4 and the discussion based on the conclusion of Theorem 4.3 for set-valued mappings, it seems that the set-valued nonlinear mappings with closed p-convex values are very strong assumptions for $0 < p < 1$, which maybe the major reason to result in trivial conclusions for the existence of fixed points, alternative principle and related approximation results as given in this paper from Sects. 4, 5, 6 and 7, thus we do expect that the most interesting results in nonlinear analysis for locally p-convex space would be given for single-valued (nonlinear) mappings instead of set-valued mappings with closed p-convex values.

On the other hand, based on the framework established in this paper (though focus on set-valued mappings), the nonlinear analysis for single-valued mappings actually can be developed by using Theorem 4.4 (instead of Theorem 4.2) as a starting tool, then we can obtain the similar result of Theorem 4.5 for single-valued mappings which are continuous condensing in Sect. 4; and then it can help us to establish the corresponding results similar to Theorems 5.1, 5.2, 5.3, 5.4, 5.5 and 5.6 for single-valued mappings under the locally p-convex spaces for $0 < p \leq 1$. In addition, the similar results of Theorems 6.1, 6.2, 6.3, 6.4, 6.5 and 6.6; and those results related to Theorems 7.1, 7.2, 7.3, 7.4, 7.5, 7.6 and 7.7 for single-valued mappings can be obtained, too under the framework of locally p-convex spaces for $0 < p \leq 1$. Though not much results in details for single-valued mappings are given here, but we do wish to share with readers that how important they are for the development of nonlinear analysis based on singe-valued mappings in p-vector spaces for $0 < p \leq 1$ in general.

Based on the framework for some key results in nonlinear analysis obtained for set-valued mappings with closed p-convex values in this paper, we conclude that the development of nonlinear analysis for singe-valued mappings in locally p-convex spaces for $0 < p \leq 1$ seem even more important, and they can be also developed by the approach ane method established in this paper.

Availability of data and materials
Not applicable.

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The authors declare no competing interests.

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