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RESEARCH

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Convergence theorems for total asymptotically nonexpansive mappings in CAT(κ) spaces

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Abstract

The purpose of this paper is to study the convergence theorems in CAT(κ) spaces with $k > 0$ for *total asymptotically nonexpansive mappings* which are essentially wider than nonexpansive mappings, asymptotically nonexpansive mapping, and asymptotically nonexpansive mappings in the intermediate sense. Our results generalize, unify, and improve several comparable results in the existing literature.

MSC: 47J05; 47H09; 49J25

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1 Introduction and preliminaries

Let C be a nonempty subset of a metric space (X, d).

Recall that a mapping $T : C \to X$ is said to be:

(i) *nonexpansive* if $d(Tx, Ty) \le d(x, y)$ for all $x, y \in C$;

(ii) *asymptotically nonexpansive* [1] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $d(T^n x, T^n y) \le k_n d(x, y)$ for all $x, y \in C$ and $n \in \mathcal{N}$, where $\mathcal{N}$ denotes the set of positive integers. The class of asymptotically nonexpansive mappings includes a class of nonexpansive mappings as a proper subclass.

(iii) In 1993, Bruck, Kuczumow, and Reich [2] introduced the concept of asymptotically nonexpansive mapping in the intermediate sense. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive in the intermediate sense* if T is uniformly continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left\{ d\left(T^n x, T^n y\right) - d(x, y) \right\} \le 0. \tag{1.1}$$

It is easy to know that the class of asymptotically nonexpansive mappings in the intermediate sense is more general than the class of asymptotically nonexpansive mappings.

Definition 1.1 ([3]) A mapping $T : C \to C$ is said to be $(\{\mu_n\}, \{\nu_n\}, \zeta)$-*total asymptotically nonexpansive* if there exist nonnegative sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\zeta : [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ such that

$$d\bigl(T^n x, T^n y\bigr) \le d(x, y) + \nu_n \zeta\bigl(d(x, y)\bigr) + \mu_n, \quad \forall n \ge 1, x, y \in C. \tag{1.2}$$

The concept of total asymptotically nonexpansive mappings is more general than that of asymptotically nonexpansive mappings in the intermediate sense. In fact, if $T : C \to C$ is an asymptotically nonexpansive mapping in the intermediate sense, denote by $\mu_n = \max\{\sup_{x,y \in C}(d(T^n x, T^n y) - d(x, y)), 0\}$. Then $\mu_n \ge 0$, $\lim_{n\to\infty} \mu_n = 0$, and

$$d\bigl(T^n x, T^n y\bigr) \le d(x, y) + \mu_n, \quad \forall x, y, \in C, n \ge 1. \tag{1.3}$$

Taking $\{\nu_n = 0\}$, $\zeta = t$, $t \ge 0$, then (1.3) can be written as

$$d\bigl(T^n x, T^n y\bigr) \le d(x, y) + \nu_n \zeta\bigl(d(x, y)\bigr) + \mu_n, \quad \forall n \ge 1, x, y \in C,$$

i.e., T is a total asymptotically nonexpansive mapping.

Definition 1.2 A mapping $T : C \to C$ is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$d\bigl(T^n x, T^n y\bigr) \le L d(x, y) \quad \forall x, y, \in C \text{ and } \forall n \ge 1.$$

In recent years, CAT(0) spaces (the precise definition of a CAT(0) space is given below) have attracted the attention of many authors because they have played a very important role in different aspects of geometry [4]. Kirk [5, 6] showed that a nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space has a fixed point.

In 2012, Chang et al. [7] studied the demiclosedness principle and Δ-convergence theorems for total asymptotically nonexpansive mappings in the setting of CAT(0) spaces. Since then the convergence of several iteration procedures for this type of mappings has been rapidly developed, and many of articles have appeared (see, e.g., [8–17]). In 2013, under some suitable assumptions, Karapinar et al. [9] obtained the demiclosedness principle, fixed point theorems, and convergence theorems for the following iteration:

Let C be a nonempty closed convex subset of a CAT(0) space X and $T : C \to C$ be a total asymptotically nonexpansive mapping. Given $x_1 \in C$, let $\{x_n\} \subset C$ be defined by

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n T^n\bigl((1 - \beta_n)x_{\oplus}\beta_n T^n(x_n)\bigr), \quad n \in \mathcal{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0; 1]$.

It is well known that any CAT(κ) space is a CAT(κ_1) space for $\kappa_1 \ge \kappa$. Thus, all results for CAT(0) spaces immediately apply to any CAT(κ) space with $\kappa \le 0$.

Very recently, Panyanak [10] obtained the demiclosedness principle, fixed point theorems, and convergence theorems for total asymptotically nonexpansive mappings on CAT(κ) space with $\kappa > 0$, which generalize the results of Chang et al. [7], Tang et al. [8], Karapinar et al. [9].

Motivated by the work going on in this direction, in this paper we aim to study the strong convergence of a sequence generated by an infinite family of total asymptotically nonexpansive mappings in CAT(κ) spaces with $\kappa > 0$. Our results are new, they extend and improve the corresponding results of Chang et al. [7], Tang et al. [8], Karapinar et al. [9], Panyanak [10], Hea et al. [18], and many others.

2 Preliminaries

In this section, we first recall some definitions, notations, and conclusions that will be needed in our paper.

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a mapping c from a closed interval $[0,l] \subset \mathscr{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t);c(t')) = |t - t'|$ for all $t,t' \in [0;l]$. In particular, c is an isometry and $d(x,y) = l$. The image $c([0,l])$ of c is called a geodesic segment joining x and y. When it is unique, this geodesic segment is denoted by $[x,y]$. This means that $z \in [x,y]$ if and only if there exists $\alpha \in [0;1]$ such that

$$d(x,z) = (1-\alpha)d(x,y), \quad \text{and} \quad d(y,z) = \alpha d(x;y).$$

In this case, we write $z = \alpha x \oplus (1-\alpha)y$.

A metric space (X,d) is said to be a *geodesic space (D-geodesic space)* if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be uniquely geodesic (D-uniquely geodesic) if there is exactly one geodesic joining x and y for each $x,y \in X$ (for $x,y \in X$ with $d(x,y) < D$). A subset C of X is said to be convex if C includes every geodesic segment joining any two of its points.

Now we introduce the concept of model spaces M_κ^n. For more details on these spaces, the reader is referred to [4, 14]. Let $n \in \mathscr{N}$. We denote by E^n the metric space $\mathscr{R}^n$ endowed with the usual Euclidean distance. We denote by $(\cdot|\cdot)$ the Euclidean scalar product in $\mathscr{R}^n$, that is,

$$(x|y) = x_1y_1 + \cdots + x_ny_n \quad \text{where } x = (x_1,\dots x_n), y = (y_1,\dots y_n).$$

Let $\mathscr{S}^n$ denote the n-dimensional sphere defined by

$$\mathscr{S}^n = \left\{x = (x_1,\dots,x_{n+1}) \in \mathscr{R}^{n+1} : (x|x) = 1\right\}$$

with metric

$$d_{\mathscr{S}^n}(x,y) = \arccos(x|y), \quad x,y \in \mathscr{S}^n.$$

Let $E^{n;1}$ denote the vector space $\mathscr{R}^{n+1}$ endowed with the symmetric bilinear form that associates to vectors $u = (u_1,\dots,u_{n+1})$ and $v = (v_1,\dots,v_{n+1})$, and the real number $\langle u|v\rangle$ is defined by

$$\langle u|v\rangle = u - u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_iv_i.$$

Let $\mathscr{H}^n$ denote the hyperbolic n-space defined by

$$\mathscr{H}^n = \left\{u = (u_1,\dots,u_{n+1)} \in E^{n;1} : \langle u|u\rangle = -1, u_{n+1} > 0\right\}$$

with metric $d_{\mathscr{H}^n}$ such that

$$\cosh d_{\mathscr{H}^n}(x,y) = -\langle x|y\rangle, \quad x,y \in \mathscr{H}^n.$$

Definition 2.1 Given $\kappa \in \mathscr{R}$, we denote by M_κ^n the following metric spaces:

(i) if $\kappa = 0$, then M_0^n is the Euclidean space E^n;

(ii) if $\kappa > 0$, then M_κ^n is obtained from the spherical space $\mathscr{S}^n$ by multiplying the distance function by the constant $\frac{1}{\sqrt{\kappa}}$;

(iii) if $\kappa < 0$, then M_κ^n is obtained from the hyperbolic space $\mathscr{H}^n$ by multiplying the distance function by the constant $\frac{1}{\sqrt{-\kappa}}$.

A *geodesic triangle* $\Delta(x,y,z)$ in a geodesic space (X,d) consists of three points x, y, z in X (the vertices of Δ) and three geodesic segments between each pair of vertices (the edges of Δ). A *comparison triangle* for a geodesic triangle $\Delta(x;y;z)$ in (X,d) is a triangle $\bar{\Delta}(\bar{x},\bar{y},\bar{z})$ in M_κ^2 such that

$$d(x;y) = d_{M_\kappa^2}(\bar{x},\bar{y}), \qquad d(y,z) = d_{M_\kappa^2}(\bar{y},\bar{z}), \qquad d(z,x) = d_{M_\kappa^2}(\bar{z},\bar{x}).$$

If $\kappa < 0$, then such a comparison triangle always exists in M_κ^2. If $\kappa > 0$, then such a triangle exists whenever $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$, where $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$.

A point $\bar{p} \in [\bar{x},\bar{y}]$ is called a *comparison point* for $p \in [x,y]$ if $d(x,p) = d_{M_\kappa^2}(\bar{x},\bar{p})$.

A geodesic triangle $\Delta(x,y,z)$ in X is said to satisfy the CAT(κ) inequality if for any $p,q \in \Delta(x,y,z)$ and for their comparison points $\bar{p},\bar{q} \in \Delta(\bar{x},\bar{y},\bar{z})$, one has

$$d(p,q) \leq d_{M_\kappa^2}(\bar{p},\bar{q}).$$

Definition 2.2 A metric space (X,d) is called a CAT(0) space if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(κ) inequality.

If $\kappa > 0$, then X is called a CAT(κ) space if X is D_κ-geodesic and any geodesic triangle $\Delta(x,y,z)$ in X with $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$ satisfies the CAT(κ) inequality.

Definition 2.3 A geodesic space (X,d) is said to be R-convex with $R \in (0,2]$ (see [16]) if for any three points $x,y,z \in X$, we have

$$d^2\bigl(x,(1-\alpha)y \oplus \alpha z\bigr) \leq (1-\alpha)d^2(x,y) + \alpha d^2(x,z) - \frac{R}{2}\alpha(1-\alpha)d^2(y,z). \tag{2.1}$$

Notice that if (X,d) is a geodesic space, then the following statements are equivalent:

(i) (X,d) is a CAT(0) space;

(ii) (X,d) is R-convex with $R = 2$, i.e., it satisfies the following inequality:

$$d^2\bigl(x,(1-\alpha)y \oplus \alpha z\bigr) \leq (1-\alpha)d^2(x,y) + \alpha d^2(x,z) - \alpha(1-\alpha)d^2(y;z) \tag{2.2}$$

for all $\alpha \in (0,1]$ and $x,y,z \in X$.

The following lemma is a consequence of Proposition 3.1 in [19].

Lemma 2.4 *Let $\kappa > 0$ and (X,d) be a CAT(κ) space with* $\operatorname{diam}(X) \leq \frac{\frac{\pi}{2}-\epsilon}{\sqrt{\kappa}}$ *for some $\epsilon \in (0, \frac{\pi}{2})$. Then (X,d) is R-convex with $R = (\pi - 2\epsilon)\tan(\epsilon)$.*

Lemma 2.5 ([20, page 176]) *Let $\kappa > 0$ and (X,d) be a complete CAT(κ) space with* $\operatorname{diam}(X) \leq \frac{\frac{\pi}{2}-\epsilon}{\sqrt{\kappa}}$ *for some $\epsilon \in (0, \frac{\pi}{2})$. Then*

$$d\bigl((1-\alpha)x \oplus \alpha y, z\bigr) \leq (1-\alpha)d(x,z) + \alpha d(y,z) \tag{2.3}$$

for all $x, y, z \in X$ and $\alpha \in [0,1]$.

We now collect some elementary facts about CAT(κ) spaces, $\kappa > 0$.

Let $\{x_n\}$ be a bounded sequence in a CAT(κ) space (X,d). For $x \in X$, we set

$$r\bigl(x, \{x_n\}\bigr) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\bigl(\{x_n\}\bigr) = \inf\bigl\{r\bigl(x, \{x_n\}\bigr) : x \in X\bigr\}.$$

The *asymptotic radius* $r(\{x_n\})$ *of* $\{x_n\}$ is given by

$$r\bigl(\{x_n\}\bigr) = \inf\bigl\{r\bigl(x, \{x_n\}\bigr) : x \in X\bigr\}. \tag{2.4}$$

The *asymptotic center* $A(\{x_n\})$ *of* $\{x_n\}$ is the set

$$A\bigl(\{x_n\}\bigr) = \bigl\{x \in X : r\bigl(x, \{x_n\}\bigr) = r\bigl(\{x_n\}\bigr)\bigr\}. \tag{2.5}$$

It is known from Proposition 4.1 of [21] that in a CAT(κ) space X with $\operatorname{diam}(X) < \frac{\pi}{2\sqrt{\kappa}}$, $A(\{x_n\})$ consists of exactly one point.

We now give the concept of Δ-convergence and collect some of its basic properties.

Definition 2.6 ([22, 23]) A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_{n\to\infty} x_n = x$, and x is called the Δ-limit of $\{x_n\}$.

Lemma 2.7 *Let (X,d) be a complete CAT(κ) space with $\kappa > 0$ and* $\operatorname{diam}(X) \leq \frac{\frac{\pi}{2}-\epsilon}{\sqrt{\kappa}}$ *for some $\epsilon \in (0, \frac{\pi}{2})$. Then the following statements hold:*

(i) [17, *Corollary* 4.4] *Every sequence in X has a Δ-convergence subsequence;*

(ii) [17, *Proposition* 4.5] *If $\{x_n\} \subset X$ and $\Delta - \lim x_n = x$, then*

$$x \in \bigcap_{n=1}^{\infty} \overline{\operatorname{conv}}\{x_n, x_{n+1}, \dots\},$$

where $\overline{\operatorname{conv}}(A) = \bigcap\{B : B \supseteq AandBisclosedandconvex\}$.

By the uniqueness of asymptotic centers, we can obtain the following lemma (cf. [24, Lemma 2.8]).

Lemma 2.8 *Let (X,d) be a complete CAT(κ) space with $\kappa > 0$ and $\operatorname{diam}(X) \leq \frac{\frac{\pi}{2}-\epsilon}{\sqrt{\kappa}}$ for some $\epsilon \in (0,\frac{\pi}{2})$. If $\{x_n\}$ is a sequence in X with $A(\{x_n\}) = \{x\}$ and if $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n,u)\}$ converges, then $x = u$.*

In the sequel, we use $F(T)$ to denote the fixed point set of a mapping T.

Definition 2.9 ([25]) (1) A triple (X,d,W) is called a hyperbolic space if (X,d) is a metric space and $W : X \times X \times [0,1] \to X$ is a mapping such that $\forall x,y,z,w \in X$, $\alpha,\beta, \in [0,1]$, the following hold:

(W1) $d(z, W(x,y,\alpha) \leq \alpha d(z,x) + (1-\alpha)d(z,y)$;

(W2) $d(W(x,y,\alpha), W(x,y,\beta) = |\alpha-\beta| d(x,y)$;

(W3) $W(x,y,\alpha) = W(y,x,1-\alpha)$;

(W4) $d(W(x,z,\alpha), W(y,w,\alpha) \leq \alpha d(x,y) + (1-\alpha)d(z,w)$.

(2) A hyperbolic space (X,d,W) is called *uniformly convex* if for any $r > 0$ and $\epsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that, for all $x,y,z \in X$,

$$\left.\begin{aligned} &d(x,z) \leq r \\ &d(y,z) \leq r \\ &d(x,y) \geq \epsilon \cdot r \end{aligned}\right\} \Rightarrow d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \leq (1-\delta)r. \tag{2.6}$$

A mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$ providing such $\delta := \eta(r,\epsilon)$ for given $r > 0$ and $\epsilon \in (0,2]$ is called a *modulus of uniform convexity*.

Lemma 2.10 ([25]) *Let (X,d,W) be a uniformly convex hyperbolic space with modulus of uniform convexity η. For any $r > 0$, $\epsilon \in (0,2]$, $\lambda \in [0,1]$, and $x,y,z \in X$,*

$$\left.\begin{aligned} &d(x,z) \leq r \\ &d(y,z) \leq r \\ &d(x,y) \geq \epsilon \cdot r \end{aligned}\right\} \Rightarrow d\big((1-\lambda)x \oplus \lambda y, z\big) \leq \big(1 - 2\lambda(1-\lambda)\eta(r,\epsilon)\big)r. \tag{2.7}$$

Proposition 2.11 *Let (X,d) be a complete uniformly convex CAT(κ) space $\kappa > 0$ with modulus of uniform convexity η and $\operatorname{diam}(X) \leq \frac{\frac{\pi}{2}-\epsilon}{\sqrt{\kappa}}$ for some $\epsilon \in (0,\frac{\pi}{2})$. Let $x \in X$ be a given point and $\{t_n\}$ be a sequence in $[b,c]$ with $b,c \in (0,1)$ and $0 < b(1-c) \leq \frac{1}{2}$. Let $\{x_n\}$ and $\{y_n\}$ be any sequences in X such that*

$$\limsup_{n\to\infty} d(x_n,x) \leq r, \qquad \limsup_{n\to\infty} d(y_n,x) \leq r \quad \textit{and}$$
$$\lim_{n\to\infty} d\big((1-t_n)x_n \oplus t_n y_n\big),x) = r \quad \textit{for some } r \geq 0.$$

Then

$$\lim_{n\to\infty} d(x_n,y_n) = 0. \tag{2.8}$$

Proof By the assumption that (X,d) is a complete CAT(κ) space $\kappa > 0$ and $\operatorname{diam}(X) \leq \frac{\frac{\pi}{2}-\epsilon}{\sqrt{\kappa}}$ for some $\epsilon \in (0,\frac{\pi}{2})$, it follows from Lemma 2.5 that for all $x,y,z \in X$ and $\alpha \in [0,1]$

$$d\big((1-\alpha)x \oplus \alpha y, z\big) \leq (1-\alpha)d(x,z) + \alpha d(y,z).$$

Letting $W(x,y,\alpha) := (1-\alpha)x \oplus \alpha y$. It is easy to prove that $W(x,y,\alpha)$ satisfies conditions $(W1)-(W4)$. Hence (X,d,W) is a hyperbolic space. Again, since (X,d) is uniformly convex with modulus of uniform convexity η, this implies that (X,d,W) is a uniformly convex hyperbolic space with modulus of uniform convexity η.

Now we consider two cases.

1. If $r = 0$, then the conclusion of Proposition 2.11 is obvious.

2. The case of $r > 0$. If it is not the case that $d(x_n,y_n) \to 0$ as $n \to \infty$, then there are subsequences (denoted by $\{x_n\}$ and $\{y_n\}$ again) such that

$$\inf_n d(x_n,y_n) > 0. \tag{2.9}$$

Choose $\epsilon \in (0,1]$ such that

$$d(x_n,y_n) \geq \epsilon(r+1) > 0, \quad \forall n \in \mathcal{N}. \tag{2.10}$$

Since $0 < b(1-c) < \frac{1}{2}$ and $0 < \eta(r,\epsilon) \leq 1$, $0 < 2b(1-c)\eta(r,\epsilon) \leq 1$. This implies $0 \leq 1-2b(1-c)\eta(r,\epsilon) < 1$. Choose $R \in (r,r+1)$ such that

$$(1-2b(1-c)\eta(r,\epsilon)R < r. \tag{2.11}$$

Since

$$\limsup_n d(x_n,x) \leq r, \qquad \limsup_n d(y_n,x) \leq r, \quad r < R, \tag{2.12}$$

there are further subsequences again denoted by $\{x_n\}$ and $\{y_n\}$ such that

$$d(x_n,x) \leq R, \qquad d(y_n,x) \leq R, \qquad d(x_n,y_n) \geq \epsilon R, \quad \forall n \in \mathcal{N}. \tag{2.13}$$

Then, by Lemma 2.10 and (2.11),

$$\begin{aligned} d\big((1-t_n)x_n, t_n y_n, x\big) &\leq \big(1-2t_n(1-t_n)\eta(R,\epsilon)\big)R \\ &\leq \big(1-2b(1-c)\eta(r,\epsilon)\big)R < r \end{aligned} \tag{2.14}$$

for all $n \in \mathcal{N}$. Taking $n \to \infty$, we obtain

$$\lim_{n\to\infty} d\big((1-t_n)x_n \oplus t_n y_n, x\big) < r, \tag{2.15}$$

which contradicts the hypothesis.

The conclusion of Proposition 2.11 is proved. □

Lemma 2.12 *Let $\{a_n\}$, $\{\lambda_n\}$, and $\{c_n\}$ be the sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1+\lambda_n)a_n + c_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In addition, if there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $\lim_{n\to\infty} a_n = 0$.

3 Strong convergence theorems for total asymptotically nonexpansive mappings in CAT(κ) spaces

Lemma 3.1 ([26]) (1) *For each positive integer $n \geq 1$, the unique solutions $i(n)$ and $k(n)$ with $k(n) \geq i(n)$ to the following positive integer equation*

$$n = i(n) + \frac{(k(n)-1)k(n)}{2} \tag{3.1}$$

are as follows:

$$\begin{cases} i(n) = n - \frac{(k(n)-1)k(n)}{2}, \\ k(n) = [\frac{1}{2} + \sqrt[2]{2n - \frac{7}{4}}], \quad k(n) \geq i(n), \end{cases}$$

and $k(n) \to \infty$ (as $n \to \infty$), where $[x]$ denotes the maximal integer that is not larger than x.

(2) *For each $i \geq 1$, denote by*

$$\begin{cases} \Gamma_i := \{n \in \mathscr{N} : n = i + \frac{(k(n)-1)k(n)}{2}, k(n) \geq i\}, \quad \textit{and} \\ K_i := \{k(n) : n \in \Gamma_i, n = i + \frac{(k(n)-1)k(n)}{2}, k(n) \geq i\}, \end{cases}$$

then $k(n) + 1 = k(n+1)$, $\forall n \in \Gamma_i$.

In this section we prove some strong convergence theorems for the following iterative scheme:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T_{i(n)}^{k(n)} y_n, \quad n \geq 1, \\ y_n = (1-\beta_n)x_n \oplus \beta_n T_{i(n)}^{k(n)} x_n, \end{cases} \tag{3.2}$$

where C is a nonempty closed and convex subset of a complete CAT(κ) space X, $\kappa > 0$, for each $i \geq 1$, $T_i : C \to C$ is uniformly L_i-Lipschitzian and $(\{\nu_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive mappings defined by (1.2); and for each positive integer $n \geq 1$, $i(n)$ and $k(n)$ are the unique solutions of the positive integer equation (3.1).

Theorem 3.2 *Let (X,d) be a complete uniformly convex CAT(κ) space with $\kappa > 0$ and* $\operatorname{diam}(X) \leq \frac{\frac{\pi}{2}-\epsilon}{\sqrt{\kappa}}$ *for some $\epsilon \in (0, \frac{\pi}{2})$. Let C be a nonempty closed and convex subset of X and, for each $i \geq 1$, let $T_i : C \to C$ be uniformly L_i-Lipschitzian and $(\{\nu_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive mappings defined by (1.2) such that*

(i) $\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \nu_n^{(i)} < \infty$, $\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu_n^{(i)} < \infty$,
(ii) *there exists a constant $M_* > 0$ such that $\zeta^{(i)}(r) \leq M_* r$, $\forall r \geq 0$, $i = 1, 2, \ldots$;*
(iii) *there exist constants $a, b \in (0,1)$ with $0 < b(1-a) \leq \frac{1}{2}$ such that $\{\alpha_n\}, \{\beta_n\} \subset [a,b]$.*

If $\mathscr{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exist a mapping $T_{n_0} \in \{T_i\}_{i=1}^{\infty}$ and a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with f(0)= 0 and $f(r) > 0$ $\forall r > 0$ such that

$$f\big(d(x_n, \mathscr{F})\big) \leq d(x_n, T_{n_0} x_n), \quad \forall n \geq 1, \tag{3.3}$$

then the sequence $\{x_n\}$ defined by (3.2) converges strongly (i.e., in metric topology) to some point $p^* \in \mathscr{F}$.

Proof First we observe that for each $i \geq 1$, $T_i : C \to C$ is a $(\{\nu_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive mapping. By condition (ii), for each $n \geq 1$ and any $x, y \in C$, we have

$$d\big(T_i^n x, T_i^n y\big) \leq d(x,y) + \nu_n^{(i)} \zeta^i\big(d(x,y)\big) + \mu_n^{(i)} \leq \big(1 + \nu_n^{(i)} M_*\big) d(x,y) + \mu_n^{(i)}. \tag{3.4}$$

(I) *First we prove that the following limits exist:*

$$\lim_{n\to\infty} d(x_n, \mathscr{F}), \quad \text{and} \quad \lim_{n\to\infty} d(x_n, p) \quad \text{for each } p \in \mathscr{F}. \tag{3.5}$$

In fact, since $p \in \mathscr{F}$ and T_i, $i \geq 1$ is a total asymptotically nonexpansive mapping, it follows from (3.4) and Lemma 2.5 that

$$\begin{aligned} d(y_n, p) &= d\big((1-\beta_n)x_n \oplus \beta_n T_{i(n)}^{k(n)} x_n, p\big) \\ &\leq (1-\beta_n) d(x_n, p) + \beta_n d\big(T_{i(n)}^{k(n)} x_n, p\big) \\ &= (1-\beta_n) d(x_n, p) + \beta_n \big\{ d(x_n, p) + \nu_{k(n)}^{i(n)} \zeta^{i(n)}\big(d(x_n, p)\big) + \mu_{k(n)}^{i(n)} \big\} \\ &\leq d(x_n, p) + \nu_{k(n)}^{i(n)} \zeta^{i(n)}\big(d(x_n, p)\big) + \mu_{k(n)}^{i(n)} \\ &\leq \big(1 + \nu_{k(n)}^{i(n)} M_*\big) d(x_n, p) + \mu_{k(n)}^{i(n)} \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d\big((1-\alpha_n)x_n \oplus \alpha_n T_{i(n)}^{k(n)} y_n, p\big) \\ &\leq (1-\alpha_n) d(x_n, p) + \alpha_n d\big(T_{i(n)}^{k(n)} y_n, p\big) \\ &= (1-\alpha_n) d(x_n, p) + \alpha_n \big\{ d(y_n, p) + \nu_{k(n)}^{i(n)} \zeta^{i(n)}\big(d(y_n, p)\big) + \mu_{k(n)}^{i(n)} \big\} \\ &\leq (1-\alpha_n) d(x_n, p) + \alpha_n \big\{ \big(1 + \nu_{k(n)}^{i(n)} M_*\big) d(y_n, p) + \mu_{k(n)}^{i(n)} \big\}. \end{aligned} \tag{3.7}$$

Substituting (3.6) into (3.7) and simplifying it, we have

$$d(x_{n+1}, p) \leq (1 + \sigma_n) d(x_n, p) + \xi_n, \quad \forall n \geq 1 \text{ and } p \in \mathscr{F}, \tag{3.8}$$

and so

$$d(x_{n+1}, \mathscr{F}) \leq (1 + \sigma_n) d(x_n, \mathscr{F}) + \xi_n, \quad \forall n \geq 1, \tag{3.9}$$

where $\sigma_n = b\nu_{k(n)}^{i(n)} M_*(2 + \nu_{k(n)}^{i(n)} M_*)$, $\xi_n = b(2 + \nu_{k(n)}^{i(n)} M_*)\mu_{k(n)}^{i(n)}$. By virtue of condition (i),

$$\sum_{n=1}^{\infty} \sigma_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \xi_n < \infty. \tag{3.10}$$

By Lemma 2.12, the limits $\lim_{n\to\infty} d(x_n, \mathscr{F})$ and $\lim_{n\to\infty} d(x_n, p)$ exist for each $p \in \mathscr{F}$.

(II) *Next we prove that for each $i \geq 1$ there exists some subsequence $\{x_{m(\in \Gamma_i)}\} \subset \{x_n\}$ such that*

$$\lim_{m(\in \Gamma_i) \to \infty} d(x_m, T_i x_m) = 0, \tag{3.11}$$

where Γ_i is the set of positive integers defined by Lemma 3.1(2).

In fact, it follows from (3.5) that for each given $p \in \mathscr{F}$, the limit $\lim_{n\to\infty} d(x_n, p)$ exists. Without loss of generality, we can assume that

$$\lim_{n\to\infty} d(x_n, p) = r \geq 0. \tag{3.12}$$

From (3.6) we have

$$\limsup_{n\to\infty} d(y_n, p) \leq \lim_{n\to\infty} \left\{ \left(1 + \nu_{k(n)}^{i(n)} M_*\right) d(x_n, p) + \mu_{k(n)}^{i(n)} \right\} = r. \tag{3.13}$$

Since

$$\begin{aligned} d\left(T_{i(n)}^{k(n)} y_n, p\right) &\leq d(y_n, p) + \nu_{k(n)}^{i(n)} \zeta^{i(n)}\left(d(y_n, p)\right) + \nu_{k(n)}^{i(n)} \\ &\leq \left(1 + \nu_{k(n)}^{i(n)} M_*\right) d(y_n, p) + \mu_{k(n)}^{i(n)}, \quad \forall n \geq 1, \end{aligned}$$

from (3.13) we have

$$\limsup_{n\to\infty} d\left(T_{i(n)}^{k(n)} y_n, p\right) \leq r. \tag{3.14}$$

In addition, it follows from (3.8) that

$$\begin{aligned} d(x_{n+1}, p) &= d\left((1-\alpha_n) x_n \oplus \alpha_n T_{i(n)}^{k(n)} y_n, p\right) \\ &\leq (1 + \sigma_n) d(x_n, p) + \xi_n. \end{aligned}$$

This implies that

$$\lim_{n\to\infty} d\left((1-\alpha_n) x_n \oplus \alpha_n T_{i(n)}^{k(n)} y_n, p\right) = r. \tag{3.15}$$

From (3.12), (3.14), (3.15), and Proposition 2.11, we have

$$\lim_{n\to\infty} d\left(x_n, T_{i(n)}^{k(n)} y_n\right) = 0. \tag{3.16}$$

Since

$$\begin{aligned} d(x_n, p) &\leq d\left(x_n, T_{i(n)}^{k(n)} y_n\right) + d\left(T_{i(n)}^{k(n)} y_n, p\right) \\ &\leq d\left(x_n, T_{i(n)}^{k(n)} y_n\right) + \left\{ d(y_n, p) + \nu_{k(n)}^{i(n)} \zeta^{i(n)}\left(d(y_n, p)\right) + \mu_{k(n)}^{i(n)} \right\} \\ &\leq d\left(x_n, T_{i(n)}^{k(n)} y_n\right) + \left(1 + \nu_{k(n)}^{i(n)} M_*\right) d(y_n, p) + \mu_{k(n)}^{i(n)}. \end{aligned}$$

Taking lim inf on both sides of the above inequality, from (3.16) we have

$$\liminf_{n\to\infty} d(y_n, p) \geq r.$$

This together with (3.13) shows that

$$\lim_{n\to\infty} d(y_n, p) = r. \tag{3.17}$$

Using (3.6) we have

$$\begin{aligned} r = \lim_{n\to\infty} d(y_n, p) &= \lim_{n\to\infty} \left\{ d\left((1-\beta_n)x_n \oplus \beta_n T_{i(n)}^{k(n)} x_n, p\right)\right\} \\ &\le \lim_{n\to\infty} \left\{\left(1 + \nu_{k(n)}^{i(n)} M_*\right) d(x_n, p) + \mu_{k(n)}^{i(n)}\right\} = r. \end{aligned} \tag{3.18}$$

This implies that

$$\lim_{n\to\infty} \left\{ d\left((1-\beta_n)x_n \oplus \beta_n T_{i(n)}^{k(n)} x_n, p\right)\right\} = r. \tag{3.19}$$

Similarly, we can also prove that

$$\limsup_{n\to\infty} d\left(T_{i(n)}^{k(n)} x_n, p\right) \le \limsup_{n\to\infty} \left\{ d(x_n, p) + \nu_{k(n)}^{i(n)} \zeta^{i(n)}\left(d(x_n, p)\right) + \mu_{k(n)}^{i(n)}\right\} \le r.$$

This together with (3.12), (3.19), and Lemma 2.11 gives that

$$\lim_{n\to\infty} d\left(x_n, T_{i(n)}^{k(n)} x_n\right) = 0. \tag{3.20}$$

Therefore we have

$$\begin{aligned} d(x_n, y_n) &= d\left(x_n, (1-\beta_n)x_n \oplus \beta_n T_{i(n)}^{k(n)} x_n\right) \\ &\le \beta_n d\left(x_n, T_{i(n)}^{k(n)} x_n\right) \to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{3.21}$$

Furthermore, it follows from (3.16) that

$$\begin{aligned} d(x_{n+1}, x_n) &= d((1-\alpha_n)x_n \oplus \alpha_n d\left(T_{i(n)}^{k(n)} y_n, x_n\right) \\ &\le \alpha_n d\left(T_{i(n)}^{k(n)} y_n, x_n\right) \to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{3.22}$$

This together with (3.21) shows that

$$d(x_{n+1}, y_n) \le d(x_{n+1}, x_n) + d(x_n, y_n) \to 0 \quad (\text{as } n \to \infty). \tag{3.23}$$

From Lemma 3.1, (3.16), (3.20), (3.22), and (3.23), we have that for each given positive integer $i \ge 1$, there exist subsequences $\{x_m\}_{m\in\Gamma_i}$, $\{y_m\}_{m\in\Gamma_i}$, and $\{k(m)\}_{m\in\Gamma_i} \subset K_i := \{k(m) : m \in \Gamma_i, m = i + \frac{(k(m)-1)k(m)}{2}, k(m) \ge i\}$ such that

$$\begin{aligned} d(x_m, T_i x_m) &\le d\left(x_m, T_i^{k(m)} x_m\right) + d\left(T_i^{k(m)} x_m, T_i^{k(m)} y_{m-1}\right) + d\left(T_i^{k(m)} y_{m-1}, T_i x_m\right) \\ &\le d\left(x_m, T_i^{k(m)} x_m\right) + \left\{ d(x_m, y_{m-1}) + \nu_{k(m)}^{(i)} \zeta^{(i)}\left(d(x_m, y_{m-1})\right) + \mu_{k(m)}^{(i)}\right\} \\ &\quad + L_i(d\left(T_i^{k(m)-1} y_{m-1}, x_m\right) \\ &\le d\left(x_m, T_i^{k(m)} x_m\right) + \left\{ d(x_m, y_{m-1}) + \nu_{k(m)}^{(i)} \zeta^{(i)}\left(d(x_m, y_{m-1})\right) + \mu_{k(m)}^{(i)}\right\} \\ &\quad + L_i(d\left(T_i^{k(m-1)} y_{m-1}, x_{m-1}\right) + L_i(d(x_{m-1}, x_m) \to 0 (\text{as} m \to \infty). \end{aligned} \tag{3.24}$$

The conclusion (3.11) is proved.

(III) *Now we prove that* $\{x_n\}$ *converges strongly (i.e., in the metric topology) to some point* $p^* \in \mathscr{F}$.

In fact, it follows from (3.11) and (3.24) that for given mapping T_{n_0} there exists some subsequence $\{x_m\}_{m\in\Gamma_{n_0}}$ of $\{x_n\}$ such that

$$\lim_{m(\in\Gamma_{n_0})\to\infty} d(x_m, T_{n_0}x_m) = 0.$$

By (3.3) we have

$$f\bigl(d(x_m, \mathscr{F})\bigr) \le d(x_m, T_{n_0}x_m) \quad \forall m \ge 1.$$

Let $m \to \infty$, and taking $\limsup$ on the above inequality, we have $\lim_{m\to\infty} f(d(x_m, \mathscr{F})) = 0$. By the property of f, this implies that

$$\lim_{m(\in\Gamma_{n_0})\to\infty} d(x_m, \mathscr{F}) = 0. \tag{3.25}$$

Next we prove that $\{x_m\}_{m\in\Gamma_{n_0}}$ is a Cauchy sequence in C. In fact, it follows from (3.8) that for any $p \in \mathscr{F}$

$$d(x_{m+1}, p) \le (1 + \sigma_m)d(x_m, p) + \xi_m, \quad \forall m(\in \Gamma_{n_0}) \ge 1,$$

where $\sum_{m=1}^{\infty} \sigma_m < \infty$ and $\sum_{m=1}^{\infty} \xi_m < \infty$. Hence, for any positive integers $j, n \in \Gamma_{n_0}$, $n > j$, and $n = m + j$ for some positive integer m, we have

$$\begin{aligned} d(x_n, x_j) &= d(x_{j+m}, x_j) \le d(x_{j+m}, p) + d(x_j, p) \\ &\le (1 + \sigma_{j+m-1})d(x_{j+m-1}, p) + \xi_{j+m-1} + d(x_j, p). \end{aligned}$$

Since for each $x \ge 0$, $1 + x \le e^x$, we have

$$\begin{aligned} d(x_n, x_j) &= d(x_{j+m}, x_j) \\ &\le e^{\sigma_{j+m-1}} d(x_{j+m-1}, p) + \xi_{j+m-1} + d(x_j, p) \\ &\le e^{\sigma_{j+m-1}+\sigma_{j+m-2}} d(x_{j+m-2}, p) + e^{\sigma_{j+m-1}}\xi_{j+m-2} + \xi_{j+m-1} + d(x_j, p) \\ &\le \cdots \\ &\le e^{\sum_{i=j}^{j+m-1}\sigma_i} d(x_j, p) + e^{\sum_{i=j+1}^{j+m-1}\sigma_i}\xi_j + e^{\sum_{i=j+2}^{j+m-2}\sigma_i}\xi_{j+1} + \cdots \\ &\quad + e^{\sigma_{j+m-1}}\xi_{j+m-2} + \xi_{j+m-1} + d(x_j, p) \\ &\le (1 + M)d(x_j, p) + M\sum_{i=j}^{j+m-1}\xi_i \\ &= (1 + M)d(x_j, p) + M\sum_{i=j}^{n-1}\xi_i, \quad \text{for each } p \in \mathscr{F}. \end{aligned}$$

Therefore we have

$$d(x_n,x_j) = d(x_{j+m},x_j) \le (1+M)d(x_j,\mathscr{F}) + M\sum_{i=j}^{n-1}\xi_i,$$

where $M = e^{\sum_{i=1}^{\infty}\sigma_i} < \infty$. By (3.25) we have

$$d(x_n,x_j) \le (1+M)d(x_j,\mathscr{F}) + M\sum_{i=j}^{n-1}\xi_i \to 0 \quad \bigl(\text{as } n,j (\in \Gamma_{n_0}) \to \infty\bigr).$$

This shows that the subsequence $\{x_m\}_{m\in\Gamma_{n_0}}$ is a Cauchy sequence in C. Since C is a closed subset in a complete CAT(κ) space X, it is complete. Without loss of generality, we can assume that the subsequence $\{x_m\}$ converges strongly (i.e., in metric topology in X) to some point $p^* \in C$. It is easy to know that $\mathscr{F}$ is a closed subset in C. Since $\lim_{m\to\infty} d(x_m,\mathscr{F}) = 0$, $p^* \in \mathscr{F}$. By using (3.5), it yields that the whole sequence $\{x_n\}$ converges strongly to $p^* \in \mathscr{F}$.

This completes the proof of Theorem 3.2. □

Remark 3.3 It should be pointed out that if (X,d) is a CAT(0) space, then X is uniformly convex, its modulus of uniform convexity $\eta(r,\epsilon) = \frac{\epsilon^2}{8}$ [25, 27] and all of its geodesic triangles satisfy the CAT(κ) inequality. These imply that if (X,d) is a CAT(0) space, then the conditions that appeared in Theorem 3.2 "(X,d) is uniformly convex and $\operatorname{diam}(X) \le \frac{\frac{\pi}{2}-\epsilon}{\sqrt{\kappa}}$ for some $\epsilon \in (0,\frac{\pi}{2})$" are of no use here. Therefore from Theorem 3.2 we can obtain the following.

Theorem 3.4 *Let (X,d) be a complete CAT(0) space. Let C be a nonempty closed and convex subset of X, and for each $i \ge 1$, let $T_i : C \to C$ be uniformly L_i-Lipschitzian and $(\{\nu_n^{(i)}\},\{\mu_n^{(i)}\},\zeta^{(i)})$-total asymptotically nonexpansive mappings defined by (1.2) such that*

(i) $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\nu_n^{(i)} < \infty$, $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\mu_n^{(i)} < \infty$,

(ii) *there exists a constant $M_* > 0$ such that $\zeta^{(i)}(r) \le M_* r$, $\forall r \ge 0$, $i = 1,2,\dots$;*

(iii) *there exist constants $a,b \in (0,1)$ with $0 < b(1-a) \le \frac{1}{2}$ such that $\{\alpha_n\},\{\beta_n\} \subset [a,b]$.*

If $\mathscr{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exist a mapping $T_{n_0} \in \{T_i\}_{i=1}^{\infty}$ and a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ $\forall r > 0$ such that

$$f\bigl(d(x_n,\mathscr{F})\bigr) \le d(x_n, T_{n_0}x_n), \quad \forall n \ge 1,$$

then the sequence $\{x_n\}$ defined by (3.2) converges strongly (i.e., in metric topology) to some point $p^* \in \mathscr{F}$.

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Availability of data and materials
The data sets used and/or analyzed during the current study are available from the corresponding author on reasonable request. All data generated or analyzed during this study are included in this manuscript.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
S.C. L.Z, M.L. and J.T.wrote the main manuscript text. All authors reviewed the manuscript.

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