Fixed Point Theory and Algorithms
for Sciences and Engineering

RESEARCH Open Access

A Tseng-type algorithm for approximating zeros of monotone inclusion and J-fixed-point problems with applications

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Abstract

In this paper, a Halpern–Tseng-type algorithm for approximating zeros of the sum of two monotone operators whose zeros are J-fixed points of relatively J-nonexpansive mappings is introduced and studied. A strong convergence theorem is established in Banach spaces that are uniformly smooth and 2-uniformly convex. Furthermore, applications of the theorem to convex minimization and image-restoration problems are presented. In addition, the proposed algorithm is used in solving some classical image-recovery problems and a numerical example in a Banach space is presented to support the main theorem. Finally, the performance of the proposed algorithm is compared with that of some existing algorithms in the literature.

MSC: 47H20; 49M20; 49M25; 49M27; 47J25; 47H05

Keywords: Zeros; J-fixed point; Convex minimization; Image restoration

1 Introduction

Let E be a real Banach space with dual space, E^*. Let $A : E \to E^*$ and $B : E \to 2^{E^*}$ be single-valued and multivalued monotone operators, respectively. The following monotone inclusion problem:

$$\text{find } u \in E \quad \text{such that } 0 \in (A + B)u, \tag{1}$$

has been of interest to several authors due to its numerous applications in solving problems arising from image restoration, signal recovery, and machine learning. One of the early methods used for approximating solutions of the inclusion problem (1) is the forward–backward algorithm (FBA); which was introduced by Passty [37] and studied extensively by many authors (see, e.g., [1, 2, 6, 15, 16, 20, 21, 28, 49]).

Recently, there is growing interest in the study of the monotone inclusion problem (1) whose solutions are fixed points of some nonexpansive-type mappings. In general, the problem is stated as follows:

$$\text{find } u \in E \quad \text{such that } 0 \in (A + B)u \text{ and } Tu = u, \tag{2}$$

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where $T : E \to E$ is a *nonexpansive-type mapping.*

In 2010, Takahashi et al. [44] introduced and studied an iterative algorithm that approximates solutions of problem (2) in the setting of real Hilbert spaces. They proved the following strong convergence theorem:

Theorem 1.1 *Let C be a closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$ and let T be a nonexpansive mapping of C into itself, such that $F(T) \cap (A + B)^{-1}0 \neq \emptyset$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)T\big(\alpha_n x + (1 - \alpha_n)J_{\lambda_n}(x_n - \lambda_n A x_n)\big), \tag{3}$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq b < 2\alpha, \qquad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n\to\infty} (\lambda_n - \lambda_{n+1}) = 0, \qquad \lim_{n\to\infty} \alpha_n = 0 \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, $\{x_n\}$ converges strongly to a point of $F(T) \cap (A + B)^{-1}0$.

In recent years, many authors have exploited the inertial technique in order to accelerate the convergence of sequences generated by existing algorithms in the literature. The inertial extrapolation technique was first introduced by Polyak [39] as an acceleration process in solving smooth, convex minimization problems. An algorithm of inertial type is an iterative procedure in which subsequent terms are obtained using the preceding two terms. Many authors have shown numerically that adding the inertial extrapolation term in many existing algorithms improves its performance (see, e.g., [3, 12, 17, 18, 25, 30, 36, 38, 42, 43]).

In 2021, Adamu et al. [4] introduced and studied the following inertial algorithm that approximates solutions of problem (2) in real Hilbert spaces. They proved the following strong convergence theorem:

Theorem 1.2 *Let H be a real Hilbert space. Let $A : H \to H$ be α-inverse strongly monotone, $B : H \to 2^H$ be a set-valued maximal monotone operator, and $T : H \to H$ be a nonexpansive mapping. Assume $F(T) \cap (A + B)^{-1}0 \neq \emptyset$. Let $x_0, x_1, u \in H$ and let $\{x_n\} \subset H$ be a sequence generated by*:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = \gamma_n w_n + (1 - \gamma_n)(I + \lambda_n B)^{-1}(I - \lambda_n A)w_n, \\ y_n = s_n w_n + (1 - s_n)(I + \lambda_n B)^{-1}(I - \lambda_n A)z_n, \\ x_{n+1} = \tau_n u + \sigma_n w_n + \mu_n T y_n, \end{cases} \tag{4}$$

where the control parameters satisfy some appropriate conditions. Then, $\{x_n\}$ converges strongly to a point in $F(T) \cap (A + B)^{-1}0$.

Remark 1 We recall that in Algorithms (3) and (4) the operator A is required to be α-inverse strongly monotone, i.e., A satisfies the following inequality:

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2.$$

This requirement rules out some important applications (see, e.g., Sect. 4 of [45]).

To dispense with the α-inverse strong monotonicity assumption on A, using the idea of the extragradient method of Korpelevic [27] for monotone variational inequalities, Tseng [45] introduced the following algorithm in real Hilbert spaces:

$$\begin{cases} x_1 \in C; \\ y_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)x_n; \\ x_{n+1} = P_C(y_n - \lambda_n(Ay_n - Ax_n)), \end{cases} \tag{5}$$

where $C \subset H$ is nonempty, closed, and convex such that $C \cap (A + B)^{-1}0 \neq \emptyset$, A is maximal monotone and Lipschitz continuous with constant $L > 0$ and B is maximal monotone. He proved weak convergence of the sequence generated by his algorithm to a solution of problem (1).

Remark 2 We note here that the class of monotone operators that are Lipschitz continuous contain, properly, the class of monotone operators that are α-inverse strongly monotone, since every α-inverse strongly monotone operator is $\frac{1}{\alpha}$-Lipschitz continuous.

Recently, in 2021, Padcharoen et al. [35] proposed an inertial version of Tseng's Algorithm (5) in the setting of real Hilbert spaces. They proved the following theorem:

Theorem 1.3 *Let H be a real Hilbert space. Let $A : H \to H$ be an L-Lipschitz continuous and monotone mapping and $B : H \to 2^H$ be a maximal monotone map. Assume that the solution set $(A + B)^{-1}$ is nonempty. Given $x_0, x_1 \in H$, let $\{x_n\}$ be a sequence defined by*:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)w_n, \\ x_{n+1} = y_n - \lambda_n(Ay_n - Aw_n), \end{cases} \tag{6}$$

where the control parameters satisfy some appropriate conditions. Then, the sequence $\{x_n\}$ generated by (6) *converges weakly to a solution of problem* (1).

In 2019, Shehu [41] extended the inclusion problem (1) involving monotone operators to Banach spaces. He introduced and studied a modified version of Tseng's algorithm and proved the following theorem:

Theorem 1.4 *Let E be a uniformly smooth and 2-uniformly convex real Banach space. Let $A : E \to E^*$ be a monotone and L-Lipschitz continuous mapping and $B : E \to 2^{E^*}$ be a maximal monotone mapping. Suppose the solution set $(A + B)^{-1}0$ is nonempty and the normalized duality mapping J on E is weakly sequentially continuous. Let $\{x_n\}$ be a sequence* *in E generated by*:

$$\begin{cases} x_1 \in E, \\ y_n = (J + \lambda_n B)^{-1}(Jx_n - \lambda_n A x_n), \\ x_{n+1} = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)), \end{cases} \tag{7}$$

where the control parameters satisfy some appropriate conditions. Then, the sequence $\{x_n\}$ *generated by* (7) *converges weakly to a point* $x \in (A+B)^{-1}0$.

To obtain a strong convergence theorem and dispense with the weak sequential continuity assumption on the normalized duality mapping J in Theorem 1.4, in the same paper [41], Shehu introduced and studied a Halpern modification of Algorithm (7). He proved the following theorem:

Theorem 1.5 *Let E be a uniformly smooth and* 2*-uniformly convex real Banach space. Let* $A : E \to E^*$ *be a monotone and L-Lipschitz continuous mapping and* $B : E \to 2^{E^*}$ *be a maximal monotone mapping. Suppose the solution set* $(A+B)^{-1}0$ *is nonempty. Let* $\{x_n\}$ *be a sequence in E generated by*:

$$\begin{cases} x_1 \in E, \\ y_n = (J + \lambda_n B)^{-1}(Jx_n - \lambda_n A x_n), \\ w_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)), \\ x_{n+1} = J^{-1}(\alpha_n J x_1 + (1-\alpha_n) J w_n), \end{cases} \tag{8}$$

where the control parameters satisfy some appropriate conditions. Then, the sequence $\{x_n\}$ *generated by* (8) *converges strongly to a point* $x \in (A+B)^{-1}0$.

Recently, Cholamjiak et al. [23] introduced and studied a Halpern–Tseng-type algorithm for approximating solutions of the inclusion problem (2) in the setting of Banach spaces. They proved the following theorem:

Theorem 1.6 *Let E be a uniformly smooth and* 2*-uniformly convex real Banach space. Let* $A : E \to E^*$ *be a monotone and L-Lipschitz continuous mapping and* $B : E \to 2^{E^*}$ *be a maximal monotone mapping and* $T : E \to E$ *be relatively nonexpansive. Suppose the solution set* $\Omega = (A+B)^{-1}0 \cap F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence in E generated by*:

$$\begin{cases} u, x_1 \in E, \\ y_n = (J + \lambda_n B)^{-1}(Jx_n - \lambda_n A x_n), \\ z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)), \\ x_{n+1} = J^{-1}(\alpha_n J u + (1-\alpha_n)(\beta_n J z_n + (1-\beta_n) J T z_n)), \end{cases} \tag{9}$$

where $\{\lambda_n\} \subset (0, \frac{\sqrt{c}}{\sqrt{\kappa} L})$, *for some* $c, \kappa > 0$; $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *with* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$; $0 < a \leq \beta_n \leq b < 1$. *Then, the sequence* $\{x_n\}$ *generated by* (9) *converges strongly to a solution of problem* (2).

In 2016, Chidume and Idu [22] reintroduced a fixed-point notion for operators mapping a uniformly convex and uniformly smooth real Banach space, E to its dual space, E^*. Given a map $T : E \to E^*$ let J be the normalized duality mapping on E. Chidume and Idu [22] called a point $u \in E$ a J-fixed point of T if $Tu = Ju$ and denoted the set of by $F_J(T) := \{x \in E : Tx = Jx\}$. An intriguing property of a J-fixed point is its connection with optimization problems, see, e.g., [22] for the connection. Currently, there is a growing interest in the study of J-fixed points (see, e.g., [11, 13, 33, 34], for some interesting results concerning J-fixed points in the literature).

Remark 3 We note here that this notion has also been defined by Zegeye [50] who called it a *semifixed point*. Also, Liu [29] called it *a duality fixed point*.

In line with the current interest on the inclusion problems (1) and (2) involving monotone operators on Banach spaces, J-fixed points and the inertial acceleration technique, it is our purpose in this paper to propose an inertial Halpern–Tseng-type algorithm for approximating solutions of the inclusion problem (1) that are J-fixed points of a relatively J-nonexpansive mapping. Furthermore, we prove the strong convergence of the sequence generated by our algorithm in the setting of real Banach spaces that are uniformly smooth and 2-uniformly convex. In addition, we present applications of our theorem to convex minimization and use our algorithm to solve some classical problems arising from image restoration. Finally, we present a numerical example on a real Banach space to support our main theorem.

2 Preliminaries

In this section, we define some notions and state some results that will be needed in our subsequent analysis.

Let E be a real normed space and let $J : E \to 2^{E^*}$ be the *normalized duality map* (see, e.g., [8] for the explicit definition of J and its properties on certain Banach spaces). The following functional $\phi : E \times E \to \mathbb{R}$ defined on a smooth real Banach space by

$$\phi(x,y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E, \tag{10}$$

will be needed in our estimations in the following. The functional ϕ was first introduced by Alber [8] and has been extensively studied by many authors (see, for example, [8, 14, 26, 32] and the references contained in them). Observe that on a real Hilbert space H, the definition of ϕ above reduces to $\phi(x,y) = \|x - y\|^2$, $\forall x, y \in H$. Furthermore, given $x, y, z \in E$ and $\tau \in [0,1]$, using the definition of ϕ, one can easily deduce the following (see, e.g., [22, 32]):

D1: $(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2$,

D2: $\phi(x, J^{-1}(\tau Jy + (1-\tau)Jz) \leq \tau\phi(x,y) + (1-\tau)\phi(x,z)$,

D3: $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle z - x, Jy - Jz\rangle$,

where J and J^{-1} are the duality maps on E and E^*, respectively.

We shall use interchangeably ϕ and $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, y^*) := \|x\|^2 - 2\langle x, y^*\rangle + \|y\|^2, \quad \forall x \in E, y^* \in E^*,$$

since $V(x,y^*) = \phi(x, J^{-1}y^*)$.

The following ideas will be used in the subsequent discussion.

Definition 2.1 Let $T : E \to E^*$ be a map. A point $x^* \in E$ is called an *asymptotic J-fixed point of T* if there exists a sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup x^*$ and $\|Jx_n - Tx_n\| \to 0$, as $n \to \infty$. Let $\widehat{F}_J(T)$ be the set of asymptotic J-fixed points of T.

Definition 2.2 A map $T : E \to E^*$ is said to be *relatively J-nonexpansive* if

(*i*) $\widehat{F}_J(T) = F_J(T) \neq \emptyset$,

(*ii*) $\phi(u, J^{-1}Tx) \leq \phi(u, x)$, $\forall x \in E$, $u \in F_J(T)$.

Remark 4 See Chidume *et al.* [19] for a nontrivial example of a relatively J-nonexpansive mapping. One can easily verify from the definition above, that if an operator T is relatively J-nonexpansive then the operator $J^{-1}T$ is relatively nonexpansive in the usual sense and vice versa. Furthermore, $x^* \in F_J(T) \Leftrightarrow x^* \in F(J^{-1}T)$.

Definition 2.3 Let E be a smooth, strictly convex, and reflexive real Banach space and let C be a nonempty, closed, and convex subset of E. Following Alber [8], the *generalized projection map*, $\Pi_C : E \to C$ is defined by

$$\Pi_C(u) = \inf_{v \in C} \phi(v, u), \quad \forall u \in E.$$

Clearly, in a real Hilbert space, the generalized projection Π_C coincides with the metric projection P_C from E onto C.

Definition 2.4 Let E be a reflexive, strictly convex, and smooth real Banach space and let $B : E \to 2^{E^*}$ be a maximal monotone operator. Then, for any $\lambda > 0$ and $u \in E$, there exists a unique element $u_\lambda \in E$ such that $Ju \in (Ju_\lambda + \lambda Bu_\lambda)$. The element u_λ is called the resolvent of B and it is denoted by $J_\lambda^B u$. Alternatively, $J_\lambda^B = (J + \lambda B)^{-1}J$, for all $\lambda > 0$. It is easy to verify that $B^{-1}0 = F(J_\lambda^B)$, $\forall \lambda > 0$, where $F(J_\lambda^B)$ denotes the set of fixed points of J_λ^B.

Now, we recall some fundamental and useful results that will be needed in the proof of our main theorem and its corollaries.

Lemma 2.5 ([7]) *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive real Banach space E. For any $x \in E$ and $y \in C$, $\tilde{x} = \Pi_C x$ if and only if $\langle \tilde{x} - y, Jx - J\tilde{x} \rangle \geq 0$, for all $y \in C$.*

Lemma 2.6 ([8]) *Let E be a reflexive, strictly convex, and smooth Banach space with E^* as its dual. Then,*

$$V\big(u, u^*\big) + 2\big\langle J^{-1}u^* - u, v^*\big\rangle \leq V\big(u, u^* + v^*\big), \tag{11}$$

for all $u \in E$ and $u^*, v^* \in E^*$.

Lemma 2.7 ([10]) *Let E be a reflexive Banach space. Let $A : E \to E^*$ be a monotone, hemicontinuous, and bounded mapping. Let $B : E \to 2^{E^*}$ be a maximal monotone mapping. Then, $A + B$ is a maximal monotone mapping.*

Lemma 2.8 *Let* $\frac{1}{p}+\frac{1}{q}=1, p,q>1$. *The space* E *is* q*-uniformly smooth if and only if its dual space* E^* *is* p*-uniformly convex.*

Lemma 2.9 ([48]) *Let* E *be a* 2*-uniformly smooth, real Banach space. Then, there exists a constant* $\rho>0$ *such that* $\forall x,y\in E$

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, Jx\rangle + \rho\|y\|^2.$$

In a real Hilbert space, $\rho=1$.

Lemma 2.10 ([46]) *Let* E *be a* 2*-uniformly convex and smooth real Banach space. Then, there exists a positive constant* μ *such that*

$$\mu\|x-y\|^2 \le \phi(x,y), \quad \forall x,y\in E. \tag{12}$$

Lemma 2.11 ([26]) *Let* E *be a uniformly convex and smooth real Banach space, and let* $\{u_n\}$ *and* $\{v_n\}$ *be two sequences of* E. *If either* $\{u_n\}$ *or* $\{v_n\}$ *is bounded and* $\phi(u_n,v_n)\to 0$ *then* $\|u_n-v_n\|\to 0$.

Lemma 2.12 ([32]) *Let* E *be a uniformly smooth Banach space and* $r>0$. *Then, there exists a continuous, strictly increasing, and convex function* $g:[0,2r]\to[0,1)$ *such that* $g(0)=0$ *and*

$$\phi\big(u, J^{-1}\big[\beta Jx + (1-\beta)Jy\big]\big) \le \beta\phi(u,x) + (1-\beta)\phi(u,y) - \beta(1-\beta)g\big(\|Jx-Jy\|\big),$$

for all $\beta\in[0,1]$, $u\in E$ *and* $x,y\in B_r := \{z\in E: \|z\|\le r\}$.

Lemma 2.13 ([47]) *Let* $\{a_n\}$ *be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\beta_n + c_n, \quad n\ge 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{c_n\}$ *are sequences of real numbers such that*

$$(i) \quad \{\alpha_n\}\subset[0,1] \quad s.t. \sum_{n=0}^{\infty}\alpha_n = \infty; \qquad (ii) \quad \limsup_{n\to\infty}\beta_n \le 0;$$

$$(iii) \quad c_n\ge 0, \sum_{n=0}^{\infty} c_n < \infty.$$

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.14 ([31]) *Let* Γ_n *be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence* $\{\Gamma_{n_j}\}_{j\ge 0}$ *of* $\{\Gamma_n\}$ *that satisfies* $\Gamma_{n_j} < \Gamma_{n_j+1}$ *for all* $j\ge 0$. *Also, consider the sequence of integers* $\{\tau(n)\}_{n\ge n_0}$ *defined by*

$$\tau(n) = \max\{k\le n \mid \Gamma_k < \Gamma_{k+1}\}.$$

Then, $\{\tau(n)\}_{n\ge n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty}\tau(n)=\infty$ and, for all $n\ge n_0$, it holds that $\Gamma_{\tau(n)}\le\Gamma_{\tau(n)+1}$ and we have

$$\Gamma_n\le\Gamma_{\tau(n)+1}.$$

Lemma 2.15 ([9]) *Let $\{\Gamma_n\}$, $\{\delta_n\}$, and $\{\alpha_n\}$ be sequences in $[0,\infty)$ such that*

$$\Gamma_{n+1}\le\Gamma_n+\alpha_n(\Gamma_n-\Gamma_{n-1})+\delta_n,$$

for all $n\ge 1$, $\sum_{n=1}^{\infty}\delta_n<+\infty$ and there exists a real number α with $0\le\alpha_n\le\alpha<1$, for all $n\in\mathbb{N}$. Then, the following hold:

(i) $\sum_{n\ge1}[\Gamma_n-\Gamma_{n-1}]_+<+\infty$, *where* $[t]_+=\max\{t,0\}$;
(ii) *there exists* $\Gamma^*\in[0,\infty)$ *such that* $\lim_{n\to\infty}\Gamma_n=\Gamma^*$.

Lemma 2.16 ([5]) *Let E be a 2-uniformly convex and uniformly smooth real Banach space and let $x_0,x_1,w\in E$. Let $\{v_n\}\subset E$ be a sequence defined by $v_n:=J^{-1}(Jx_n+\mu_n(Jx_n-Jx_{n-1}))$. Then,*

$$\begin{aligned}\phi(w,v_n)\le{}&\phi(w,x_n)+\rho\mu_n^2\|Jx_n-Jx_{n-1}\|^2+\mu_n\phi(x_n,x_{n-1})\\&+\mu_n\bigl(\phi(w,x_n)-\phi(w,x_{n-1})\bigr),\end{aligned}$$

where $\{\mu_n\}\subset(0,1)$ and ρ is the constant appearing in Lemma 2.9. For completeness, we shall give the proof here.

Proof Using property D3, we have

$$\begin{aligned}\phi(w,v_n)&=\phi(w,x_n)+\phi(x_n,v_n)+2\langle x_n-w,Jv_n-Jx_n\rangle\\&=\phi(w,x_n)+\phi(x_n,v_n)+2\mu_n\langle x_n-w,Jx_n-Jx_{n-1}\rangle \qquad (13)\\&=\phi(w,x_n)+\phi(x_n,v_n)+\mu_n\phi(x_n,x_{n-1})+\mu_n\phi(w,x_n)-\mu_n\phi(w,x_{n-1}). \qquad (14)\end{aligned}$$

Also, by Lemma 2.9, one can estimate v_n as follows:

$$\begin{aligned}\phi(w,v_n)&=\phi\bigl(w,J^{-1}\bigl(Jx_n+\mu_n(Jx_n-Jx_{n-1})\bigr)\bigr)\\&=\|w\|^2+\bigl\|Jx_n+\mu_n(Jx_n-Jx_{n-1})\bigr\|^2-2\bigl\langle w,Jx_n+\mu_n(Jx_n-Jx_{n-1})\bigr\rangle\\&=\|w\|^2+\bigl\|Jx_n+\mu_n(Jx_n-Jx_{n-1})\bigr\|^2-2\langle w,Jx_n\rangle-2\mu_n\langle w,Jx_n-Jx_{n-1}\rangle\\&\le\phi(w,x_n)+\rho\mu_n^2\|Jx_n-Jx_{n-1}\|^2+2\mu_n\langle x_n-w,Jx_n-Jx_{n-1}\rangle. \qquad (15)\end{aligned}$$

Putting together equation (13) and inequality (15), we obtain

$$\phi(x_n,v_n)\le\rho\mu_n^2\|Jx_n-Jx_{n-1}\|^2.$$

From (14), this implies that

$$\begin{aligned}\phi(w,v_n)\le{}&\phi(w,x_n)+\rho\mu_n^2\|Jx_n-Jx_{n-1}\|^2+\mu_n\phi(x_n,x_{n-1})\\&+\mu_n\bigl(\phi(w,x_n)-\phi(w,x_{n-1})\bigr). \qquad (16)\end{aligned}$$

□

3 Main result

The Setting for Algorithm 3.1.

1. The space E is a 2-uniformly convex and uniformly smooth real Banach space with dual space, E^*.
2. The operator $A : E \to E^*$ is monotone and L-Lipschitz continuous, and $B : E \to 2^{E^*}$ is maximal monotone and $T : E \to E^*$ is relatively J-nonexpansive.
3. The solution set $\Omega = (A + B)^{-1}0 \cap F_J(T)$ is nonempty.
4. The control parameters $\{\beta_n\} \subset (0,1)$, $\{\gamma_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$, $\{\epsilon_n\} \subset (0,1)$ such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$, and $\{\lambda_n\} \subset (\lambda, \frac{\sqrt{\mu}}{\sqrt{\rho L}})$, where $\lambda \in (0, \frac{\sqrt{\mu}}{\sqrt{\rho L}})$, ρ and μ are the constants appearing in Lemmas 2.9 and 2.10, respectively.

Algorithm 3.1 Inertial Halpern–Tseng-type algorithm:

Step 0. (Initialization) Choose arbitrary points $u, x_0, x_1 \in E$, $\theta \in (0,1)$ and set $n = 1$,

Step 1. Choose θ_n such that $0 \le \theta_n \le \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|Jx_n - Jx_{n-1}\|^2}, \frac{\epsilon_n}{\phi(x_n,x_{n-1})}\}, & x_n \ne x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$\begin{cases} w_n = J^{-1}(Jx_n + \theta_n(Jx_n - Jx_{n-1})), \\ y_n = J^B_{\lambda_n} J^{-1}(Jw_n - \lambda_n Aw_n), \\ z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Aw_n)), \\ u_n = J^{-1}(\beta_n Jz_n + (1-\beta_n)Tz_n), \\ x_{n+1} = J^{-1}(\gamma_n Ju + (1-\gamma_n)Ju_n). \end{cases}$$

Step 3. Set $n = n + 1$ and go to Step 1.

Lemma 3.2 *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then, $\{x_n\}$ is bounded.*

Proof Let $x \in \Omega$. Using Lemma 2.9 and D3, we have

$$\begin{aligned}
\phi(x, z_n) &\le \phi\big(x, J^{-1}\big(Jy_n - \lambda_n(Ay_n - Aw_n)\big)\big) \\
&= \|x\|^2 - 2\big\langle x, Jy_n - \lambda_n(Ay_n - Aw_n)\big\rangle + \big\|Jy_n - \lambda_n(Ay_n - Aw_n)\big\|^2 \\
&\le \|x\|^2 - 2\langle x, Jy_n\rangle + 2\lambda_n\langle x, Ay_n - Aw_n\rangle + \|Jy_n\|^2 \\
&\quad - 2\lambda_n\langle y_n, Ay_n - Aw_n\rangle + \rho\big\|\lambda_n(Ay_n - Aw_n)\big\|^2 \\
&= \phi(x, y_n) - 2\lambda_n\langle y_n - x, Ay_n - Aw_n\rangle + \rho\big\|\lambda_n(Ay_n - Aw_n)\big\|^2 \\
&= \phi(x, w_n) + \phi(w_n, y_n) + 2\langle w_n - x, Jy_n - Jw_n\rangle \\
&\quad - 2\lambda_n\langle y_n - x, Ay_n - Aw_n\rangle + \rho\big\|\lambda_n(Ay_n - Aw_n)\big\|^2 \\
&= \phi(x, w_n) + \phi(w_n, y_n) - 2\langle y_n - w_n, Jy_n - Jw_n\rangle + 2\langle y_n - x, Jy_n - Jw_n\rangle \\
&\quad - 2\lambda_n\langle y_n - x, Ay_n - Aw_n\rangle + \rho\big\|\lambda_n(Ay_n - Aw_n)\big\|^2
\end{aligned}$$

$$\begin{aligned}&\leq \phi(x,w_n)-\phi(y_n,w_n)+\rho\lambda_n^2L^2\|y_n-w_n\|^2\\&\quad-2\big\langle y_n-x,Jw_n-Jy_n-\lambda_n(Aw_n-Ay_n)\big\rangle.\end{aligned}\tag{17}$$

Claim.

$$\big\langle y_n-x,Jw_n-Jy_n-\lambda_n(Aw_n-Ay_n)\big\rangle\geq 0.\tag{18}$$

Proof of claim. Observe that $y_n = J_{\lambda_n}^B J^{-1}(Jw_n - \lambda_n Aw_n)$ implies $(Jw_n - \lambda_n Aw_n) \in (Jy_n + \lambda_n By_n)$. Since B is maximal monotone, there exists $b_n \in By_n$ such that $Jw_n - \lambda_n Aw_n = Jy_n + \lambda_n b_n$. Thus,

$$b_n=\frac{1}{\lambda_n}(Jw_n-Jy_n-\lambda_nAw_n).\tag{19}$$

Furthermore, since $0 \in (A + B)x$ and $(Ay_n + b_n) \in (A + B)y_n$, by the monotonicity of $(A + B)$, we have

$$\langle y_n-x,Ay_n+b_n\rangle\geq 0.$$

Substituting equation (19) into this inequality, we obtain

$$\big\langle y_n-x,Jw_n-Jy_n-\lambda_n(Aw_n-Ay_n)\big\rangle\geq 0,$$

which justifies our claim.

Now, substituting inequality (18) into inequality (17) and using Lemma 2.10, we deduce that

$$\phi(x,z_n)\leq\phi(x,w_n)-\left(1-\frac{\rho\lambda_n^2L^2}{\mu}\right)\phi(y_n,w_n).\tag{20}$$

Since $\lambda_n \in (0, \frac{\sqrt{\mu}}{\sqrt{\rho}L})$, $1 - \frac{\rho\lambda_n^2L^2}{\mu} > 0$. Thus,

$$\phi(x,z_n)\leq\phi(x,w_n).\tag{21}$$

Also, using D2 and the fact that T is relatively J-nonexpansive, we have

$$\begin{aligned}\phi(x,u_n)&\leq\beta_n\phi(x,z_n)+(1-\beta_n)\phi\big(x,J^{-1}Tz_n\big)\\&\leq\beta_n\phi(x,z_n)+(1-\beta_n)\phi(x,z_n)=\phi(x,z_n).\end{aligned}\tag{22}$$

Next, using D2, inequalities (22) and (21), Lemma 2.16, and the fact that $\{\theta_n\} \subset (0, 1)$, we obtain

$$\begin{aligned}\phi(x,x_{n+1})&=\phi\big(x,J^{-1}\big(\gamma_nJu+(1-\gamma_n)Ju_n\big)\big)\\&\leq\gamma_n\phi(x,u)+(1-\gamma_n)\phi(x,u_n)\\&\leq\gamma_n\phi(x,u)+(1-\gamma_n)\phi(x,z_n)\end{aligned}$$

$$
\begin{aligned}
&\le \gamma_n\phi(x,u) + (1-\gamma_n)\phi(x,w_n) \\
&\le \gamma_n\phi(x,u) + (1-\gamma_n)\big(\phi(x,x_n) + \theta_n\big(\phi(x,x_n) - \phi(x,x_{n-1})\big) \\
&\quad + \rho\theta_n^2\|Jx_n - Jx_{n-1}\|^2 + \theta_n\phi(x_n,x_{n-1})\big) \\
&\le \max\big\{\phi(x,u), \phi(x,x_n) + \theta_n\big(\phi(x,x_n) - \phi(x,x_{n-1})\big) \\
&\quad + \rho\theta_n\|Jx_n - Jx_{n-1}\|^2 + \theta_n\phi(x_n,x_{n-1})\big\}.
\end{aligned}
\tag{23}
$$

If the maximum is $\phi(x,u)$, for all $n \ge 1$, we are done. Else, there exists an $n_0 \ge 1$ such that for all $n \ge n_0$, we have that

$$
\begin{aligned}
\phi(x,x_{n+1}) \le{}& \phi(x,x_n) + \theta_n\big(\phi(x,x_n) - \phi(x,x_{n-1})\big) + \rho\theta_n\|Jx_n - Jx_{n-1}\|^2 \\
& + \theta_n\phi(x_n,x_{n-1}).
\end{aligned}
$$

From Step 1 and the setting for Algorithm 3.1 (4), we obtain

$$
\rho\theta_n\|Jx_n - Jx_{n-1}\|^2 \le \rho\epsilon_n, \qquad \theta_n\phi(x_n,x_{n-1}) \le \epsilon_n \quad \text{and} \quad \sum_{n=1}^{\infty}\epsilon_n < \infty.
$$

Hence, by Lemma 2.15, $\{\phi(x,x_n)\}$ is convergent and thus, bounded. Furthermore, by D1, $\{x_n\}$ is bounded. This implies that $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ are bounded. □

Now, we are ready to state our main convergence theorem.

Theorem 3.3 *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then, $\{x_n\}$ converges strongly to $x \in \Omega$.*

Proof Let $x \in \Omega$. First, we estimate $\phi(x,u_n)$ using Lemma 2.12, the fact that T is relatively J-nonexpansive, and inequalities (20) and (21). Now,

$$
\begin{aligned}
\phi(x,u_n) &\le \beta_n\phi(x,z_n) + (1-\beta_n)\phi\big(x,J^{-1}Tz_n\big) - \beta_n(1-\beta_n)g\big(\|Jz_n - Tz_n\|\big) \\
&\le \beta_n\phi(x,z_n) + (1-\beta_n)\phi(x,z_n) - \beta_n(1-\beta_n)g\big(\|Jz_n - Tz_n\|\big) \\
&\le \beta_n\phi(x,z_n) + (1-\beta_n)\left[\phi(x,w_n) - \left(1 - \frac{\rho\lambda_n^2L^2}{\mu}\right)\phi(y_n,w_n)\right] \\
&\quad - \beta_n(1-\beta_n)g\big(\|Jz_n - Tz_n\|\big) \\
&\le \phi(x,w_n) - (1-\beta_n)\left(1 - \frac{\rho\lambda_n^2L^2}{\mu}\right)\phi(y_n,w_n) \\
&\quad - \beta_n(1-\beta_n)g\big(\|Jz_n - JTz_n\|\big).
\end{aligned}
\tag{24}
$$

Next, we estimate $\phi(x,x_{n+1})$ using Lemma 2.12 and inequality (24). Hence,

$$
\begin{aligned}
\phi(x,x_{n+1}) &\le \gamma_n\phi(x,u) + (1-\gamma_n)\phi(x,u_n) \\
&\le \gamma_n\phi(x,u) + (1-\gamma_n)\left[\phi(x,w_n) - (1-\beta_n)\left(1 - \frac{\rho\lambda_n^2L^2}{\mu}\right)\phi(y_n,w_n)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \beta_n(1-\beta_n)g\big(\|Jz_n - JTz_n\|\big)\Big] \\
&= \gamma_n\phi(x,u) + (1-\gamma_n)\phi(x,w_n) - (1-\gamma_n)(1-\beta_n)\left(1 - \frac{\rho\lambda_n^2L^2}{\mu}\right)\phi(y_n,w_n) \\
&\quad - (1-\gamma_n)\beta_n(1-\beta_n)g\big(\|Jz_n - JTz_n\|\big).
\end{aligned} \tag{25}
$$

Set $\eta_n = (1-\gamma_n)(1-\beta_n)(1-\frac{\rho\lambda_n^2L^2}{\mu})$ and $\zeta_n = (1-\gamma_n)\beta_n(1-\beta_n)$. By rearranging the terms in inequality (25) and using Lemma 2.15, we obtain

$$
\begin{aligned}
&\eta_n\phi(y_n,w_n) + \zeta_n g\big(\|Jz_n - Tz_n\|\big) \\
&\quad \le \gamma_n\big(\phi(x,u) - \phi(x,w_n)\big) + \phi(x,w_n) - \phi(x,x_{n+1}) \\
&\quad \le \gamma_n\big(\phi(x,u) - \phi(x,w_n)\big) + \phi(x,x_n) + \rho\theta_n\|Jx_n - Jx_{n-1}\|^2 \\
&\qquad + \theta_n\phi(x_n,x_{n-1}) + \theta_n\big(\phi(x,x_n) - \phi(x,x_{n-1})\big) - \phi(x,x_{n+1}) \\
&\quad = \gamma_n\big(\phi(x,u) - \phi(x,w_n)\big) + \phi(x,x_n) - \phi(x,x_{n+1}) + \theta_n\phi(x_n,x_{n-1}) \\
&\qquad + \rho\theta_n\|Jx_n - Jx_{n-1}\|^2 + \theta_n\big(\phi(x,x_n) - \phi(x,x_{n-1})\big).
\end{aligned} \tag{26}
$$

To complete the proof, we consider the following two cases:

Case 1. Assume there exits an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$
\phi(x,x_{n+1}) \le \phi(x,x_n), \quad \forall n \ge n_0.
$$

Then, $\{\phi(x,x_n)\}$ is convergent.

From inequality (26), and using the fact that $\lim_{n\to\infty}\gamma_n = 0$, the boundedness $\{w_n\}$, the existence of $\lim_{n\to\infty}\phi(x,x_n)$, and the fact that $\lim_{n\to\infty}\rho\theta_n\|Jx_n - Jx_{n-1}\|^2 = 0 = \lim_{n\to\infty}\theta_n \times \phi(x_n,x_{n-1})$, we obtain the following:

$$
\lim_{n\to\infty}\phi(y_n,w_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} g\big(\|Jz_n - Tz_n\|\big) = 0.
$$

This implies by Lemma 2.11 and the properties of g that

$$
\lim_{n\to\infty}\|y_n - w_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|Jz_n - Tz_n\| = 0. \tag{27}
$$

Furthermore, since

$$
\|Jx_n - Jw_n\| = \theta_n\|Jx_n - Jx_{n-1}\|, \qquad \lim_{n\to\infty}\|Jx_n - Jw_n\| = 0.
$$

Moreover, by the uniform continuity of J^{-1} on bounded sets, $\lim_{n\to\infty}\|x_n - w_n\| = 0$. This and equation (27) imply that $\lim_{n\to\infty}\|x_n - y_n\| = 0$. By the uniform continuity of J on bounded sets, this implies $\lim_{n\to\infty}\|Jx_n - Jy_n\| = 0$. Also, the Lipschitz continuity of A and equation (27) imply that $\lim_{n\to\infty}\|Aw_n - Ay_n\| = 0$. Therefore,

$$
\lim_{n\to\infty}\|Jz_n - Jy_n\| = \lim_{n\to\infty}\lambda_n\|Aw_n - Ay_n\| = 0. \tag{28}
$$

By the uniform continuity of J^{-1}, equation (28) implies that $\lim_{n\to\infty}\|z_n-y_n\|=0$. Thus,

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{29}$$

Now, observe that

$$\begin{aligned}\|Jx_{n+1}-Jx_n\| &\le \|Jx_{n+1}-Ju_n\|+\|Ju_n-Jz_n\|+\|Jz_n-Jx_n\|\\ &\le \gamma_n\|Ju-Ju_n\|+(1-\beta_n)\|Tz_n-Jz_n\|+\|Jz_n-Jx_n\|\\ &\le \gamma_n\|Ju-Ju_n\|+(1-\beta_n)\|Tz_n-Jz_n\|+\|Jz_n-Jy_n\|\\ &\quad +\|Jy_n-Jx_n\|.\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\|Jx_{n+1}-Jx_n\|=0. \tag{30}$$

Now, we prove that $\Omega_w(x_n)\subset\Omega$, where $\Omega_w(x_n)$ denotes the set of weak subsequential limits of $\{x_n\}$. Since $\{x_n\}$ is bounded, $\Omega_w(x_n)\neq\emptyset$. Let $x^*\in\Omega_w(x_n)$. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x^*$. From equation (29), we have $z_{n_k}\rightharpoonup x^*$. This and (27) imply that $x^*\in\widehat{F}_J(T)$. Since T is relatively J-nonexpansive, $x^*\in F_J(T)$.

Next, we show that $x^*\in(A+B)^{-1}0$. Let $(v,w)\in G(A+B):=\{(x,y)\in E\times E^*: y\in(Ax+Bx)\}$. Then, $(w-Av)\in Bv$. By the definition of y_n in Algorithm 3.1, we have that $(Jw_{n_k}-\lambda_{n_k}Aw_{n_k})\in(Jy_{n_k}+\lambda_{n_k}By_{n_k})$. Thus, $\frac{1}{\lambda_{n_k}}(Jw_{n_k}-Jy_{n_k}-\lambda_{n_k}Aw_{n_k})\in By_{n_k}$. By the monotonicity of B, we have

$$\left\langle v-y_{n_k}, w-Av-\frac{1}{\lambda_{n_k}}(Jw_{n_k}-Jy_{n_k}-\lambda_{n_k}Aw_{n_k})\right\rangle\ge 0.$$

Using the fact that A is monotone, we estimate this as follows

$$\begin{aligned}\langle v-y_{n_k},w\rangle &\ge \left\langle v-y_{n_k}, Av+\frac{1}{\lambda_{n_k}}(Jw_{n_k}-Jy_{n_k}-\lambda_{n_k}Aw_{n_k})\right\rangle\\ &= \langle v-y_{n_k},Av-Aw_{n_k}\rangle+\frac{1}{\lambda_{n_k}}\langle v-y_{n_k},Jw_{n_k}-Jy_{n_k}\rangle\\ &= \langle v-y_{n_k},Av-Ay_{n_k}\rangle+\langle v-y_{n_k},Ay_{n_k}-Aw_{n_k}\rangle\\ &\quad+\frac{1}{\lambda_{n_k}}\langle v-y_{n_k},Jw_{n_k}-Jy_{n_k}\rangle\\ &\ge \langle v-y_{n_k},Ay_{n_k}-Aw_{n_k}\rangle+\frac{1}{\lambda_{n_k}}\langle v-y_{n_k},Jw_{n_k}-Jy_{n_k}\rangle.\end{aligned}$$

Since $\lim_{n\to\infty}\|Aw_n-Ay_n\|=\lim_{n\to\infty}\|Jy_n-Jw_n\|=0$, $\{\frac{1}{\lambda_n}\}$ is bounded and $y_{n_k}\rightharpoonup x^*$, it follows that

$$\langle v-x^*,w\rangle\ge 0.$$

By Lemma 2.7, $A+B$ is maximal monotone. This implies that $0\in(A+B)x^*$, i.e., $x^*\in(A+B)^{-1}0$. Hence, $x^*\in\Omega=F_J(T)\cap(A+B)^{-1}0$.

Now, we show that $\{x_n\}$ converges strongly to the point $x = \Pi_\Omega u$. Observe that if $x = x^*$, we are done. Suppose $x \neq x^*$. Using the boundedness of $\{x_n\}$, Lemma 2.5, and the fact that Ω is closed and convex (see, e.g., [23]), there exits a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x_n - x, Ju - Jx\rangle = \lim_{k\to\infty} \langle x_{n_k} - x, Ju - Jx\rangle = \langle x^* - x, Ju - Jx\rangle \leq 0.$$

Using (30) and the uniform boundedness of J^{-1}, we deduce that

$$\limsup_{n\to\infty} \langle x_{n+1} - x, Ju - Jx\rangle \leq 0.$$

Next, using Lemma 2.6, D2, inequalities (22), (21), and Lemma 2.15, we have

$$\begin{aligned}
\phi(x, x_{n+1}) &= \phi\bigl(x, J^{-1}\bigl(\gamma_n Ju + (1-\gamma_n)Ju_n\bigr)\bigr) \\
&= V\bigl(x, \gamma_n Ju + (1-\gamma_n)Ju_n\bigr) \\
&\leq V\bigl(x, \gamma_n Ju + (1-\gamma_n)Ju_n - \gamma_n(Ju - Jx)\bigr) \\
&\quad + 2\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle \\
&= V\bigl(x, \gamma_n Jx + (1-\gamma_n)Ju_n\bigr) + 2\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle \\
&= \phi\bigl(x, J^{-1}\bigl(\gamma_n Jx + (1-\gamma_n)Ju_n\bigr)\bigr) + 2\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle \\
&\leq \gamma_n\phi(x,x) + (1-\gamma_n)\phi(x,u_n) + 2\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle \\
&\leq (1-\gamma_n)\bigl(\phi(x,x_n) + \rho\theta_n^2\|Jx_n - Jx_{n-1}\|^2 + \theta_n\phi(x_n,x_{n-1}) \\
&\quad + \theta_n\bigl(\phi(x,x_n) - \phi(x,x_{n-1})\bigr)\bigr) + 2\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle \qquad (31)\\
&\leq (1-\gamma_n)\phi(x,x_n) + 2\gamma_n\langle x_{n+1} - x, Ju - Jx\rangle \\
&\quad + \rho\theta_n\|Jx_n - Jx_{n-1}\|^2 + \theta_n\phi(x_n,x_{n-1}). \qquad (32)
\end{aligned}$$

By Lemma 2.13, inequality (32) implies that $\lim_{n\to\infty}\phi(x,x_n) = 0$. Using Lemma 2.11, we obtain that $\lim_{n\to\infty} x_n = x$.

Case 2. If Case 1 does not hold, then, there exists a subsequence $\{x_{m_j}\} \subset \{x_n\}$ such that

$$\phi(x, x_{m_j+1}) > \phi(x, x_{m_j}), \quad \forall j \in \mathbb{N}.$$

By Lemma 2.14, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$, such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold

$$\phi(x, x_{m_k}) \leq \phi(x, x_{m_k+1}) \quad \text{and} \quad \phi(x, x_k) \leq \phi(x, x_{m_k}), \quad \forall k \in \mathbb{N}.$$

From inequality (26) we have

$$\begin{aligned}
&\eta_{m_k}\phi(y_{m_k}, x_{m_k}) + \zeta_{m_k} g\bigl(\|Jz_{m_k} - Tz_{m_k}\|\bigr) \\
&\quad \leq \gamma_{m_k}\bigl(\phi(x,u) - \phi(x, w_{m_k})\bigr) + \phi(x, x_{m_k}) \\
&\qquad - \phi(x, x_{m_k+1}) + \theta_{m_k}\phi(x_{m_k}, x_{m_k+1}) + \rho\theta_{m_k}\|Jx_{m_k} - Jx_{m_k-1}\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + \theta_{m_k}\big(\phi(x,x_{m_k}) - \phi(x,x_{m_k-1})\big)\\
&\le \gamma_{m_k}\big(\phi(x,u) - \phi(x,w_{m_k})\big) + \theta_{m_k}\phi(x_{m_k},x_{m_k+1})\\
&\quad + \rho\theta_{m_k}\|Jx_{m_k} - Jx_{m_k-1}\|^2.
\end{aligned}
$$

Following a similar argument as in Case 1, one can establish the following

$$
\begin{aligned}
&\lim_{k\to\infty}\|y_{m_k} - x_{m_k}\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\|Jz_{m_k} - Tz_{m_k}\| = 0,\\
&\lim_{k\to\infty}\|x_{m_k+1} - x_{m_k}\| = 0 \quad \text{and} \quad \limsup_{k\to\infty}\langle x_{m_k+1} - x, Ju - Jx\rangle \le 0.
\end{aligned}
$$

From (31) we have

$$
\begin{aligned}
\phi(x,x_{m_k+1}) &\le (1-\gamma_{m_k})\big(\phi(x,x_{m_k}) + \rho\theta_{m_k}^2\|Jx_{m_k} - Jx_{m_k-1}\|^2\\
&\quad + \theta_{m_k}\phi(x_{m_k},x_{m_k-1}) + \theta_{m_k}\big(\phi(x,x_{m_k}) - \phi(x,x_{m_k-1})\big)\big)\\
&\quad + 2\gamma_{m_k}\langle x_{m_k+1} - x, Ju - Jx\rangle\\
&\le (1-\gamma_{m_k})\phi(x,x_{m_k}) + 2\gamma_{m_k}\langle x_{m_k+1} - x, Ju - Jx\rangle\\
&\quad + \rho\theta_{m_k}\|Jx_{m_k} - Jx_{m_k-1}\|^2 + \theta_{m_k}\phi(x_{m_k},x_{m_k-1})\\
&\quad + \big(\phi(x,x_{m_k}) - \phi(x,x_{m_k-1})\big).
\end{aligned} \tag{33}
$$

By Lemma 2.13, inequality (33) implies that $\lim_{n\to\infty}\phi(x,x_{m_k}) = 0$. Thus,

$$
\limsup_{k\to\infty}\phi(x,x_k) \le \lim_{k\to\infty}\phi(x,x_{m_k}) = 0.
$$

Therefore, $\limsup_{k\to\infty}\phi(x,x_k) = 0$ and so, by Lemma 2.11, $\lim_{k\to\infty}x_k = x$. This completes the proof. □

4 Applications and numerical illustrations

In this section, we give applications of Theorem 3.3 to a structured, nonsmooth, and convex minimization problem, image denoising, and deblurring problems and a numerical illustration on the classical Banach space $l_{\frac{3}{2}}$. Finally, we will compare the performance of Algorithm 3.1 with Algorithms (3) and (9).

4.1 Application to a convex minimization problem

In this subsection, we shall give an application of our theorem to the structured nonsmooth convex minimization problem that is to

$$
\text{find } x^* \in E \quad \text{with } f\big(x^*\big) + g\big(x^*\big) = \min_{x\in E}\big\{f(x) + g(x)\big\}, \tag{34}
$$

where f is a real-valued function on E that is smooth and convex and g is an extended real-valued function that is convex and lower-semicontinuous (E is a real Banach space). Problem (34) can be recast as:

$$
\text{find } x^* \in E \quad \text{with } 0 \in \big(\nabla f\big(x^*\big) + \partial g\big(x^*\big)\big), \tag{35}
$$

where ∇f is the gradient of f and ∂g is the subdifferential of g. Suppose ∇f is monotone and Lipschitz continuous. Then, setting $A = \nabla f$ and $B = \partial g$, in Algorithm 3.1 and assuming that the solution set $\Omega := F_J(T) \cap (\nabla f + \partial g)^{-1}0 \neq \emptyset$, it follows from Theorem 3.3 that $\{x_n\}$ converges strongly to a point $x \in \Omega$.

4.2 Application to image-restoration problems

The general image-recovery problem can be modeled as the following undetermined linear equation system:

$$y = Dx + \varrho, \tag{36}$$

where $x \in \mathbb{R}^N$ is an original image, $y \in \mathbb{R}^M$ is the observed image with noise ϱ, and $D : \mathbb{R}^N \to \mathbb{R}^M$ $(M < N)$ is a bounded linear operator. It is well known that solving (36) can be viewed as solving the LASSO problem:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|Dx - y\|_2^2 + \lambda\|x\|_1, \tag{37}$$

where $\lambda > 0$. Following [24], we define $Ax := \nabla(\frac{1}{2}\|Dx - y\|_2^2) = D^T(Dx - y)$ and $Bx := \partial(\lambda\|x\|_1)$. It is known that A is $\|D\|^2$-Lipschitz continuous and monotone. Moreover, B is maximal monotone (see [40]).

Remark 5 For the purpose of existence, one can take

$$D = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

to see that indeed, there exists a matrix D such that

$$Ax := \nabla\left(\frac{1}{2}\|Dx - y\|_2^2\right) = D^T(Dx - y) \quad \text{is Lipschitz continuous and monotone.}$$

In Algorithm (3) of Takahashi et al., [44], we set $\alpha_n = \frac{1}{1000n}$, $\beta_n = \frac{n}{2n+1}$, and $\lambda_n = 0.001$, and $Sx = \frac{nx}{n+1}$, in Algorithm (9) of Cholamjiak [23], we set $\lambda_n = 0.03$, $\beta_n = 0.999$, $\gamma_n = \frac{1}{(n+1)^2}$, $\theta = 0.999$, $\varepsilon_n = \frac{1}{(n+5)^2}$, $\theta_n = 0.95$, and $Tx := \frac{nx}{n+1}$ and in our proposed Algorithm 3.1, we set $\lambda_n = 0.03$, $\beta_n = 0.999$, $\alpha_n = \frac{1}{(n+1)^2}$, $\theta = 0.999$, $\varepsilon_n = \frac{1}{(n+5)^2}$, $\theta_n = 0.95$, and $Tx := \frac{nx}{n+1}$. The test images were degraded using the following MATLAB blur functions "fspecial('motion',9,15)" and "fspecial('gaussian', 5,5)" and then we added random noise. Finally, we used a tolerance of 10^{-4} and the maximum number of iterations (n) of 300, for all the algorithms. The results are presented in Figs. 1 and 2, and Table 1.

Looking at the restored images in Fig. 2, it is difficult to tell which algorithm performs better in the restoration process. To distinguish this, there is a powerful tool that is used to measure the quality of restored images. The tool is called SNR, meaning signal-to-noise ratio. The higher the SNR value for a restored image, the better the restoration process via

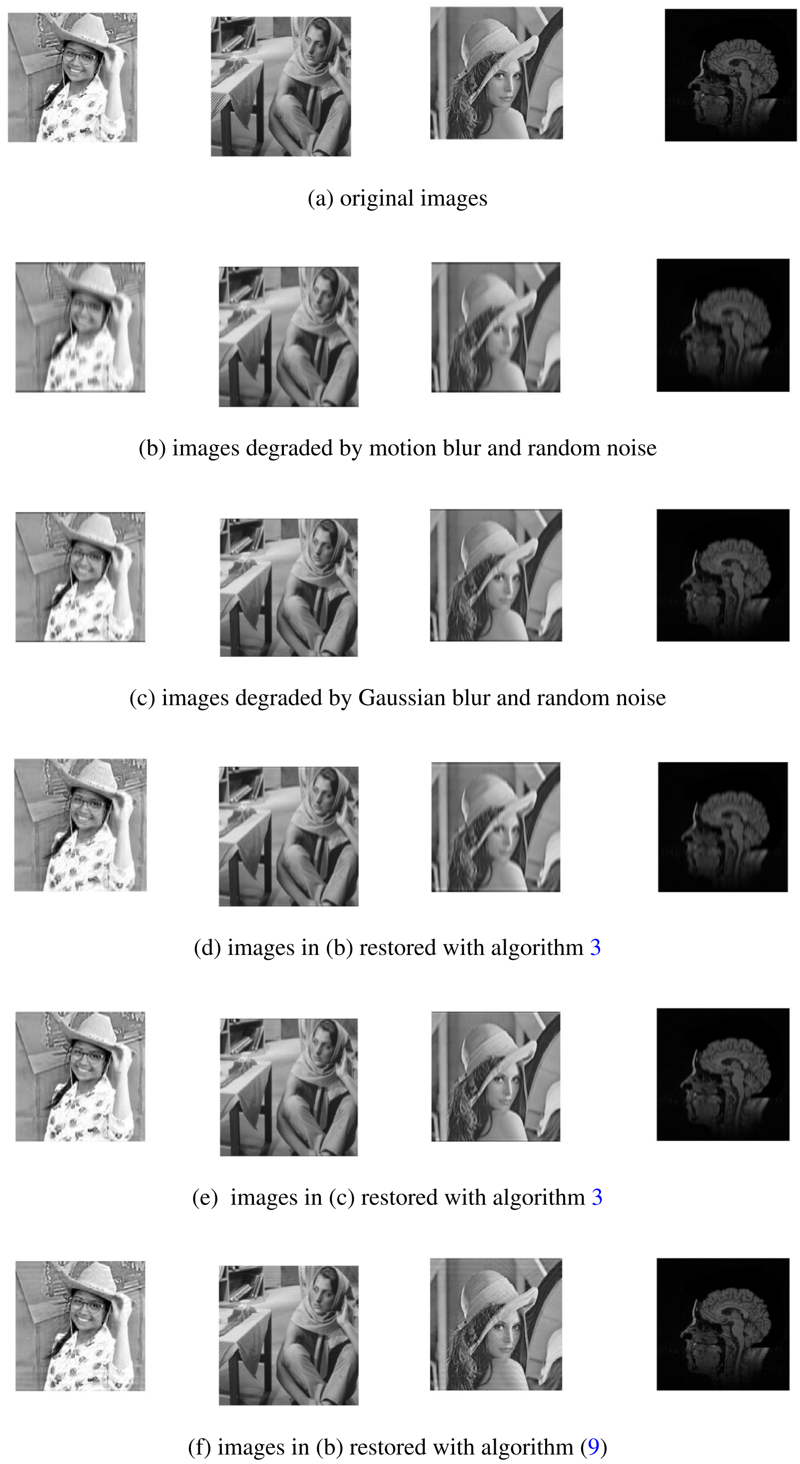


Figure 1 Degradation of the test images and their restorations via Algorithms (3) and (9)

the algorithm. The SNR is defined as follows:

$$\mathrm{SNR} = 10 \log \frac{\|x\|_2^2}{\|x - x_n\|_2^2},$$

(a) images in Figure 1 (c) restored with algorithm (9)

(b) images in Figure 1 (b) restored with algorithm 3.1

(c) images in Figure 1 (c) restored with algorithm 3.1

Figure 2 Test images and their restorations via Algorithms (9) and 3.1

Table 1 SNR values of the restored images in Figs. 1 and 2

Figure	Blur	Iter.	SNR of (3)	Iter.	SNR of (9)	Iter.	SNR of Algorithm 3.1
Kitkuan	Motion	192	63.801416	162	61.123542	107	65.629686
	Gaussian	127	59.672691	125	56.124635	81	61.465479
Lena	Motion	202	62.323217	160	60.284472	108	63.786493
	Gaussian	135	57.504927	113	56.570624	83	59.168384
Barbara	Motion	219	60.073402	171	57.910650	116	61.336816
	Gaussian	160	53.912694	134	53.087088	89	54.800387
Brain	Motion	300	47.262300	277	46.395405	131	47.612237
	Gaussian	300	43.230982	237	42.640876	105	43.353247

where x and x_n are the original image and estimated image at iteration n, respectively. The SNR values for the restored images via Algorithms (3), (9), and 3.1 are presented in Table 1.

Discussion of the numerical results. For the restored images in Figs. 1 and 2, with regards to the number of iterations and the quality of the restored images (SNR values) our proposed Algorithm 3.1 outperforms Algorithm (3) of Takahashi et al. [44] and Algorithm (9) of Cholamjiak et al. [23]. In particular, for the brain image, Algorithms (3) failed to restore the image before the maximum number of iterations was exhausted, however, it took our proposed Algorithm 3.1 just 131 iterations to restore the brain image degraded by motion blur and 105 iterations to restore the brain image degraded by Gaussian blur. From the above experiment, our proposed method appears to be competitive and promising.

4.3 An example in $l_{\frac{3}{2}}$

In this subsection we present a numerical implementation of our proposed Algorithm 3.1 on the Banach space

$$l_p = \left\{ \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\} \quad \text{with norm } \|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}.$$

It is well known that for $1 < p \leq 2$, l_p spaces are uniformly smooth and 2-uniformly convex. Now, let $p = \frac{3}{2}$. Since we cannot sum to infinity on a computer, for the purpose of numerical illustration, we considered the subspace of $l_{\frac{3}{2}}$ consisting of finite, nonzero terms

$$S^k_{\frac{3}{2}} := \big\{ \{x_n\} \subset \mathbb{R} : \{x_n\} = \{x_1, x_2, \ldots, x_k, 0, 0, 0, \ldots\} \big\}, \quad \text{for some } k \geq 1.$$

Example 1 Consider the space $S^3_{\frac{3}{2}}$ with dual space S^3_3. Let $A, B, T : S^3_{\frac{3}{2}} \to S^3_3$ be defined by

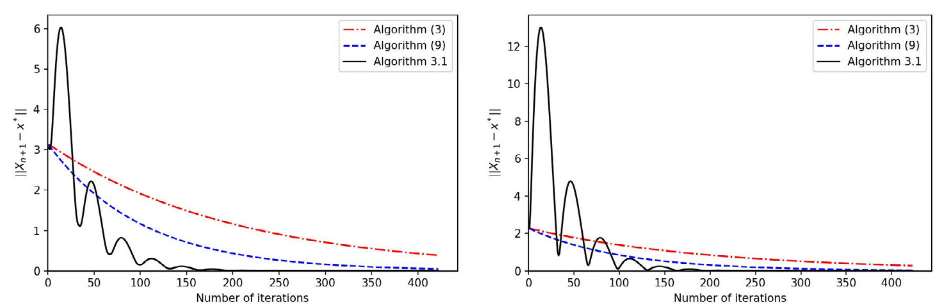
$$Ax := 3x + (1, 0.5, 0.25, 0, 0, \ldots), \qquad Bx := 2x, \qquad Tx := x.$$

It is not difficult to verify the map A is 3-Lipschitz, B is maximal monotone, and T is nonexpansive, at the same time it is relatively nonexpansive and relatively J-nonexpansive. Furthermore, the point $x^* = (0.2, 0.1, 0.125, 0, 0, \ldots)$ is the only point in the solution set $\Omega = (A + B)^{-1}0 \cap F_J(T)$. In the numerical experiment, we compared the performance of Algorithms (3), (9), and 3.1. For a fair comparison, since these algorithms have similar control parameters we used the same values for each parameter appearing in all the algorithms. For the step-size λ_n, we used 0.02 for all the algorithms. For α_n defined Algorithms (3) and (9) that was required to satisfy same conditions with γ_n defined in our Algorithm 3.1, we used $\frac{1}{(50{,}000 \times n)+1}$ for all the three algorithms. Next, for β_n appearing in all the algorithms with the same condition, in Algorithms (9) and 3.1, we used 0.999; however, for Algorithm (3), the choice of $\beta_n = 0.5$ gave a better approximation so we used it for the algorithm and finally, to obtain the inertial parameter in our Algorithm 3.1 we chose $\theta = 0.999$, $\varepsilon_n = \frac{1}{(n+5)^2}$. We set the Halpern-vector (x or u) to be zero in all the algorithms. The iteration process was started with the initial points $x_0 = (2, 1, 3, 0, 0, 0, \ldots)$ and we observed the behavior of the algorithms as we varied x_1 to be: First $x_1 = (1, 1, 3, 0, 0, 0, \ldots)$ and Second $x_1 = (2, 0, 1, 0, 0, 0 \cdots)$. The iteration process was terminated when $\|x_n - x^*\| > 10^{-6}$ or $n > 1999$. The results of the experiment are presented in Table 2 and Fig. 3.

Discussion of the numerical results. From the numerical illustrations presented in Example 1, we observe that the iterates generated by Algorithm (3) of Takahashi et al. [44] fail to satisfy the stopping criterion before the prescribed maximum number of iterations was exhausted. While Algorithm (9) of Cholamjiak et al. [23] took 1275 iterations to satisfy the tolerance for the First initial point x_1 and 1244 for the Second x_1, it took our proposed Algorithm 3.1 just 422 for the First initial point x_1 and 423 for the Second x_1. Thus, in this example our proposed algorithm outperforms the algorithms of Takahashi et al. [44] and Cholamjiak et al. [23].

Table 2 Numerical results for the varied initial point x_1

n	$\|x_{n+1} - x^*\|$ for the First and Second x_1					
	Algorithm (3)		Algorithm (9)		Algorithm 3.1	
	First x_1	Second x_1	First x_1	Second x_1	First x_1	Second x_1
2	3.1401	2.2657	3.1245	2.2544	3.0667	2.2589
10	3.0021	2.1661	2.8568	2.0612	5.4641	12.1081
40	2.5844	1.8647	2.1195	1.5293	1.6811	3.7584
90	2.0135	1.4528	1.2887	0.9298	0.4827	0.9486
150	1.4923	1.0767	0.7094	0.5118	0.0977	0.2025
220	1.0521	0.7591	0.3535	0.2551	0.0093	0.0184
300	0.7057	0.5091	0.1594	0.1151	0.0005	0.0011
400	0.4283	0.3091	0.0589	0.0425	3.04E–05	6.97E–05
422	0.3837	0.2769	0.0473	0.0341	8.34E–06	1.19E–05
423	0.3818	0.2755	0.0469	0.0338	successful	7.29E–06
434	0.3799	0.2741	0.0464	0.0335	successful	successful
500	0.2600	0.1876	0.0218	0.0157	successful	successful
1000	0.0214	0.0154	0.0001	0.0001	successful	successful
1244	0.0063	0.0045	1.34E–05	9.94E–06	successful	successful
1275	0.0054	0.0039	9.92E–06	successful	successful	successful
1700	0.0006	0.0004	successful	successful	successful	successful
1999	0.0001	0.0001	successful	successful	successful	successful



(a) Graph of iterates generated using First and Second x_1, respectively

Figure 3 First 423 iterations of Algorithms (3), (9), and 3.1 illustrated graphically

4.4 Conclusion

This paper presents a modified inertial extension of the theorem of Cholamjiak [23] whose solutions are J-fixed points of relatively J-nonexpansive mappings. Applications of the theorem to convex minimization and image restoration are presented. Furthermore, some interesting numerical implementations of our proposed algorithm in solving image-recovery problems and an example on $l_{\frac{3}{2}}$ are presented. Finally, the performance of our proposed method is compared with that of Takahashi et al. [44] and Cholamjiak et al. [23] and from the numerical illustrations our proposed Algorithm 3.1 appears to be competitive and promising.

Acknowledgements

The authors will like to thank the referees for their esteemed comments and suggestions. The authors would like to dedicate this manuscript to the memory of the late Professor Charles Ejike Chidume who was part of the original draft of the manuscript. He passed away before we compiled the final version submitted to this journal. The first author acknowledges with thanks, the King Mongkut's University of Technology Thonburi's Postdoctoral Fellowship and Center of Excellence in Theoretical and Computational Science (TaCS-CoE) for their financial support.

Funding
This research was supported by the King Mongkut's University of Technology Thonburi's Postdoctoral Fellowship and the National Research Council of Thailand (NRCT) under Research Grants for Talented Mid-Career Researchers (Contract no. N41A640089).

Availability of data and materials
Not applicable.

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
The authors undersigned and gave their consent for the publication of their personal images used to be published in the above Journal and Article.

Competing interests
The authors declare no competing interests.

Author contributions
AA and PK formulated the problem and discussed the formulation with DK and AP. Analysis, proof of the main theorem, and the draft manuscript was written jointly by AA, Pk, DK, and AP. Proofreading and writing of the original manuscript were done jointly by AA and PK. Software and numerical simulations were done jointly by Dk and AP. Finally, PK secured the grant for the research.

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Publisher's Note

Received: 12 May 2022 Accepted: 31 January 2023 Published online: 14 April 2023

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