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# A remark on Jleli–Samet's best proximity point theorems for $\alpha$-$\psi$-contraction mappings

Pinya Ardsalee[1*]

*Correspondence: ardsalee.p@msu.ac.th
[1] Department of Mathematics, Mahasarakham University, Mahasarakham, Thailand

**Abstract**

Inspired by the work of Jachymski, we slightly extend some fixed point theorems with a graph and show that some best proximity point theorems for $\alpha$-$\psi$-contraction mappings of Jleli and Samet can be deduced by our results.

**Keywords:** Fixed point; Best proximity point; $\alpha$-proximal admissible

## 1 Introduction

Let $f$ be a mapping on a nonempty set $X$. We say that $z \in X$ is a *fixed point* of $f$ if $z = fz$. In 1922, Banach established one of the most famous fixed point theorems, namely the Banach contraction principle (see [2]), which has been generalized in many directions (for examples, see [1, 3, 6, 7]).

Let $A, B$ be nonempty subsets of a metric space $(X, d)$ and $T : A \to B$. We say that $z \in A$ is a *best proximity point* of $T$ if $d(z, Tz) = d(A, B)$. Note that if $A \cap B \neq \emptyset$, then a best proximity point becomes a fixed point.

In this paper, we slightly extend some fixed point theorems with a graph, which were introduced by Jachymski [4], and show that some best proximity point theorems for $\alpha$-$\psi$-contraction mappings of Jleli and Samet (see Theorems 3.1, 3.2 and 3.3 in [5]) can be deduced by our results.

The following two theorems were proved by Jachymski in 2008.

**Theorem J1** *Let $(X, d)$ be a complete metric space, $f : X \to X$ be a mapping and $G$ be a directed graph. Suppose that*

(A1) *for any sequence $\{x_n\}$ in $X$ if $\lim_n x_n = x$ for some $x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \geq 1$.*

*Assume that $f$ satisfies the followings*:

(1) *For all $x, y \in X$ if $(x, y) \in E(G)$, then $(fx, fy) \in E(G)$;*

(2) *There is $\lambda \in (0, 1)$ such that for all $x, y \in X$ if $(x, y) \in E(G)$, then $d(fx, fy) \leq \lambda d(x, y)$.*

*Then* $\mathrm{Fix}(f) := \{x : x = fx\} \neq \emptyset$ *if and only if* $X_f := \{x : (x, fx) \in E(G)\} \neq \emptyset$.

**Theorem J2** *Let $(X,d)$ be a complete metric space, $f : X \to X$ be a mapping and $G$ be a directed graph. Suppose that*

(A2) *$f$ is orbitally $G$-continuous.*

*Assume that $f$ satisfies the followings*:

(1) *For all $x, y \in X$ if $(x,y) \in E(G)$, then $(fx, fy) \in E(G)$;*

(2) *There is $\lambda \in (0,1)$ such that for all $x, y \in X$ if $(x,y) \in E(G)$, then $d(fx, fy) \le \lambda d(x,y)$.*

*Then* $\mathrm{Fix}(f) \ne \emptyset$ *if and only if* $X_f \ne \emptyset$.

## 2 Basics concepts and notations

Let $X$ be a nonempty set and $\Delta := \{(x,x) : x \in X\}$. In this paper, a directed graph $G$ on $X$ means the set of its vertices $V(G)$ is $X$ and the set of its edges $E(G)$ is a subset of $X \times X$ and we assume that $\Delta \subset E(G)$ and $G$ has no parallel edges.

Let $G$ be a directed graph. The *conversion* of $G$, denoted by $G^{-1}$, is the graph such that $V(G^{-1}) = V(G)$ and $E(G^{-1}) = \{(x,y) : (y,x) \in E(G)\}$. The undirected graph obtained from $G$, denoted by $\widetilde{G}$, is the graph such that $V(\widetilde{G}) = V(G)$ and $E(\widetilde{G}) = E(G) \cup E(G^{-1})$.

For $x, y \in V(G)$, a *path* in a directed graph $G$ from $x$ to $y$ of length $N$ is a sequence $\{x_i\}_{i=0}^N$ such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \ldots, N$.

A directed graph $G$ is *connected* if every pair of vertices has a path. A directed graph $G$ is *weakly connected* if $\widetilde{G}$ is connected.

The following definition was introduced by Jachymski.

**Definition 2.1** ([4]) Let $(X,d)$ be a metric space and $G$ be a directed graph. A mapping $f : X \to X$ is called *orbitally $G$-continuous* if for all $x, y \in X$ and any sequence $\{k_n\}$ of positive integers

$$\left.\begin{array}{l} \lim_n f^{k_n} x = y \\ \left(f^{k_n} x, f^{k_{n+1}} x\right) \in E(G) \quad \text{for all } n > 0 \end{array}\right\} \quad \Rightarrow \quad \lim_n f\left(f^{k_n} x\right) = fy.$$

By using the concept of orbitally $G$-continuity of $f$, we slightly extend Theorems J1 and J2 by weakening the continuity of $f$ and the completeness of $(X,d)$.

**Definition 2.2** Let $(X,d)$ be a metric space and $G$ be a directed graph. A mapping $f : X \to X$ is called *weakly orbitally $G$-continuous* if for all $x, y \in X$,

$$\left.\begin{array}{l} \lim_n f^{n} x = y \\ \left(f^{n} x, f^{n+1} x\right) \in E(G) \quad \text{for all } n > 0 \end{array}\right\} \quad \Rightarrow \quad \lim_n f\left(f^{n} x\right) = fy.$$

The following example shows that there is a weakly orbitally $G$-continuous mapping which is not orbitally $G$-continuous.

*Example* 2.1 Let $X = [0, \infty)$ with the usual metric $|\cdot|$. Let $G$ be a directed graph on $X$ with $E(G) = \Delta \cup \{(x,y) : x, y \in (0,1)\}$. Suppose $f : X \to X$ is a mapping defined by

$$f(x) = \begin{cases} 1/2 & \text{if } x = 0 \text{ or } x = 1, \\ 1/x & \text{if } 0 < x < 1, \\ 1/x^2 & \text{if } x > 1. \end{cases}$$

Since there is no $x \in X$ such that $(f^n x, f^{n+1}x) \in E(G)$ for all $n > 0$, we have that $f$ is weakly orbitally $G$-continuous.

Note that $\lim_n f^{2n+1}(1) = 0$ and $(f^{2n+1}(1), f^{2n+3}(1)) \in E(G)$ for all $n > 0$ but $\lim_n f(f^{2n+1}(1))$ does not exist. That is, $f$ is not orbitally $G$-continuous.

**Definition 2.3** Let $(X,d)$ be a metric space and $f : X \to X$ be a mapping. Let $G$ be a directed graph. We say that $(X,d)$ is *weakly $(f,G)$-orbitally complete* if for all $x \in X$,

$$\left.\begin{array}{l}\{f^n x\} \text{ is Cauchy} \\ (f^n x, f^{n+1}x) \in E(G) \quad \text{for all } n > 0\end{array}\right\} \quad \Rightarrow \quad \lim_n f^n x = y \quad \text{for some } y \in X.$$

## 3 Main results

We denote by $\Psi$ the set of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$.

**Lemma 3.1** *Let $\psi \in \Psi$. Then the followings hold*:

(1) $\psi(t) < t$ *for all* $t > 0$;

(2) $\psi(0) = 0$.

*Proof* (2) follows immediately from (1) and a proof of (1) can be found in [8]. □

**Lemma 3.2** *Let $(X,d)$ be a metric space with a directed graph $G$ and $f$ be a self-mapping on $X$. If* $\mathrm{Fix}(f) \neq \emptyset$, *then* $X_f \neq \emptyset$.

*Proof* Assume $\mathrm{Fix}(f) \neq \emptyset$. Let $z \in X$ such that $z = fz$. Since $E(G)$ contains all loops, we get $(z, fz) \in E(G)$, that is, $z \in X_f$. □

**Theorem 3.1** *Let $(X,d)$ be a metric space and $G$ be a directed graph. Suppose that*

(A1) *for any sequence $\{x_n\}$ in $X$ if $\lim_n x_n = x$ for some $x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \geq 1$.*

*Suppose that $f : X \to X$ and $\psi \in \Psi$ satisfy the followings*:

(1) *For all $x, y \in X$ if $(x,y) \in E(G)$, then $(fx, fy) \in E(G)$;*

(2) *For all $x, y \in X$ if $(x,y) \in E(G)$, then $d(fx, fy) \leq \psi(d(x,y))$.*

*Suppose that $X$ is weakly $(f,G)$-orbitally complete. Then* $\mathrm{Fix}(f) \neq \emptyset$ *if and only if* $X_f \neq \emptyset$.

*Proof* It follows immediately from Lemma 3.2 that $\mathrm{Fix}(f) \neq \emptyset$ implies $X_f \neq \emptyset$. On the other hand, we assume that there is $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$. For each $n \geq 1$, we have $(f^n x_0, f^{n+1}x_0) \in E(G)$ which implies that $d(f^{n+1}x_0, f^{n+2}x_0) \leq \psi(d(f^n x_0, f^{n+1}x_0))$. Since $\psi$ is nondecreasing, $d(f^n x_0, f^{n+1}x_0) \leq \psi^n(d(x_0, fx_0))$. Then

$$\sum_{n=1}^{\infty} d(f^n x_0, f^{n+1}x_0) \leq \sum_{n=1}^{\infty} \psi^n(d(x_0, fx_0)) < \infty$$

which implies that $\{f^n x_0\}$ is a Cauchy sequence. Note that $X$ is weakly $(f,G)$-orbitally complete. Therefore, there is $z \in X$ such that $\lim_n f^n x_0 = z$. By Condition (A1), there exists a subsequence $\{f^{n_k}x_0\}$ of $\{f^n x_0\}$ such that $(f^{n_k}x_0, z) \in E(G)$ for all $k \geq 1$. By Condition (2) and Lemma 3.1, it follows that

$$d(f^{n_k+1}x_0, fz) \leq \psi(d(f^{n_k}x_0, z)) \leq d(f^{n_k}x_0, z).$$

Therefore, $d(z, fz) = \lim_k d(f^{n_k+1}x_0, fz) \leq \lim_k d(f^{n_k}x_0, z) = 0$, that is, $z \in \mathrm{Fix}(f)$. □

**Theorem 3.2** *Let $(X, d)$ be a metric space and $G$ be a directed graph. Let $f : X \to X$ be a mapping such that*

(A2*) *$f$ is weakly orbitally $G$-continuous.*

*Let $\psi \in \Psi$. Suppose that $f$ and $\psi$ satisfy the followings*:

(1) *For all $x, y \in X$ if $(x, y) \in E(G)$, then $(fx, fy) \in E(G)$;*

(2) *For all $x, y \in X$ if $(x, y) \in E(G)$, then $d(fx, fy) \leq \psi(d(x, y))$.*

*Suppose that $X$ is weakly $(f, G)$-orbitally complete. Then $\mathrm{Fix}(f) \neq \emptyset$ if and only if $X_f \neq \emptyset$.*

*Proof* Let $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$. By following the proof of Theorem 3.1, we have that $\lim_n f^n x_0 = z$ for some $z \in X$. Since the condition (A2*) holds, we get $z = \lim_n f^{n+1}x_0 = fz$ and this completes the proof. □

*Remark* 3.1 Theorems 3.1 and 3.2 extend Theorems J1 and J2 as follows:

1. The orbitally $G$-continuity of $f$ in Theorem J2 implies that $f$ is weakly orbitally $G$-continuous in Theorem 3.2;
2. The completeness of $(X, d)$ implies that $X$ is weakly $(f, G)$-orbitally complete;
3. In Theorems J1 and J2, if we put $\psi(t) = \lambda t$ for all $t \in [0, \infty)$, then $\psi \in \Psi$.

Inspired by the work of Jachymski (see [4, Theorem 3.1]), the following theorem characterizes the uniqueness of a fixed point (if it exists) of a mapping in a metric space with a directed graph.

**Theorem 3.3** *Let $(X, d)$ be a metric space and $G$ be a directed graph. The followings are equivalent*:

(1) *$G$ is weakly connected;*

(2) *For any $f : X \to X$ with $\lim_n d(f^n x, f^n y) = 0$ whenever $(x, y) \in E(G)$, $\mathrm{card}(\mathrm{Fix}(f)) \leq 1$.*

*Proof* (1)⇒(2): We assume that $G$ is weakly connected. Let $f : X \to X$ be a mapping such that $\lim_n d(f^n x, f^n y) = 0$ whenever $(x, y) \in E(G)$. Let $u$ and $v$ be two fixed points of $f$, that is, $u = fu$ and $v = fv$. Since $G$ is weakly connected, there is a path $\{x_i\}_{i=0}^N$ of length $N$ such that

$$x_0 = u, \qquad x_N = v \quad \text{and} \quad (x_{i-1}, x_i) \in E(\widetilde{G}) \quad \text{for all } i = 1, 2, \ldots, N.$$

Therefore,

$$d(u, v) = \lim_n d(f^n u, f^n v) \leq \lim_n \sum_{i=1}^{N} d(f^n x_{i-1}, f^n x_i) = 0.$$

That is, $\mathrm{card}(\mathrm{Fix}(f)) \leq 1$.

(2)⇒(1): Suppose that $G$ is not weakly connected. Then there are $x, y \in X$ such that there is no path from $x$ to $y$ in $\widetilde{G}$. Note that $x \neq y$. We define $f : X \to X$ by for all $w \in X$

$$fw = \begin{cases} x & \text{if there exists a path in } \widetilde{G} \text{ from } w \text{ to } x; \\ y & \text{otherwise.} \end{cases}$$

Note that $x$ and $y$ are two different fixed points of $f$. Let $(u, v) \in E(G)$. Then $fu = fv = x$ or $fu = fv = y$ which implies that $\lim_n d(f^n u, f^n v) = 0$. This completes the proof. □

## 4 Discussion on best proximity fixed point results for $\alpha$-$\psi$-proximal contrative type mappings

In this section, we discuss some best proximity fixed point results for $\alpha$-$\psi$-proximal contrative type mappings of of Jleli and Samet in [5] that can be deduced by our results.

Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$. We recall the following notations:

$$d(A,B) := \inf\{d(a,b) : a \in A, b \in B\};$$
$$A_0 := \{a \in A : d(a,b) = d(A,B) \text{ for some } b \in B\};$$
$$B_0 := \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A\}.$$

**Definition 4.1** ([5]) Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$. Then the pair $(A, B)$ is said to have the *P-property* if

$$d(x_1, y_1) = d(x_2, y_2) = d(A,B) \quad \Rightarrow \quad d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

**Definition 4.2** ([5]) Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$. Let $T : A \to B$ and $\alpha : A \times A \to [0, \infty)$. We say that $T$ is *$\alpha$-proximal admissible* if

$$\left.\begin{array}{l} \alpha(x_1, x_2) \geq 1 \\ d(u_1, Tx_1) = d(u_2, Tx_2) = d(A,B) \end{array}\right\} \quad \Rightarrow \quad \alpha(u_1, u_2) \geq 1$$

for all $x_1, x_2, u_1, u_2 \in A$.

**Definition 4.3** ([5]) Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$. Let $T : A \to B$ and $\alpha : A \times A \to [0, \infty)$ and $\psi \in \Psi$. We say that $T$ is an $\alpha$-$\psi$-proximal contraction if

$$\alpha(x,y) d(Tx, Ty) \leq \psi\big(d(x,y)\big) \quad \text{for all } x, y \in A.$$

**Definition 4.4** ([5]) Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$. Let $T : A \to B$ and $\alpha : A \times A \to [0, \infty)$. We say that $T$ is *$(\alpha, d)$-regular* if for all $(x, y) \in \alpha^{-1}([0, 1))$, there exists $z \in A_0$ such that

$$\alpha(x,z) \geq 1 \quad \text{and} \quad \alpha(y,z) \geq 1.$$

The following three theorems were proved by Jleli and Samet in 2013.

**Theorem JS1** *Let $A$, $B$ be nonempty closed subsets of a complete metric space $(X,d)$ such that $A_0$ is nonempty. Let $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$. Suppose that $T : A \to B$ satisfies the followings*:

(B1) *$T(A_0) \subset B_0$ and $(A,B)$ satisfies the P-property*;
(B2) *$T$ is $\alpha$-proximal admissible*;
(B3) *$T$ is an $\alpha$-$\psi$-proximal contraction*;
(B4) *There are $u, v \in A_0$ such that $d(v,Tu) = d(A,B)$ and $\alpha(u,v) \geq 1$*;
(B5) *If $\{x_n\}$ is a sequence in $A$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n > 0$ and $\lim_n x_n \to x \in A$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k > 0$.*

*Then there is $z \in A_0$ such that $d(z,Tz) = d(A,B)$.*

**Theorem JS2** *Let $A$, $B$ be nonempty closed subsets of a complete metric space $(X,d)$ such that $A_0$ is nonempty. Let $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$. Suppose that $T : A \to B$ satisfies the followings*:

(B1) *$T(A_0) \subset B_0$ and $(A,B)$ satisfies the P-property*;
(B2) *$T$ is $\alpha$-proximal admissible*;
(B3) *$T$ is an $\alpha$-$\psi$-proximal contraction*;
(B4) *There are $u, v \in A_0$ such that $d(v,Tu) = d(A,B)$ and $\alpha(u,v) \geq 1$*;
(B5′) *$T$ is continuous.*

*Then there is $z \in A_0$ such that $d(z,Tz) = d(A,B)$.*

**Theorem JS3** *Let $A$, $B$ be nonempty subsets of a metric space $(X,d)$ such that $A_0$ is nonempty. Let $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$. Suppose that $T : A \to B$ satisfies the followings*:

(B1) *$T(A_0) \subset B_0$ and $(A,B)$ satisfies the P-property*;
(B2) *$T$ is $\alpha$-proximal admissible*;
(B3) *$T$ is an $\alpha$-$\psi$-proximal contraction.*

*If $T$ is $(\alpha,d)$-regular, then $T$ has at most one best proximity point.*

To show that Theorems JS1, JS2 and JS3 are the consequences of our Theorems 3.1, 3.2 and 3.3, respectively, we need the following lemmas.

**Lemma 4.1** *Let $(X,d)$ be a metric space. Let $A, B \subset X$ such that $A_0$ is nonempty and $(A,B)$ has the P-property. Suppose that $T : A \to B$ is a mapping such that $T(A_0) \subset B_0$. Then, for each $x \in A_0$, the set $\{u \in A_0 : d(u,Tx) = d(A,B)\}$ is a singleton set.*

*Proof* Let $x \in A_0$. Put $P := \{u \in A_0 : d(u,Tx) = d(A,B)\}$. Since $T(A_0) \subset B_0$, $P$ is nonempty. Let $u_1, u_2 \in P$. Then $d(u_1,Tx) = d(u_2,Tx) = d(A,B)$. Since $(A,B)$ satisfy the P-property, we get $d(u_1,u_2) = d(Tx,Tx) = 0$, that is, $u_1 = u_2$. □

**Lemma 4.2** *Let $A, B$ be nonempty subsets of a metric space $(X,d)$ such that $A_0$ is nonempty. Let $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$. Suppose that $T : A \to B$ satisfies the followings*:

(B1) *$T(A_0) \subset B_0$ and $(A,B)$ satisfies the P-property*;
(B2) *$T$ is $\alpha$-proximal admissible*;
(B3) *$T$ is an $\alpha$-$\psi$-proximal contraction.*

*Let $f : A_0 \to A_0$ be defined by for each $x \in A_0$*

$$fx = u \quad \text{where } u \in A_0 \text{ with } d(u, Tx) = d(A, B)$$

*and $G_0$ a directed graph defined by $V(G_0) = A_0$ and*

$$E(G_0) = \{(x, y) \in A_0 \times A_0 : \alpha(x, y) \geq 1\} \cup \{(x, x) : x \in A_0\}.$$

*Then $f$ and $G_0$ satisfy the followings*:

(1) *$f$ is well-defined*;

(2) *For all $x, y \in X$ if $(x, y) \in E(G_0)$, then $(fx, fy) \in E(G_0)$*;

(3) *For all $x, y \in X$ if $(x, y) \in E(G_0)$, then $d(fx, fy) \leq \psi(d(x, y))$*.

*Proof* It follows from Lemma 4.1 that $f$ is well-defined.

To see (2), let $(x, y) \in E(G_0)$. If $x = y$, then $fx = fy$ which implies that $(fx, fy) \in E(G_0)$. Otherwise, we assume that $\alpha(x, y) \geq 1$. Note that $d(fx, Tx) = d(fy, Ty) = d(A, B)$ and $T$ is $\alpha$-proximal admissible. We have $\alpha(fx, fy) \geq 1$, which implies that $(fx, fy) \in E(G_0)$.

To see (3), let $(x, y) \in E(G_0)$. If $x = y$, then $d(fx, fy) = 0 = \psi(d(x, y))$. Otherwise, we assume that $\alpha(x, y) \geq 1$. Note that $d(fx, Tx) = d(fy, Ty) = d(A, B)$. Since $T$ is an $\alpha$-$\psi$-proximal contraction and $(A, B)$ satisfies the $P$-property, we get $d(fx, fy) = d(Tx, Ty) \leq \alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$, as desired. □

**Lemma 4.3** *Let $A$, $B$ be nonempty closed subsets of a metric space $(X, d)$ such that $A_0$ is nonempty. Suppose that $T : A \to B$ satisfies $T(A_0) \subset B_0$ and $(A, B)$ satisfies the $P$-property. Let $f : A_0 \to A_0$ be defined by for each $x \in A_0$*

$$fx = u \quad \text{where } u \in A_0 \text{ with } d(u, Tx) = d(A, B).$$

*Then for each $z \in A_0$, $z = fz$ if and only if $d(z, Tz) = d(A, B)$.*

*Proof* It obtains immediately by the definition of $f$. □

The following is a proof of Theorem JS1 using our Theorem 3.1.

*Proof of Theorem* JS1 Let all assumptions in Theorem JS1 be satisfied. We define a mapping $f : A_0 \to A_0$ and a graph $G_0$ as in Lemma 4.2. Then $f$ and $G_0$ satisfy the conditions (1) and (2) in Theorem 3.1.

To see that the metric space $(A_0, d)$ with the directed graph $G_0$ satisfies Condition (A1) in Theorem 3.1, let $\{x_n\}$ be a sequence in $A_0$ with $\lim_n x_n = x$ for some $x \in A_0$ and $(x_n, x_{n+1}) \in E(G_0)$ for all $n \geq 1$. If $\{n : \alpha(x_n, x_{n+1}) \geq 1\}$ is finite, then there is $N \geq 1$ such that, for each $n \geq N$, $x_n = x$, that is, $(x_n, x) \in E(G_0)$. Otherwise, we assume that $\{n : \alpha(x_n, x_{n+1}) \geq 1\}$ is infinite. Note that $\alpha(x_n, x_{n+1}) \geq 1$ whenever $x_n \neq x_{n+1}$. Then there is a subsequence $\{\widetilde{x}_n\}$ of $\{x_n\}$ such that $\alpha(\widetilde{x}_n, \widetilde{x}_{n+1}) \geq 1$ for all $n \geq 1$. By the condition (B5), there is a subsequence $\{\widetilde{x}_{n_k}\}$ of $\{\widetilde{x}_n\}$ such that, for each $k \geq 1$, $\alpha(\widetilde{x}_{n_k}, x) \geq 1$, that is, $(\widetilde{x}_{n_k}, x) \in E(G_0)$.

We now show that $(A_0, d)$ is weakly $(f, G_0)$-orbitally complete. Let $x \in A_0$. Assume that $\{f^n x\}$ is Cauchy and $(f^n x, f^{n+1} x) \in E(G_0)$ for all $n \geq 1$. Since $X$ is complete and $A$ is closed, there is $z \in A$ such that $\lim_n f^n x = z$. If $\{n : \alpha(f^n x, f^{n+1} x) \geq 1\}$ is finite, then there is $N \geq 1$ such that $f^n x = z$ for all $n \geq N$, that is, $z \in A_0$. Otherwise,we assume that $\{n : \alpha(f^n x, f^{n+1}x) \geq 1\}$ is infinite. Using Condition (B5), there is a subsequence $\{f^{n_k}x\}$ of $\{f^n x\}$ such that $\alpha(f^{n_k}x, z) \geq 1$ for all $k \geq 1$. Then we have

$$d\big(Tf^{n_k}x, Tz\big) \leq \alpha\big(f^{n_k}x, z\big)d\big(Tf^{n_k}x, Tz\big) \leq \psi\big(d\big(f^{n_k}x, z\big)\big) \leq d\big(f^{n_k}x, z\big).$$

We get $\lim_k Tf^{n_k}x = Tz$. By the definition of $f$, $d(f^{n_k+1}x, Tf^{n_k}x) = d(A, B)$ for all $k \geq 1$. Therefore,

$$d(z, Tz) = \lim_k d\big(f^{n_k+1}x, Tf^{n_k}x\big) = d(A, B).$$

That is $z \in A_0$.

Finally, we show that $\{x : (x, fx) \in E(G_0)\} \neq \emptyset$. Since $d(v, Tu) = d(A, B)$ and $\alpha(u, v) \geq 1$, we have $v = fu$ and hence $(u, fu) \in E(G_0)$. By using our Theorem 3.1, there is $p \in A_0$ such that $p = fp$. By Lemma 4.3, we have $d(p, Tp) = d(A, B)$, as desired. □

The following is a proof of Theorem JS2 via Theorem 3.2.

*Proof of Theorem* JS2 Let all assumptions in Theorem JS2 be satisfied. We define a mapping $f : A_0 \to A_0$ and a graph $G_0$ as in Lemma 4.2. Then we have $f$ and $G_0$ satisfy (1) and (2) in Theorem 3.2.

To see that $f$ satisfies Condition (A2*) in Theorem 3.2, let $x, y \in A_0$ such that $\lim_n f^n x = y$ and $(f^n x, f^{n+1}x) \in E(G_0)$ for all $n \geq 1$. Note that $d(fw, Tw) = d(A, B)$ for all $w \in A_0$. Since $(A, B)$ has the P-property and $T$ is continuous,

$$\lim_n d\big(f\big(f^n x\big), fy\big) = \lim_n d\big(T\big(f^n x\big), Ty\big) = 0.$$

Finally, we show that $(A_0, d)$ is weakly $(f, G_0)$-orbitally complete. Let $x \in A_0$. Assume that $\{f^n x\}$ is Cauchy and $(f^n x, f^{n+1}x) \in E(G_0)$ for all $n \geq 1$. Since $X$ is complete and $A$ is closed, there is $z \in A$ such that $\lim_n f^n x = z$. By the continuity of $T$, we have $\lim_n T(f^n x) = Tz$. This implies that

$$d(z, Tz) = \lim_n d\big(f\big(f^n x\big), T\big(f^n x\big)\big) = d(A, B).$$

That is $z \in A_0$. Note that Condition (B4) implies $(u, fu) \in E(G_0)$. By using Theorem 3.2, there is $p \in A_0$ such that $p = fp$ which implies that $d(p, Tp) = d(A, B)$. □

Finally, we show a proof of Theorem JS3 by using our Theorem 3.3.

*Proof of Theorem* JS3 We assume that all the assumptions hold and suppose that $T$ is $(\alpha, d)$-regular. We define the mapping $f : A_0 \to A_0$ and the directed graph $G_0$ as in Lemma 4.2. Since $T$ is $(\alpha, d)$-regular, we obtain immediately that $G_0$ is weakly connected. Note that for each $(x, y) \in E(G_0)$,

$$(fx, fy) \in E(G_0) \quad \text{and} \quad d(fx, fy) \leq \psi\big(d(x, y)\big).$$

Therefore,

$$\lim_{n} d\left(f^{n}x, f^{n}y\right) \leq \lim_{n} \psi^{n}\left(d(x,y)\right) = 0.$$

By Theorem 3.3, we have $\operatorname{card}(\operatorname{Fix}(f)) \leq 1$. Using Lemma 4.3, $T$ has at most one best proximity point. The proof is complete. □

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