Wong et al. *Fixed Point Theory Algorithms Sci Eng* (2023) 2023:8
https://doi.org/10.1186/s13663-023-00746-x

Fixed Point Theory and Algorithms for Sciences and Engineering

RESEARCH Open Access

Fixed-point results for fuzzy generalized β-F-contraction mappings in fuzzy metric spaces and their applications

Koon S. Wong[1], Zabidin Salleh[1*] and Che M.I. Che Taib[1]

*Correspondence: zabidin@umt.edu.my
[1]Special Interest Group on Modelling and Data Analytics, Faculty of Ocean Engineering Technology and Informatics, University of Terengganu, 21030 Kuala Nerus, Terengganu, Malaysia

Abstract

In this paper, we introduce fuzzy generalized β-F-contractions as a generalization of fuzzy F-contractions with admissible mappings. We deduce sufficient conditions for the existence and uniqueness of fixed points for fuzzy generalized β-F-contractions in complete strong fuzzy metric spaces. Our results generalize several fixed-point results from the literature. We present an application of our main result.

Keywords: Fixed point; Fuzzy metric spaces; Fuzzy generalized β-F-contractions; β-admissible mappings

1 Introduction and preliminary results

Fixed point theory was initiated by Banach [1] with his celebrated Banach contraction principle in 1922. It has been generalized into many directions by mathematicians (see [2]) and served as an important tool to solve problems in various fields (see [3–5], and [6]). Inspired by the concept of fuzzy set by Zadeh [7], Kramosil and Michálek [8] defined fuzzy metric spaces as a generalization of probabilistic metric spaces. Later, George and Veeramani [9] modified the notion of a fuzzy metric space to obtain Hausdorff topology. The study of fixed-point theorems in fuzzy metric space was initiated by Grabiec [10]. He introduced a fuzzy version of the Banach contraction principle in a fuzzy metric space. Subsequently, mathematicians proposed various contractive conditions and studied the existence of fixed points of these contractions in fuzzy metric spaces. For example, see [11–14], and [15].

In 2012, Wardowski [16] introduced the concept of an F-contraction as a new type of contraction in complete metric spaces. After that, several mathematicians extended F-contractions to various spaces (see [17–23], and [24]). Recently, Huang et al. [25] introduced fuzzy F-contractions, a generalization of F-contractions with simpler conditions in fuzzy metric spaces and presented some fixed-point theorems for fuzzy F-contractions. Recent works related to fuzzy F-contractions can be found in [26–28], and references therein. Samet et al. [29] introduced the notion of an α-admissible contractive mapping and proved some fixed-point theorems in the setting of metric spaces. Gopal and Vetro

[30] adopted the idea and introduced α-admissible and β-admissible fuzzy contractive mappings, which generalized some fixed-point results in fuzzy metric spaces.

In this paper, motivated by the works of Gopal and Vetro [30] and Huang et al. [25], we introduce generalized fuzzy F-contractions with admissible property in a fuzzy metric space. We prove some fixed point theorems for such types of contractive mappings and provide an application to illustrate our results.

Before we proceed to our main results, we recall some definitions and notions that will be used throughout the rest of the paper.

Definition 1 ([31]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (t-norm for short) if the following conditions hold:

T1. $a * 1 = a$ for all $a \in [0,1]$;

T2. $*$ is associative and commutative;

T3. $a * b \le c * d$ whenever $a \le c$ and $b \le d$, where $a,b,c,d \in [0,1]$;

T4. $*$ is continuous.

Some commonly seen continuous t-norms are $\min\{a,b\}$, $a \cdot b$, and $\max\{a+b-1,0\}$.

Definition 2 ([9]) The 3-tuple $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t-norm, and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying the following conditions:

FMS1. $M(x,y,t) > 0$;

FMS2. $M(x,y,t) = 1$ if and only if $x = y$;

FMS3. $M(x,y,t) = M(y,x,t)$;

FMS4. $M(x,z,t+s) \ge M(x,y,t) * M(y,z,s)$;

FMS5. $M(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous

for all $x,y \in X$.

In the definition above, if we replace the triangular inequality with the condition

FMS4*. $M(x,z,t) \ge M(x,y,t) * M(y,z,t)$ for all $x,y,z \in X$ and $t > 0$,

then $(X,M,*)$ is called a strong fuzzy metric space.

Definition 3 ([9]) Let $(X,M,*)$ be a fuzzy metric space, and let $\{x_n\}$ be a sequence in X. Then

1. $\{x_n\}$ is convergent if there exists $x \in X$ such that $\lim_{n\to\infty} M(x_n,x,t) = 1$ for all $t > 0$;
2. $\{x_n\}$ is a Cauchy sequence if for all $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n,x_m,t) > 1-\varepsilon$ for all $n,m \ge n_0$;
3. $(X,M,*)$ is complete if every Cauchy sequence is convergent.

Remark 1 George and Veeramani [9] mentioned that for a fuzzy metric space $(X,M,*)$, $\{x_n\}$ converges to $x \in X$ if and only if $\lim_{n\to\infty} x_n = x$.

For the rest of this paper, we denote by $\mathcal{F}$ the class of all mappings $F : [0,1] \to \mathbb{R}$ such that for all $x,y \in [0,1]$, $x < y$ implies $Fx < Fy$. In other words, F is strictly increasing on $[0,1]$.

Definition 4 ([25]) Let $(X,M,*)$ be a fuzzy metric space, and let $F \in \mathcal{F}$. The mapping $g : X \to X$ is said to be a fuzzy F-contraction if there exists $\tau \in (0,1)$ such that

$$\tau \cdot F\big(M(gx,gy,t)\big) \geq F\big(M(x,y,t)\big)$$

for all $x,y \in X$ such that $x \neq y$ and $t > 0$.

Definition 5 ([30]) A mapping $g : X \to X$ is said to be a β-admissible mapping if there exists a function $\beta : X \times X \times (0,\infty) \to (0,\infty)$ such that for all $x,y \in X$ and $t > 0$,

$$\beta(x,y,t) \leq 1 \quad \Rightarrow \quad \beta(gx,gy,t) \leq 1.$$

The rest of the paper is organized as follows. In Sect. 2, we present our main results of fuzzy generalized β-F-contraction mappings in a fuzzy metric space together with an example. Next, we provide an application of our result in finding the solution of the integral equation in Sect. 3. Finally, Sect. 4 concludes the paper.

2 Main results

We start this section by defining fuzzy generalized β-F-contraction.

Definition 6 Let $(X,M,*)$ be a fuzzy metric space. Let $\beta : X \times X \times (0,\infty) \to (0,\infty)$ be a function, and let $F \in \mathcal{F}$. A function $g : X \to X$ is said to be a fuzzy generalized β-F-contraction if there exists $\tau \in (0,1)$ such that for all $x,y \in X$, $x \neq y$, and $t > 0$,

$$\tau \cdot \beta(x,y,t) \cdot F\big(M(gx,gy,t)\big) \geq F\big(N(x,y,t)\big), \tag{1}$$

where $N(x,y,t) = \min\{M(x,y,t), M(x,gx,t), M(y,gy,t)\}$.

Definition 7 Let $(X,M,*)$ be a fuzzy metric space. Let $\beta : X \times X \times (0,\infty) \to (0,\infty)$ be a function, and let $F \in \mathcal{F}$. A function $g : X \to X$ is said to be a fuzzy β-F-contraction if there exists $\tau \in (0,1)$ such that for all $x,y \in X$, $x \neq y$, and $t > 0$,

$$\tau \cdot \beta(x,y,t) \cdot F\big(M(gx,gy,t)\big) \geq F\big(M(x,y,t)\big). \tag{2}$$

If in Definition 6 we take $N(x,y,t) = M(x,y,t)$ for all $x,y \in X$ and $t > 0$, then we obtain Definition 7. Therefore we can say that a fuzzy β-F-contraction is a fuzzy generalized β-F-contraction. However, the converse is false. Furthermore, if we let $\beta(x,y,t) = 1$ for all $x,y \in X$, $x \neq y$, and $t > 0$, then we obtain Definition 4.

Theorem 1 *Let $(X,M,*)$ be a complete strong fuzzy metric space, and let $g : X \to X$ be a fuzzy generalized β-F-contraction, where $F \in \mathcal{F}$. Assume that the following conditions hold*:

1. *g is β-admissible*;
2. *there exists $x_0 \in X$ such that $\beta(x_0,gx_0,t) \leq 1$ for all $t > 0$*;
3. *for each sequence $\{x_n\}$ in X such that $\beta(x_n,x_{n+1},t) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$, there exists $k_0 \in \mathbb{N} \cup \{0\}$ such that $\beta(x_m,x_n,t) \leq 1$ for all $n > m \geq k_0$ and $t > 0$*;

4. *if* $\{x_n\}$ *is a sequence in* X *such that* $\beta(x_n, x_{n+1}, t) \leq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $t > 0$ *and if* $x_n \to x$ *as* $n \to \infty$, *then* $\beta(x_n, x, t) \leq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then g *has a fixed point.*

Proof Let $x_0 \in X$ be such that $\beta(x_0, gx_0, t) \leq 1$ for all $t > 0$. We define a sequence $\{x_n\}$ by $x_{n+1} = gx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If there exists $n \in \mathbb{N} \cup \{0\}$ such that $x_{n+1} = x_n$, then x_n is a fixed point of g, and the proof is complete. Now assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\beta(x_0, gx_0, t) = \beta(x_0, x_1, t) \leq 1$ for all $t > 0$ and g is β-admissible mapping, this implies that

$$\beta(gx_0, gx_1, t) = \beta(x_1, x_2, t) \leq 1.$$

Continuing this process, we conclude that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$. Since g is a fuzzy generalized β-F-contraction, there exists $\tau \in (0, 1)$ satisfying (1). It is clear that $\tau\beta(x_n, x_{n+1}, t) < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$. Now substituting $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} F\big(M(x_n, x_{n+1}, t)\big) &= F\big(M(gx_{n-1}, gx_n, t)\big) > \tau\beta(x_{n-1}, x_n, t)F\big(M(gx_{n-1}, gx_n, t)\big) \\ &\geq F\big(N(x_{n-1}, x_n, t)\big) \end{aligned}$$

for all $t > 0$, where

$$\begin{aligned} N(x_{n-1}, x_n, t) &= \min\big\{M(x_{n-1}, x_n, t), M(x_{n-1}, gx_{n-1}, t), M(x_n, gx_n, t)\big\} \\ &= \min\big\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\big\} \\ &= \min\big\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\big\}. \end{aligned}$$

If $\min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\} = M(x_n, x_{n+1}, t)$, then

$$F\big(M(x_n, x_{n+1}, t)\big) > \tau\beta(x_{n-1}, x_n, t)F\big(M(x_n, x_{n+1}, t)\big) \geq F\big(M(x_n, x_{n+1}, t)\big),$$

which leads to a contradiction. So we have $\min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\} = M(x_{n-1}, x_n, t)$ and

$$F\big(M(x_n, x_{n+1}, t)\big) > \tau\beta(x_{n-1}, x_n, t)F\big(M(x_n, x_{n+1}, t)\big) \geq F\big(M(x_{n-1}, x_n, t)\big), \tag{3}$$

which implies

$$F\big(M(x_n, x_{n+1}, t)\big) > F\big(M(x_{n-1}, x_n, t)\big).$$

Since F is a strictly increasing function, we get $M(x_n, x_{n+1}, t) > M(x_{n-1}, x_n, t)$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$. Hence for all $t > 0$, the sequence $\{M(x_n, x_{n+1}, t)\}$ is strictly increasing in the interval $[0, 1]$ and bounded above. This implies that $\{M(x_n, x_{n+1}, t)\}$ is convergent, that is, for any $t > 0$, there exists $A(t) \in [0, 1]$ such that $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = A(t)$. We claim that $A(t) = 1$ for all $t > 0$. Suppose that $A(t_0) < 1$ for some $t_0 > 0$. Taking the limit in both sides of (3), we obtain

$$F\big(A(t_0)\big) > \tau\beta(x_{n-1}, x_n, t_0)F\big(A(t_0)\big) \geq F\big(A(t_0)\big),$$

which leads to a contraction. Therefore we have

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1 \tag{4}$$

for all $t > 0$. Now we will show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then by condition 3 there exist $k_0 \in \mathbb{N} \cup \{0\}$, $\varepsilon > 0$, two subsequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ of $\{x_n\}$, and $s > 0$ such that $n_k > m_k \geq k$ for all $k \in \mathbb{N} \cup \{0\}$ with $k \geq k_0$ and

$$M(x_{n_k}, x_{m_k}, s) \leq 1 - \varepsilon \quad \text{and} \quad \beta(x_{n_k}, x_{m_k}, t) \leq 1.$$

Let n_k be the smallest integer that satisfies the above inequality. Then

$$M(x_{n_k-1}, x_{m_k}, s) > 1 - \varepsilon.$$

Since $(X, M, *)$ is a strong fuzzy metric space, by (FMS4*) we have

$$\begin{aligned} 1 - \varepsilon \geq M(x_{n_k}, x_{m_k}, s) &\geq M(x_{n_k}, x_{n_k-1}, s) * M(x_{n_k-1}, x_{m_k}, s) \\ &\geq M(x_{n_k}, x_{n_k-1}, s) * (1 - \varepsilon). \end{aligned}$$

Taking the limit as $k \to \infty$, by (4) and (T1) we obtain

$$\begin{aligned} 1 - \varepsilon &\geq \lim_{k\to\infty} M(x_{n_k}, x_{m_k}, s) \geq 1 * (1 - \varepsilon) \\ &= 1 - \varepsilon. \end{aligned}$$

This implies that $\lim_{k\to\infty} M(x_{n_k}, x_{m_k}, s) = 1 - \varepsilon$. Again, by (FMS4*) we get

$$M(x_{n_k+1}, x_{m_k+1}, s) \geq M(x_{n_k+1}, x_{n_k}, s) * M(x_{n_k}, x_{m_k}, s) * M(x_{m_k}, x_{m_k+1}, s).$$

Letting $k \to \infty$, by (4) and (T1) we have

$$\begin{aligned} \lim_{k\to\infty} M(x_{n_k+1}, x_{m_k+1}, s) &= 1 * (1 - \varepsilon) * 1 \\ &= 1 - \varepsilon. \end{aligned}$$

Since $\tau < 1$ and $\beta(x_{n_k}, x_{m_k}, t) \leq 1$ for all $k \in \mathbb{N} \cup \{0\}$ with $k \geq k_0$, this implies that $\tau\beta(x_{n_k}, x_{m_k}, t) < 1$ for all $k \in \mathbb{N} \cup \{0\}$ such that $k \geq k_0$. Using (1), we get

$$\begin{aligned} F\bigl(M(x_{n_k+1}, x_{m_k+1}, s)\bigr) &> \tau\beta(x_{n_k}, x_{m_k}, s) F\bigl(M(x_{n_k+1}, x_{m_k+1}, s)\bigr) \\ &\geq F\bigl(N(x_{n_k}, x_{m_k}, s)\bigr), \end{aligned} \tag{5}$$

where

$$N(x_{n_k}, x_{m_k}, s) = \min\bigl\{(M(x_{n_k}, x_{m_k}, s), M(x_{n_k}, x_{n_k+1}, s), M(x_{m_k}, x_{m_k+1}, s)\bigr\}.$$

Letting $k \to \infty$, we get

$$\begin{aligned}\lim_{k\to\infty} N(x_{n_k}, x_{m_k}, s) &= \min\Big\{\lim_{k\to\infty} M(x_{n_k}, x_{m_k}, s), \lim_{k\to\infty} M(x_{n_k}, x_{n_k+1}, s),\\ &\quad \lim_{k\to\infty} M(x_{m_k}, x_{m_k+1}, s)\Big\}\\ &= \min\{1-\varepsilon, 1, 1\}\\ &= 1-\varepsilon.\end{aligned}$$

Therefore letting $k \to \infty$ in both sides of (5), we obtain

$$F(1-\varepsilon) \geq \tau\beta(x_{n_k}, x_{m_k}, s)F(1-\varepsilon) \geq F(1-\varepsilon).$$

This implies that $F(1-\varepsilon) = 0$, which contradicts the fact that $F(1-\varepsilon) > 0$. Thus $\{x_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is complete, there exists $x \in X$ such that $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$ or, in other words, $\lim_{n\to\infty} x_n = x$. Since $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$, by condition 4 we have $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Now we will show that x is a fixed point of g. Suppose that $gx \neq x$, i.e., $M(x, gx, t) < 1$. Then by (1) we obtain

$$F\big(M(gx_n, gx, t)\big) > \tau\beta(x_n, x, t)F\big(M(gx_n, gx, t)\big) \geq F\big(N(x_n, x, t)\big)$$

for all $t > 0$, where

$$\begin{aligned}N(x_n, x, t) &= \min\big\{M(x_n, x, t), M(x_n, gx_n, t), M(x, gx, t)\big\}\\ &= \min\big\{M(x_n, x, t), M(x_n, x_{n+1}, t), M(x, gx, t)\big\}.\end{aligned}$$

Letting $n \to \infty$ in this inequality, we have

$$\begin{aligned}F\big(M(x, gx, t)\big) &> \tau\beta(x_n, x, t)F\big(M(x, gx, t)\big)\\ &\geq F\big(\min\big\{1, 1, M(x, gx, t)\big\}\big) = F\big(M(x, gx, t)\big),\end{aligned}$$

which leads to a contradiction. Thus $gx = x$, which means that x is a fixed point of g. □

Corollary 1 *Let $(X, M, *)$ be a complete strong fuzzy metric space, and let $g : X \to X$ be a fuzzy β-F-contraction, where $F \in \mathcal{F}$. Assume that the following conditions hold*:

1. *g is β-admissible*;
2. *there exists $x_0 \in X$ such that $\beta(x_0, gx_0, t) \leq 1$ for all $t > 0$*;
3. *for each sequence $\{x_n\}$ in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$, there exists $k_0 \in \mathbb{N} \cup \{0\}$ such that $\beta(x_m, x_n, t) \leq 1$ for all $n > m \geq k_0$ and $t > 0$*;
4. *if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$, $t > 0$ and if $x_n \to x$ as $n \to \infty$, then $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then g has a fixed point.

Proof Letting $N(x, y, t) = M(x, y, t)$ in (1), the remaining proof follows the same lines as Theorem 1. □

Remark 2 Huang et al. [25] mentioned that their results are applicable to strong fuzzy metric spaces. Therefore, if we let $\beta(x,y,t) = 1$ for all $x,y \in X$ and $t > 0$ in the corollary above, then we obtain the main results of Huang et al. [25] in the context of strong fuzzy metric spaces. Hence we successfully generalize some fixed point results in the existing literature.

Now we provide examples to illustrate Theorem 1.

Example 1 Let X be a set of positive real numbers. Let $*$ be a product continuous t-norm, that is, $a * b = a \cdot b$ for all $a, b \in X$. We define a fuzzy set as follows:

$$M(x,y,t) = \begin{cases} \frac{x}{y}, & x \leq y, \\ \frac{y}{x}, & y \leq x, \end{cases}$$

for $x, y \in X$ and $t > 0$. We can easily verify that $(X, M, *)$ is a complete strong fuzzy metric space. Let $g(x) = 1/x$ for all $x \in X$, and let $F(y) = \ln(y)$ for $0 < y \leq 1$ and $\tau = 1/2$. Also, define

$$\beta(x,y,t) = \begin{cases} \frac{1}{2}, & x \leq y, \\ 1, & y \leq x. \end{cases}$$

First of all, we show that g is a β-admissible mapping. Let $x, y \in X$ be such that $\beta(x,y,t) = 1/2 \leq 1$. Then $x \leq y$, which implies that $g(x) = 1/x \geq 1/y = g(y)$. It follows that $\beta(gx, gy, t) = 1 \leq 1$. Now suppose $\beta(x,y,t) = 1 \leq 1$, which means that $y \leq x$, and thus $g(y) = 1/y \geq 1/x = g(x)$. It follows that $\beta(gx, gy, t) = 1/2 \leq 1$. Thus g is a β-admissible mapping. Next, we proceed to check whether the contractive condition of Theorem 1 is satisfied or not. Let $x, y \in X$. We will consider the following cases.

Case 1 $x \leq y$ where $x \geq 1$. We have $1 \leq x \leq y$, $gx = \frac{1}{x} \leq x$, $gy = \frac{1}{y} \leq y$, $x^2 \leq y^2$, and

$$\begin{aligned} N(x,y,t) &= \min\bigl\{M(x,y,t), M(x,gx,t), M(y,gy,t)\bigr\} \\ &= \min\left\{\frac{x}{y}, \frac{\frac{1}{x}}{x}, \frac{\frac{1}{y}}{y}\right\} \\ &= \min\left\{\frac{x}{y}, \frac{1}{x^2}, \frac{1}{y^2}\right\} \\ &= \frac{1}{y^2}. \end{aligned}$$

This implies that

$$\begin{aligned} \tau \cdot \beta(x,y,t) \cdot F\bigl(M(gx,gy,t)\bigr) &= \tau \cdot \beta(x,y,t) \cdot F\left(\frac{\frac{1}{y}}{\frac{1}{x}}\right) \\ &= \tau \cdot \beta(x,y,t) \cdot F\left(\frac{x}{y}\right) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \ln\left(\frac{x}{y}\right) \end{aligned}$$

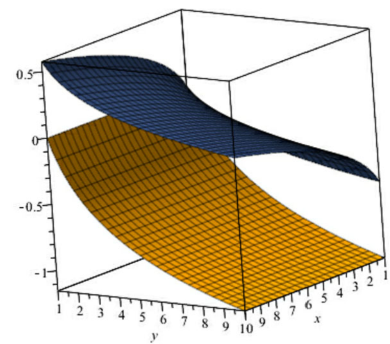


Figure 1 The plane in blue color represents the left-hand side of the inequality, and the plane in orange color represents the right-hand side of the inequality

$$
\begin{aligned}
&= \frac{1}{4}\ln\left(\frac{x}{y}\right) \\
&\geq \ln\left(\frac{x}{y}\right) \\
&\geq \ln\left(\frac{x}{y^2}\right) \\
&\geq \ln\left(\frac{1}{y^2}\right).
\end{aligned}
$$

Figure 1 shows the illustration of Case 1 on 3D view.

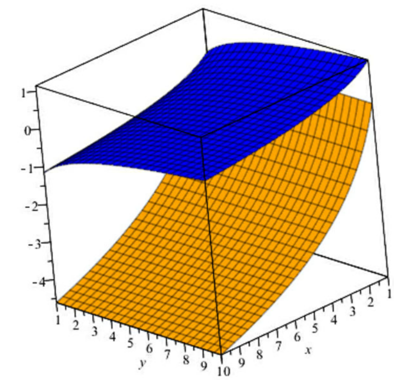
Case 2 $y \leq x$ where $y \geq 1$. We have $1 \leq y \leq x$, $gx = \frac{1}{x} \leq x$, $gy = \frac{1}{y} \leq y$, $y^2 \leq x^2$, and

$$
\begin{aligned}
N(x,y,t) &= \min\bigl\{M(x,y,t), M(x,gx,t), M(y,gy,t)\bigr\} \\
&= \min\left\{\frac{y}{x}, \frac{\frac{1}{x}}{x}, \frac{\frac{1}{y}}{y}\right\} \\
&= \min\left\{\frac{y}{x}, \frac{1}{x^2}, \frac{1}{y^2}\right\} \\
&= \frac{1}{x^2}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\tau \cdot \beta(x,y,t) \cdot F\bigl(M(gx,gy,t)\bigr) &= \tau \cdot \beta(x,y,t) \cdot F\left(\frac{\frac{1}{x}}{\frac{1}{y}}\right) \\
&= \tau \cdot \beta(x,y,t) \cdot F\left(\frac{y}{x}\right) \\
&= \frac{1}{2} \cdot 1 \cdot \ln\left(\frac{y}{x}\right) \\
&= \frac{1}{2}\ln\left(\frac{y}{x}\right) \\
&\geq \ln\left(\frac{y}{x}\right)
\end{aligned}
$$

Figure 2 The plane in blue color represents the left-hand side of the inequality, and the plane in orange color represents the right-hand side of the inequality



$$\geq \ln\left(\frac{y}{x^2}\right)$$
$$\geq \ln\left(\frac{1}{x^2}\right).$$

Figure 2 shows the illustration of Case 2 on 3D view.

Case 3 $x \leq y$ where $y < 1$. We have $x \leq y < 1$, $gx = \frac{1}{x} \geq x$, $gy = \frac{1}{y} \geq y$, $y^2 \geq x^2$, and

$$\begin{aligned} N(x,y,t) &= \min\left\{M(x,y,t), M(x,gx,t), M(y,gy,t)\right\} \\ &= \min\left\{\frac{x}{y}, \frac{x}{\frac{1}{x}}, \frac{y}{\frac{1}{y}}\right\} \\ &= \min\left\{\frac{x}{y}, x^2, y^2\right\} \\ &= x^2. \end{aligned}$$

This implies that

$$\begin{aligned} \tau \cdot \beta(x,y,t) \cdot F\big(M(gx,gy,t)\big) &= \tau \cdot \beta(x,y,t) \cdot F\left(\frac{\frac{1}{y}}{\frac{1}{x}}\right) \\ &= \tau \cdot \beta(x,y,t) \cdot F\left(\frac{x}{y}\right) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \ln\left(\frac{x}{y}\right) \\ &= \frac{1}{4} \ln\left(\frac{x}{y}\right) \\ &\geq \ln\left(\frac{x}{y}\right) \\ &\geq \ln\left(\frac{x^2}{y}\right) \\ &\geq \ln\big(x^2\big). \end{aligned}$$

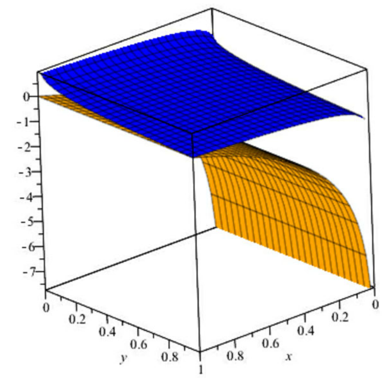


Figure 3 The plane in blue color represents the left-hand side of the inequality, and the plane in orange color represents the right-hand side of the inequality

Figure 3 shows the illustration of Case 3 on 3D view.

Case 4 $y \leq x$ where $x < 1$. We have $y \leq x < 1$, $gx = \frac{1}{x} \geq x$, $gy = \frac{1}{y} \geq y$, $x^2 \geq y^2$ and

$$\begin{aligned} N(x,y,t) &= \min\{M(x,y,t), M(x,gx,t), M(y,gy,t)\} \\ &= \min\left\{\frac{y}{x}, \frac{x}{\frac{1}{x}}, \frac{y}{\frac{1}{y}}\right\} \\ &= \min\left\{\frac{y}{x}, x^2, y^2\right\} \\ &= y^2. \end{aligned}$$

This implies that

$$\begin{aligned} \tau \cdot \beta(x,y,t) \cdot F\big(M(gx,gy,t)\big) &= \tau \cdot \beta(x,y,t) \cdot F\left(\frac{\frac{1}{x}}{\frac{1}{y}}\right) \\ &= \tau \cdot \beta(x,y,t) \cdot F\left(\frac{y}{x}\right) \\ &= \frac{1}{2} \cdot 1 \cdot \ln\left(\frac{y}{x}\right) \\ &= \frac{1}{2} \ln\left(\frac{y}{x}\right) \\ &\geq \ln\left(\frac{y}{x}\right) \\ &\geq \ln\left(\frac{y^2}{x}\right) \\ &\geq \ln(y^2). \end{aligned}$$

Figure 4 shows the illustration of Case 4 on 3D view.

Case 5 $x \leq y$ where $x < 1$ and $y \geq 1$. We have $gx = \frac{1}{x} \geq 1 > x$, $gy = \frac{1}{y} \leq 1 \leq y$, and

$$N(x,y,t) = \min\{M(x,y,t), M(x,gx,t), M(y,gy,t)\}$$

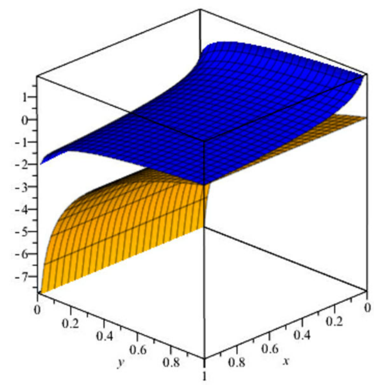


Figure 4 The plane in blue color represents the left-hand side of the inequality, and the plane in orange color represents the right-hand side of the inequality

$$\begin{aligned} &= \min\left\{\frac{x}{y}, \frac{x}{\frac{1}{x}}, \frac{\frac{1}{y}}{y}\right\} \\ &= \min\left\{\frac{x}{y}, x^2, \frac{1}{y^2}\right\}. \end{aligned}$$

This implies that

$$\begin{aligned} \tau \cdot \beta(x,y,t) \cdot F\bigl(M(gx,gy,t)\bigr) &= \tau \cdot \beta(x,y,t) \cdot F\left(\frac{\frac{1}{y}}{\frac{1}{x}}\right) \\ &= \tau \cdot \beta(x,y,t) \cdot F\left(\frac{x}{y}\right) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \ln\left(\frac{x}{y}\right) \\ &= \frac{1}{4} \ln\left(\frac{x}{y}\right) \\ &\geq \ln\left(\frac{x}{y}\right) \\ &\geq \ln\bigl(N(x,y,t)\bigr). \end{aligned}$$

Figure 5 shows the illustration of Case 5 on 3D view.

Case 6 $y \leq x$ where $y < 1$ and $x \geq 1$. We have $gx = \frac{1}{x} \leq 1 \leq x$, $gy = \frac{1}{y} \geq 1 > y$, and

$$\begin{aligned} N(x,y,t) &= \min\bigl\{M(x,y,t), M(x,gx,t), M(y,gy,t)\bigr\} \\ &= \min\left\{\frac{y}{x}, \frac{\frac{1}{x}}{x}, \frac{y}{\frac{1}{y}}\right\} \\ &= \min\left\{\frac{y}{x}, \frac{1}{x^2}, y^2\right\}. \end{aligned}$$

This implies that

$$\tau \cdot \beta(x,y,t) \cdot F\bigl(M(gx,gy,t)\bigr) = \tau \cdot \beta(x,y,t) \cdot F\left(\frac{\frac{1}{x}}{\frac{1}{y}}\right)$$

Figure 5 The plane in blue color represents the left-hand side of the inequality, and the plane in orange color represents the right-hand side of the inequality

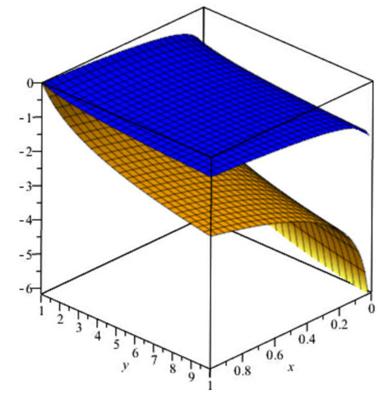
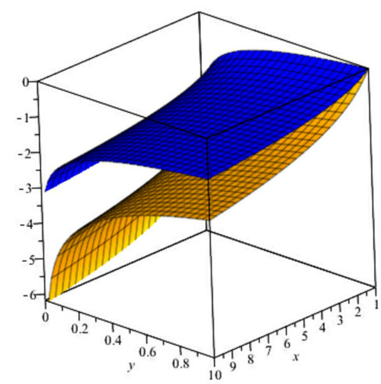


Figure 6 The plane in blue color represents the left-hand side of the inequality, and the plane in orange color represents the right-hand side of the inequality



$$
\begin{aligned}
&= \tau \cdot \beta(x,y,t) \cdot F\left(\frac{y}{x}\right) \\
&= \frac{1}{2} \cdot 1 \cdot \ln\left(\frac{y}{x}\right) \\
&= \frac{1}{2} \ln\left(\frac{y}{x}\right) \\
&\geq \ln\left(\frac{y}{x}\right) \\
&\geq \ln\bigl(N(x,y,t)\bigr).
\end{aligned}
$$

Figure 6 shows the illustration of Case 6 on 3D view.

Hence all the conditions in Theorem 1 are satisfied. Therefore g has a fixed point. In fact, we can see that $1 \in X$ is a fixed point of g.

The following example shows that the assumption "strong" in Theorem 1 is not superfluous.

Example 2 Let $X = [0, \infty)$, and let $*$ be a minimum continuous t-norm. We define the fuzzy set M by

$$
M(x,y,t) = \frac{t}{t + |x - y|}
$$

for $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a complete but not strong fuzzy metric space. For instance, take $x = 3$, $y = 0.5$, $z = 8$, and $t = 3$. Then we have $M(x,z,t) = M(3,8,3) = 0.2857$, $M(x,y,t) = M(3,0.5,3) = 0.5455$, and $M(y,z,t) = M(0.5,8,3) = 0.375$. Checking with condition FMS4*, it is clear that on the right-hand side, we get

$$\begin{aligned} M(x,y,t) * M(y,z,t) &= \min\bigl\{M(x,y,t), M(y,z,t)\bigr\} \\ &= \min\{0.5455, 0.375\} \\ &= 0.375. \end{aligned}$$

However, $0.2857 \ngeq 0.375$. Therefore $(X, M, *)$ is not strong. Now define the mapping $g : X \to X$ by

$$g(x) = \begin{cases} x+1, & x \in [0,1], \\ \sqrt{x}, & x \in (1,\infty), \end{cases}$$

for $x \in X$. Let $F(y) = y$ for $y \in [0,1]$, and let $\tau = 9/10$. Also, define $\beta : X \times X \times (0,\infty) \to (0,\infty)$ as

$$\beta(x,y,t) = \begin{cases} 2, & x,y \in [0,1], \\ 1 & \text{otherwise}, \end{cases}$$

for $x, y \in X$ and $t > 0$. It is easy to show that g is a fuzzy generalized F-contraction and conditions 1, 2, 3, and 4 of Theorem 1 hold. Observe that the mapping g does not have any fixed point.

In [25] the following theorem is obtained using Definition 4.

Theorem 2 ([25]) *Let $(X, M, *)$ be a complete fuzzy metric space such that*

$$\lim_{t \to 0^+} M(x,y,t) > 0$$

for all $x, y \in X$. If $g : X \to X$ is a continuous fuzzy F-contraction, then g has a unique fixed point.

The following example shows that Theorem 1 we obtained above is applicable but not Theorem 2 in [25].

Example 3 Let $X = [0,\infty)$, and let $*$ be a product continuous t-norm. We define the fuzzy set M by

$$M(x,y,t) = e^{\frac{-|x-y|}{t}}$$

for $x, y \in X$, and $t > 0$. Then $(X, M, *)$ is a complete strong fuzzy metric space.

Define the self-mapping $g : X \to X$ by

$$g(x) = \begin{cases} \frac{1}{4}x^2, & x \in [0,1], \\ 4x, & x \in (1,\infty), \end{cases}$$

for $x \in X$. Let $F(y) = \frac{-1}{\ln(y)}$ for $0 < y \leq 1$, and let $\tau = 1/2$. Also, let $\beta : X \times X \times (0,\infty) \to (0,\infty)$ be defined as

$$\beta(x,y,t) = \begin{cases} \frac{1}{2}, & x,y \in [0,1], \\ 3 & \text{otherwise}, \end{cases}$$

for $t > 0$. It is clear that g is a fuzzy generalized β-F-contraction. Let $x,y \in X$ be such that $\beta(x,y,t) \leq 1$. Based on the definition of β, it follows that $x,y \in [0,1]$. In the case where $x \in [0,1]$, we have $x^2 \leq x$, which implies

$$0 \leq \frac{1}{4}x^2 \leq \frac{1}{4}x \leq x \leq 1.$$

Thus by the definition of g we obtain $gx, gy \in [0,1]$, which means that $\beta(gx,gy,t) = \frac{1}{2} \leq 1$. This shows that g is β-admissible. In addition, there exists $x_0 \in X$ such that $\beta(x_0, gx_0, t) \leq 1$ for all $t > 0$. Indeed, let $x = 1/2$. Then for all $t > 0$,

$$\beta\left(\frac{1}{2}, g\left(\frac{1}{2}\right), t\right) = \beta\left(\frac{1}{2}, \frac{1}{16}, t\right) = \frac{1}{2} \leq 1.$$

Consider a sequence $\{x_n\}$ in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$ and $\lim_{n\to\infty} x_n = x$. Also, let $k_0 = 1$ where $n > m \geq k_0$. By the definition of β it follows that $x_n \in [0,1]$ for all $n \in \mathbb{N} \cup \{0\}$. Assume that $x > 1$. Then $x_n < x$, and we get

$$M(x_n, x, t) = e^{\frac{-|x_n - x|}{t}} < e^{\frac{-1}{t}} < 1$$

for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$, which contradicts the assumption that $\lim_{n\to\infty} x_n = x$. Thus we have $x \in [0,1]$. Hence $\beta(x_n, x, t) = 1/2 \leq 1$ and $\beta(x_n, x_m, t) \leq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$. Consequently, all the hypotheses in Theorem 1 are satisfied. This implies that g has a fixed point, which is $x = 0$.

However, g is not a fuzzy F-contraction. To see this, consider $x = 1$, $y = 400$, and $t = 30$. It follows that $gx = 1/4$ and $gy = 1600$. Checking the inequality in the Definition 4, we have

$$\begin{aligned} \tau \cdot F\bigl(M(gx,gy,t)\bigr) &= \frac{1}{2} \cdot \frac{-1}{\ln(e^{\frac{-|\frac{1}{4} - 1600|}{30}})} \\ &= \frac{1}{2} \cdot \frac{1599.75}{30} \\ &= 0.0094 \end{aligned}$$

on the left-hand side and

$$\begin{aligned} F\bigl(M(x,y,t)\bigr) &= \frac{-1}{\ln(e^{\frac{-|1-400|}{30}})} \\ &= \frac{399}{30} \\ &= 0.0752 \end{aligned}$$

on the right-hand side. Since $0.0094 \ngeq 0.0752$, we conclude that Theorem 2 of [25] does not hold for this example.

In the next theorem, we present a result by using a weaker condition compared with (1).

Theorem 3 *Let $(X, M, *)$ be a complete strong fuzzy metric space with $a * b = \min\{a, b\}$ for all $a, b \in [0,1]$. Moreover, let $\beta : X \times X \times (0,\infty) \to (0,\infty)$, $F \in \mathcal{F}$, and $g : X \to X$ be such that there exists $\tau \in (0,1)$ satisfying*

$$\tau \cdot \beta(x,y,t) \cdot F\bigl(M(gx,gy,t)\bigr) \geq F\bigl(N^*(x,y,t)\bigr) \tag{6}$$

for all $x, y \in X$ and $t > 0$, where

$$N^*(x,y,t) = \min\bigl\{M(x,y,t), M(x,gx,t), M(y,gy,t), M(x,gy,t), M(y,gx,t)\bigr\}.$$

In addition, assume that the following conditions hold:

1. *g is β-admissible*;
2. *there exists $x_0 \in X$ such that $\beta(x_0, gx_0, t) \leq 1$ for all $t > 0$*;
3. *for each sequence $\{x_n\}$ in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$, there exists $k_0 \in \mathbb{N} \cup \{0\}$ such that $\beta(x_m, x_n, t) \leq 1$ for all $n > m \geq k_0$ and $t > 0$*;
4. *if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$ and if $x_n \to x$ as $n \to \infty$, then $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then g has a fixed point.

Proof Let $x_0 \in X$ be such that $\beta(x_0, gx_0, t) \leq 1$ for all $t > 0$. We define the sequence $\{x_n\}$ by $x_{n+1} = gx_n$ for $n \in \mathbb{N} \cup \{0\}$. If there exists $n \in \mathbb{N} \cup \{0\}$ such that $x_{n+1} = x_n$, then x_n is a fixed point of g, and the proof is complete. Now assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\beta(x_0, gx_0, t) = \beta(x_0, x_1, t) \leq 1$ for all $t > 0$ and g is β-admissible mapping, this implies that

$$\beta(gx_0, gx_1, t) = \beta(x_1, x_2, t) \leq 1.$$

Continuing this process, we conclude that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$. By the hypothesis there exists $\tau \in (0,1)$ that satisfies (6). It is clear that $\tau\beta(x_n, x_{n+1}, t) < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$. Using (6), we have

$$\begin{aligned} F\bigl(M(x_n, x_{n+1}, t)\bigr) = F\bigl(M(gx_{n-1}, gx_n, t)\bigr) &> \tau\beta(x_{n-1}, x_n, t) F\bigl(M(gx_{n-1}, gx_n, t)\bigr) \\ &\geq F\bigl(N^*(x_{n-1}, x_n, t)\bigr) \end{aligned}$$

for all $t > 0$, where

$$\begin{aligned} N^*(x_{n-1}, x_n, t) &= \min\bigl\{M(x_{n-1}, x_n, t), M(x_{n-1}, gx_{n-1}, t), M(x_n, gx_n, t), \\ &\qquad M(x_{n-1}, gx_n, t), M(x_n, gx_{n-1}, t)\bigr\} \\ &= \min\bigl\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), \\ &\qquad M(x_{n-1}, x_{n+1}, t), M(x_n, x_n, t)\bigr\} \\ &= \min\bigl\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), M(x_{n-1}, x_{n+1}, t), 1\bigr\}. \end{aligned}$$

Since the t-norm is minimum, by (FMS4*) we obtain

$$\begin{aligned} M(x_{n-1},x_{n+1},t) &\geq M(x_{n-1},x_n,t)*M(x_n,x_{n+1},t) \\ &= \min\bigl\{M(x_{n-1},x_n,t),M(x_n,x_{n+1},t)\bigr\}. \end{aligned}$$

So we have

$$N^*(x_{n-1},x_n,t)=\min\bigl\{M(x_{n-1},x_n,t),M(x_n,x_{n+1},t)\bigr\}$$

for all $n\in\mathbb{N}\cup\{0\}$ and $t>0$. If $\min\{M(x_{n-1},x_n,t),M(x_n,x_{n+1},t)\}=M(x_n,x_{n+1},t)$, then

$$F\bigl(M(x_n,x_{n+1},t)\bigr)>\tau\beta(x_{n-1},x_n,t)F\bigl(M(x_n,x_{n+1},t)\bigr)\geq F\bigl(M(x_n,x_{n+1},t)\bigr),$$

which leads to a contradiction. So we have $\min\{M(x_{n-1},x_n,t),M(x_n,x_{n+1},t)\}=M(x_{n-1},x_n,t)$ and

$$F\bigl(M(x_n,x_{n+1},t)\bigr)>\tau\beta(x_{n-1},x_n,t)F\bigl(M(x_n,x_{n+1},t)\bigr)\geq F\bigl(M(x_{n-1},x_n,t)\bigr), \tag{7}$$

which implies

$$F\bigl(M(x_n,x_{n+1},t)\bigr)>F\bigl(M(x_{n-1},x_n,t)\bigr).$$

Since F is a strictly increasing function, we get $M(x_n,x_{n+1},t)>M(x_{n-1},x_n,t))$ for all $n\in\mathbb{N}\cup\{0\}$ and $t>0$. Hence for all $t>0$, the sequence $\{M(x_n,x_{n+1},t)\}$ is strictly increasing in the interval $[0,1]$ and bounded above. This implies that $\{M(x_n,x_{n+1},t)\}$ is convergent, that is, for any $t>0$, there exists $A(t)\in[0,1]$ such that $\lim_{n\to\infty}M(x_n,x_{n+1},t)=A(t)$. We claim that $A(t)=1$ for all $t>0$. Suppose that $A(t_0)<1$ for some $t_0>0$. Taking the limit in both sides of (7), we obtain

$$F\bigl(A(t_0)\bigr)>\tau\beta(x_{n-1},x_n,t)F\bigl(A(t_0)\bigr)\geq F\bigl(A(t_0)\bigr),$$

which leads to a contradiction. Therefore we have

$$\lim_{n\to\infty}M(x_n,x_{n+1},t)=1 \tag{8}$$

for all $t>0$. Now we will show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then by condition 3 there exist $k_0\in\mathbb{N}\cup\{0\}$, $\varepsilon>0$, two subsequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ of $\{x_n\}$, and $s>0$ such that $n_k>m_k\geq k$ for all $k\in\mathbb{N}\cup\{0\}$ with $k\geq k_0$ and

$$M(x_{n_k},x_{m_k},s)\leq 1-\varepsilon \quad\text{and}\quad \beta(x_{n_k},x_{m_k},t)\leq 1.$$

Let n_k be the smallest integer satisfying the above inequality. This means that

$$M(x_{n_k-1},x_{m_k},s)>1-\varepsilon.$$

Using (FMS4*) and the minimum t-norm, we have

$$\begin{aligned} 1-\varepsilon \geq M(x_{n_k}, x_{m_k}, s) &\geq \min\bigl\{M(x_{n_k}, x_{n_k-1}, s), M(x_{n_k-1}, x_{m_k}, s)\bigr\} \\ &\geq \min\bigl\{M(x_{n_k}, x_{n_k-1}, s), 1-\varepsilon\bigr\}. \end{aligned}$$

Taking the limit as $k \to \infty$, by (8) we obtain

$$1-\varepsilon \geq \lim_{k\to\infty} M(x_{n_k}, x_{m_k}, s) \geq \min\{1, 1-\varepsilon\} = 1-\varepsilon.$$

This implies that $\lim_{k\to\infty} M(x_{n_k}, x_{m_k}, s) = 1-\varepsilon$. Again, by (FMS4*) and minimum t-norm we get

$$M(x_{n_k+1}, x_{m_k+1}, s) \geq \min\bigl\{M(x_{n_k+1}, x_{n_k}, s), M(x_{n_k}, x_{m_k}, s), M(x_{m_k}, x_{m_k+1}, s)\bigr\}.$$

Letting $k \to \infty$, by (8) we have

$$\lim_{k\to\infty} M(x_{n_k+1}, x_{m_k+1}, s) = \min\{1, 1-\varepsilon, 1\} = 1-\varepsilon.$$

Since $\tau < 1$ and $\beta(x_{n_k}, x_{m_k}, t) \leq 1$ for all $k \in \mathbb{N} \cup \{0\}$ with $k \geq k_0$, this implies that $\tau\beta(x_{n_k}, x_{m_k}, t) < 1$ for all $k \in \mathbb{N} \cup \{0\}$ with $k \geq k_0$. Using (6), we get

$$\begin{aligned} F(M(x_{n_k+1}, x_{m_k+1}, s) &> \tau\beta(x_{n_k}, x_{m_k}, s) F\bigl(M(x_{n_k+1}, x_{m_k+1}, s)\bigr) \\ &\geq F\bigl(N^*(x_{n_k}, x_{m_k}, s)\bigr), \end{aligned} \tag{9}$$

where

$$\begin{aligned} N^*(x_{n_k}, x_{m_k}, s) = \min\bigl\{&M(x_{n_k}, x_{m_k}, s), M(x_{n_k}, x_{n_k+1}, s), M(x_{m_k}, x_{m_k+1}, s), \\ &M(x_{n_k}, x_{m_k+1}, s), M(x_{m_k}, x_{n_k+1}, s)\bigr\}. \end{aligned}$$

Since

$$M(x_{n_k}, x_{m_k+1}, s) \geq \min\bigl\{M(x_{n_k}, x_{m_k}, s), M(x_{m_k}, x_{m_k+1}, s)\bigr\}$$

and

$$M(x_{m_k}, x_{n_k+1}, s) \geq \min\bigl\{M(x_{m_k}, x_{n_k}, s), M(x_{n_k}, x_{n_k+1}, s)\bigr\},$$

we have

$$N^*(x_{n_k}, x_{m_k}, s) = \min\bigl\{(M(x_{n_k}, x_{m_k}, s), M(x_{n_k}, x_{n_k+1}, s), M(x_{m_k}, x_{m_k+1}, s)\bigr\}.$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} N^*(x_{n_k}, x_{m_k}, s) = \min\Bigl\{\lim_{k\to\infty} M(x_{n_k}, x_{m_k}, s), \lim_{k\to\infty} M(x_{n_k}, x_{n_k+1}, s),$$

$$\lim_{k\to\infty} M(x_{m_k}, x_{m_k+1}, s)\Big\}$$
$$= \min\{1-\varepsilon, 1, 1\}$$
$$= 1-\varepsilon.$$

Therefore taking the limit as $k \to \infty$ in both sides of (9), we obtain

$$F(1-\varepsilon) \geq \tau\beta(x_{n_k}, x_{m_k}, s)F(1-\varepsilon) \geq F(1-\varepsilon).$$

This implies that $F(1-\varepsilon) = 0$, which contradicts the fact that $F(1-\varepsilon) > 0$. Thus $\{x_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is complete, there exists $x \in X$ such that $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$ or, in other words, $\lim_{n\to\infty} x_n = x$. Together with the fact that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$, which we obtained above, by condition 4 we have $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Now we will show that x is a fixed point of g. Suppose that $gx \neq x$, that is, $M(x, gx, t) < 1$. Then by (6) we get

$$F\big(M(gx_n, gx, t)\big) > \tau\beta(x_n, x, t)F\big(M(gx_n, gx, t)\big) \geq F\big(N^*(x_n, x, t)\big)$$

for all $t > 0$, where

$$\begin{aligned} N^*(x_n, x, t) &= \min\big\{M(x_n, x, t), M(x_n, gx_n, t), M(x, gx, t), M(x_n, gx, t), \\ &\quad M(x, gx_n, t)\big\} \\ &= \min\big\{M(x_n, x, t), M(x_n, x_{n+1}, t), M(x, gx, t), M(x_n, gx, t), \\ &\quad M(x, x_{n+1}, t)\big\}. \end{aligned}$$

Taking the limit as $n \to \infty$ in the inequality above, we have

$$\begin{aligned} F\big(M(x, gx, t)\big) &> \tau\beta(x_n, x, t)F\big(M(x, gx, t)\big) \geq F\big(\min\big\{1, 1, M(x, gx, t), M(x, gx, t), 1\big\}\big) \\ &= F\big(M(x, gx, t)\big), \end{aligned}$$

which leads to a contradiction. Thus $gx = x$, which means that x is a fixed point of g. □

Remark 3 We can obtain Corollary 1 from Theorem 3 by letting $N^*(x, y, t) = M(x, y, t)$ for $x, y \in X$ and $t > 0$ in (6).

Remark 4 Note that Example 2 can be used to show that the assumption "strong" in Theorem 3 is not superfluous.

Note that the results above do not guarantee that the fixed points are unique. To obtain the uniqueness of the fixed point, we will consider the following condition:

(U) For any pair of fixed points $x, y \in X$, $\beta(x, y, t) \leq 1$ for all $t > 0$.

Theorem 4 *In addition to the hypotheses of Theorem* 1, *suppose that condition* (U) *holds. Then g has a unique fixed point.*

Proof The existence of a fixed point follows from Theorem 1. Assume that $x, y \in X$ are fixed points of g such that $x \neq y$, that is, $M(x, y, t) < 1$. Considering condition (U), we have $\beta(x, y, t) \leq 1$ for all $t > 0$. Since g is a fuzzy generalized β-F-contraction mapping, by (1) we obtain

$$F\bigl(M(gx, gy, t)\bigr) > \tau\beta(x, y, t)F\bigl(M(gx, gy, t)\bigr) \geq F\bigl(N(x, y, t)\bigr)$$

for all $t > 0$, where

$$N(x, y, t) = \min\bigl\{M(x, y, t), M(x, gx, t), M(y, gy, t)\bigr\}.$$

Since $gx = x$ and $gy = y$, we have

$$\begin{aligned} N(x, y, t) &= \min\bigl\{M(x, y, t), M(x, x, t), M(y, y, t)\bigr\} \\ &= \min\bigl\{M(x, y, t), 1, 1\bigr\} \\ &= \min\bigl\{M(x, y, t), 1\bigr\} \\ &= M(x, y, t). \end{aligned}$$

It follows that

$$F\bigl(M(x, y, t)\bigr) > F\bigl(M(x, y, t)\bigr),$$

which leads to a contradiction. Hence we conclude that $x = y$. □

Corollary 2 *In addition to the hypotheses of Corollary* 1, *suppose that condition* (U) *holds. Then g has a unique fixed point.*

Theorem 5 *In addition to the hypotheses of Theorem* 3, *suppose that condition* (U) *holds. Then g has a unique fixed point.*

Proof The existence of fixed point follows from Theorem 3. Assume that $x, y \in X$ are a fixed point of g such that $x \neq y$, that is, $M(x, y, t) < 1$. By condition (U) we have $\beta(x, y, t) \leq 1$ for all $t > 0$. By (6) we obtain

$$F\bigl(M(gx, gy, t)\bigr) > \tau\beta(x, y, t)F\bigl(M(gx, gy, t)\bigr) \geq F\bigl(N^*(x, y, t)\bigr)$$

for all $t > 0$, where

$$N^*(x, y, t) = \min\bigl\{M(x, y, t), M(x, gx, t), M(y, gy, t), M(x, gy, t), M(y, gx, t)\bigr\}.$$

Since $gx = x$ and $gy = y$, we have

$$\begin{aligned} N^*(x, y, t) &= \min\bigl\{M(x, y, t), M(x, x, t), M(y, y, t), M(x, y, t), M(y, x, t)\bigr\} \\ &= \min\bigl\{M(x, y, t), 1, 1, M(x, y, t), M(x, y, t)\bigr\} \\ &= \min\bigl\{M(x, y, t), 1\bigr\} \end{aligned}$$

$$= M(x,y,t).$$

It follows that

$$F\bigl(M(x,y,t)\bigr) > F\bigl(M(x,y,t)\bigr),$$

which leads to a contradiction. Hence we conclude that $x = y$. □

3 Applications

In this section, we present an application of our result in finding the solution for an integral equation. Let us define the class of functions $\Phi := \{\phi(t) : (0,\infty) \to (0,\infty)\}$ such that $\int_0^t \phi(u)\,du$ is increasing and continuous for all $t > 0$. The following theorem is our fixed-point result for the Wardowski-type contraction via β-admissible mappings of the integral type in a fuzzy metric space.

Theorem 6 *Let $(X,M,*)$ be a complete fuzzy metric space, and let $g : X \to X$ be a fuzzy generalized β-F-contraction. Let ϕ be a mapping in Φ such that*

$$\tau\beta(x,y,t)\int_0^{F(M(gy,gy,t))}\phi(t)\,dt \geq \int_0^{F(N(x,y,t))}\phi(t)\,dt,$$

where $N(x,y,t) = \min\{M(x,y,t), M(x,gx,t), M(y,gy,t)\}$ for $x,y \in X$ and $t > 0$. Suppose that the following conditions hold:

1. *g is β-admissible;*
2. *there exists $x_0 \in X$ such that $\beta(x_0,gx_0,t) \leq 1$ for all $t > 0$;*
3. *for each sequence $\{x_n\}$ in X such that $\beta(x_n,x_{n+1},t) \leq 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $t > 0$, there exists $k_0 \in \mathbb{N}\cup\{0\}$ such that $\beta(x_m,x_n,t) \leq 1$ for all $n > m \geq k_0$ and $t > 0$;*
4. *if $\{x_n\}$ is a sequence in X such that $\beta(x_n,x_{n+1},t) \leq 1$ for all $n \in \mathbb{N}\cup\{0\}$ $t > 0$ and if $x_n \to x$ as $n \to \infty$, then $\beta(x_n,x,t) \leq 1$ for all $n \in \mathbb{N}\cup\{0\}$.*
5. *For any pair of fixed points $x,y \in X$, $\beta(x,y,t) \leq 1$ for all $t > 0$.*

Then g has a unique fixed point.

Proof Let $\phi(t) = 1$. Now apply Theorem 1 followed by Theorem 4. Then we obtain the desired result. □

4 Conclusions

In this paper, we have introduced fuzzy generalized β-F-contractions as an extension of fuzzy F-contractions. We proved several fixed-point results for this contraction and its variation under the setting of complete strong fuzzy metric spaces. In addition, we provided a sufficient condition to obtain the uniqueness of a fixed point for this contraction. At the end of this paper, we showed an application of our main result in finding the solution for an integral equation. As mentioned in Remark 2, our work extends and generalizes the results of [25]. This is supported by Example 3, which shows that our result is applicable, but not the result in [25].

Since the debut of F-contractions in metric spaces, they received considerable attention from other researchers. Lately, Gautam et al. [26] introduced Kannan F-contractions as an extension of F-contractions. They obtained common fixed-point results for Kannan

F-contractions in quasi-partial b-metric spaces and gave an application of their result in functional equations. Now, if we consider fuzzy a metric space, it is interesting to see if we can incorporate Kannan's contractive condition for fuzzy generalized β-F-contractions. We will end this paper with an open problem: "Can we obtain the existence and uniqueness of a fixed point for a Kannan-type fuzzy generalized F-contractive mapping under a fuzzy metric space setting?".

Acknowledgements
The authors appreciate the support of their institutions. They gratefully acknowledge the Ministry of Higher Education Malaysia, Universiti Malaysia Terengganu, and Fundamental Research Grant Scheme (FRGS) for their financial support.

Funding
This work was supported by the Ministry of Higher Education Malaysia and Universiti Malaysia Terengganu under the Fundamental Research Grant Scheme (FRGS) Project Code FRGS/1/2021/STG06/UMT/02/1 and Vote no. 59659.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Conceptualization: Zabidin Salleh, Koon Sang Wong; Methodology: Koon Sang Wong; Formal analysis and investigation: Koon Sang Wong, Zabidin Salleh, Che Mohd Imran Che Taib; Writing - original draft preparation: Koon Sang Wong; Writing - review and editing: Zabidin Salleh, Che Mohd Imran Che Taib; Funding acquisition: Zabidin Salleh; Resources: Zabidin Salleh, Che Mohd Imran Che Taib; Validation and Visualization: Che Mohd Imran Che Taib; Supervision: Zabidin Salleh. All authors reviewed the manuscript.

Received: 10 April 2022 Accepted: 3 May 2023 Published online: 12 June 2023

References

1. Banach, S.: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundam. Math. **3**, 131–181 (1922)
2. Gopal, D., Kumam, P., Abbas, M.: Background and Recent Developments of Metric Fixed Point Theory. CRC Press, New York (2017)
3. Agarwal, R.P., Meehan, M., O'regan, D.: Fixed Point Theory and Applications. Cambridge University Press, Cambridge (2001)
4. Bucur, A.: About applications of the fixed point theory. Sci. Bull. **22**(1), 13–17 (2017)
5. Cho, Y.J.: In: Ruzhansky, M., Cho, Y., Agarwal, P., Area, I. (eds.) Survey on Metric Fixed Point Theory and Applications, pp. 183–241. Birkhäuser, Singapore (2017)
6. Farajzadeh, A.P., Kaewcharoen, A., Plubtieng, S.: An application of fixed point theory to a nonlinear differential equation. Abstr. Appl. Anal. **2014**, 605405 (2014)
7. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–353 (1965)
8. Kramosil, I., Michálek, J.: Fuzzy metric and statistical metric spaces. Kybernetika **11**(5), 326–334 (1975)
9. George, A., Veeramani, P.: On some results on fuzzy metric spaces. Fuzzy Sets Syst. **64**(3), 395–399 (1994)
10. Grabiec, M.: Fixed points in fuzzy metric spaces. Fuzzy Sets Syst. **27**(3), 385–389 (1988)
11. Beg, I., Gopal, D., Došenović, T., Rakić, D.: α-Type fuzzy H-contractive mappings in fuzzy metric spaces. Fixed Point Theory **19**(2), 463–474 (2018)
12. Gregori, V., Sapena, A.: On fixed-point theorems in fuzzy metric spaces. Fuzzy Sets Syst. **125**(2), 245–252 (2002)
13. Rakić, D., Mukheimer, A., Došenović, T., Mitrović, Z.D., Radenović, S.: On some new fixed point results in fuzzy b-metric spaces. J. Inequal. Appl. **2020**, 99 (2020)
14. Sedghi, S., Shobkolaei, N., Došenović, T., Radenović, S.: Suzuki-type of common fixed point theorems in fuzzy metric spaces. Math. Slovaca **68**(2), 451–462 (2018)
15. Zheng, D., Wang, P.: Meir–Keeler theorems in fuzzy metric spaces. Fuzzy Sets Syst. **370**, 120–128 (2019)
16. Wardowski, D.: Fixed points of a new type of contractive mappings in complete metric spaces. Fixed Point Theory Appl. **2012**, 94 (2012)
17. Chauhan, S.S., Imdad, M., Kaur, G., Sharma, A.: Some fixed point theorems for S_F-contraction in complete fuzzy metric spaces. Afr. Math. **30**, 651–662 (2019)
18. Dey, L.K., Kumam, P., Senapati, T.: Fixed point results concerning α-F-contraction mappings in metric space. Appl. Gen. Topol. **20**(1), 81–95 (2019)
19. Fabiano, N., Parvaneh, V., Mirković, D., Paunović, L., Radenović, S.: On W-contractions of Jungck–Ćirić–Wardowski-type in metric spaces. Cogent Math. Stat. **7**(1), 1792699 (2020)
20. Karapınar, E., Fulga, A., Agarwal, R.P.: A survey: F-contractions with related fixed point results. J. Fixed Point Theory Appl. **22**, 69 (2020)

21. Qawaqneh, H., Noorani, M.S., Shatanawi, W.: Fixed point results for Geraghty type generalized F-contraction for weak α-admissible mappings in metric-like spaces. Eur. J. Pure Appl. Math. **11**(3), 702–716 (2018)
22. Rasham, T., Marino, G., Shahzad, A., Park, C., Shoaib, A.: Fixed point results for a pair of fuzzy mappings and related applications in *b*-metric like spaces. Adv. Differ. Equ. **2021**, 259 (2021)
23. Saleem, N., Abbas, M., Raza, Z.: Fixed fuzzy point results of generalized Suzuki type *F*-contraction mappings in ordered metric spaces. Georgian Math. J. **27**(2), 307–320 (2020)
24. Vujaković, J.Z., Radenović, S.N.: On some *F*-contraction of Piri–Kumam–Dung-type mappings in metric spaces. Vojnotehnički glasnik **68**(4), 697–714 (2020)
25. Huang, H., Carić, B., Došenović, T., Rakić, D., Brdar, M.: Fixed-point theorems in fuzzy metric spaces via fuzzy *F*-contraction. Mathematics **9**(6), 641 (2021)
26. Gautam, P., Kumar, S., Verma, S., Gupta, G.: Existence of common fixed point in Kannan *F*-contractive mappings in quasi-partial *b*-metric space with an application. Fixed Point Theory Algorithms Sci. Eng. **2202**, 23 (2022)
27. Vujaković, K., Mitrović, S., Mitrović, Z.D., Radenović, S.: On *F*-contractions for weak α-admissible mappings in metric-like spaces. Mathematics **2020**(8), 1629 (2020)
28. Patel, U.D., Radenović, S.: An application to nonlinear fractional differential equation via α-ΓF-fuzzy contractive mappings in a fuzzy metric space. Mathematics **2022**(10), 2831 (2022)
29. Samet, B., Vetro, C., Vetro, P.: Fixed point theorems for α-ψ-contractive type mappings. Nonlinear Anal., Theory Methods Appl. **75**(4), 2154–2165 (2012)
30. Gopal, D., Vetro, C.: Some new fixed point theorems in fuzzy metric spaces. Iran. J. Fuzzy Syst. **11**(3), 95–107 (2014)
31. Schweizer, B., Sklar, A.: Statistical metric spaces. Pac. J. Math. **10**(1), 314–334 (1960)

Publisher's Note