Fixed Point Theory and Algorithms for Sciences and Engineering

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Fixed point theorems and applications in p-vector spaces

George Xianzhi Yuan[1,2,3,4,5*]

This paper is dedicated to the memory of Professor Wataru Takahashi (1944–2020), Professor Kazimierz Goebel (1940–2022), and Professor William Art Kirk (1936–2022) for their significant contribution on the development of fixed point theory and applications for nonlinear functional analysis in mathematics and related disciplines.

*Correspondence:
george_yuan99@yahoo.com
[1]Chongqing University of Technology, Chongqing, 400054, China
[2]Chengdu University, Chengdu 610601, China
Full list of author information is available at the end of the article

Abstract

The goal of this paper is to develop new fixed points for quasi upper semicontinuous set-valued mappings and compact continuous (single-valued) mappings, and related applications for useful tools in nonlinear analysis by applying the best approximation approach for classes of semiclosed 1-set contractive set-valued mappings in locally p-convex and p-vector spaces for $p \in (0, 1]$. In particular, we first develop general fixed point theorems for quasi upper semicontinuous set-valued and single-valued condensing mappings, which provide answers to the Schauder conjecture in the affirmative way under the setting of locally p-convex spaces and topological vector spaces for $p \in (0, 1]$; then the best approximation results for quasi upper semicontinuous and 1-set contractive set-valued mappings are established, which are used as tools to establish some new fixed points for nonself quasi upper semicontinuous set-valued mappings with either inward or outward set conditions under various boundary situations. The results established in this paper unify or improve corresponding results in the existing literature for nonlinear analysis, and they would be regarded as the continuation of the related work by Yuan (Fixed Point Theory Algorithms Sci. Eng. 2022:20, 2022)–(Fixed Point Theory Algorithms Sci. Eng. 2022:26, 2022) recently.

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1 Introduction

It is known that the class of p-seminorm spaces ($0 < p \leq 1$) is an important generalization of the usual normed spaces with rich topological and geometrical structures, and related studies have received a lot of attention (e.g., see Alghamdi et al. [5], Balachandran [7], Bayoumi [8], Bayoumi et al. [9], Bernuées and Pena [13], Chang et al. [26], Ding [34], Ennassik and Taoudi [38], Ennassik et al. [37], Gal and Goldstein [45], Gholizadeh et al. [46], Jarchow [61], Kalton [62, 63], Kalton et al. [64], Machrafi and Oubbi [82], Park [100], Qiu and Rolewicz [109], Rolewicz [114], Sezer et al. [119], Silva et al. [123], Simons [124], Tabor et al. [127], Tan [128], Wang [131], Xiao and Lu [134], Xiao and Zhu [135], Yuan [142–145], and many others). However, to the best of our knowledge, the corresponding basic tools and associated results in the category of nonlinear functional analysis have not been well developed, thus the goal of this paper is to develop some important tools in nonlinear analysis for semiclosed 1-set contractive mappings under the framework of p-vector spaces, in particular, in locally p-convex spaces by including nonexpansive set-valued mappings as a special class under uniformly convex Banach spaces or locally convex spaces with Opial condition.

In particular, we first develop the general fixed point theorems for upper semicontinuous (USC) set-valued 1-set contractive mappings, which provide answer to Schauder conjecture since 1930s in the affirmative under the general framework of locally p-convex spaces (when $p = 1$ being locally convex spaces), then the best approximation results for upper semicontinuous and 1-set contractive mappings are given with various boundary condition, which are used as tools to establish fixed points for nonself set-valued mappings with either inward or outward set conditions; and finally, we give existence results for solutions of Birkhoff–Kellogg problems, the general principle of nonlinear alternative by including Leray–Schauder alternative, and related results as special classes. The results given in this paper do not only include the corresponding results in the existing literature as special cases, but also are expected to be useful for the study of nonlinear problems arising from social science, engineering, applied mathematics, and related topics and areas.

Before discussing the study of best approximations and related nonlinear analysis tools under the framework of p-vector spaces, we would like first to share with readers that though most of results in nonlinear analysis are normally highly associated with the convexity hypotheses under the locally convex spaces by including normed spaces, Banach spaces, and metric spaces special classes, it seems that p-vector spaces provide some nice properties for p-convex subsets, which would play very important roles for us to describe Birkhoff and Kellogg problems, and related nonlinear problems such as fixed point problem comparing with convexity in topological vector spaces (TVS) for p in $(0, 1)$ (see the properties given by Remark 2.1(1), and Lemma 2.1(ii) in Sect. 2 in detail).

Here, we would also like to recall that the first Birkhoff–Kellogg theorem was proved by Birkhoff and Kellogg [14] in 1922 in discussing the existence of solutions for the equation $x = \lambda F(x)$, where λ is a real parameter and F is a general nonlinear nonself mapping defined on an open convex subset U of a topological vector space E. Thus the general form of the Birkhoff–Kellogg problem is to find an invariant direction for the nonlinear set-valued mappings F, i.e., to find $x_0 \in \overline{U}$ and $\lambda > 0$ such that $\lambda x_0 \in F(x_0)$.

Since the Birkhoff and Kellogg theorem given by Birkhoff and Kellogg in 1920s, the study on Birkhoff–Kellogg problem has received a lot of attention from scholars. For example, in 1934, one of the fundamental results in nonlinear functional analysis, famously called the Leray–Schauder alternative, by Leray and Schauder [76] was established via topological degree theory, and thereafter, certain other types of Leray–Schauder alternatives were proved using different techniques other than by using the topological degree approach (see the works by Granas and Dugundji [53], Furi and Pera [44] in the Banach space setting and applications to the boundary value problems for ordinary differential equations in noncompact cases, a general class of mappings for nonlinear alternative of Leray–Schauder type in normal spaces, and Birkhoff–Kellogg type theorems for general class mappings in topological vector spaces by Agarwal et al. [1], Agarwal and O'Regan [2, 3], Park [98], and O'Regan [91] (see the related references therein).

In this paper, based on the application of our best approximation as a tool for quasi upper semicontinuous 1-set contractive set-valued mappings, we first establish general principles for the existence of solutions for Birkhoff–Kellogg problems and related nonlinear alternatives, which then also allows us to give general existence of Leray–Schauder type and related fixed point theorems for nonself mappings in general vector p-spaces, in particular, locally p-convex spaces for $p \in (0, 1]$. The results established in this paper not only include the corresponding results in the existing literature as special cases, but are also expected to be useful tools for the study of nonlinear problems arising from theory to practice under the framework of p-vector spaces. In particular, the work in this paper can be regarded as the continuation of related work established by Yuan [144, 145] recently.

Now we give a brief discussion and background on the best approximation method related to the study of nonlinear analysis.

We all know that the best approximation method is related to fixed points for nonself mappings, which tightly links with the classical Leray–Schauder alternative based on the Leray–Schauder continuation theorem by Leray and Schauder [76], which is a remarkable result in nonlinear analysis; in addition, there exist several continuation theorems, which have many applications in the study of nonlinear functional equations (see O'Regan and Precup [93]). Historically, it seems that the continuation theorem is based on the idea of obtaining a solution of a given equation, starting from one solution for a simpler equation, the essential part of this theorem is the "Leray–Schauder boundary condition". But indeed, it seems that "continuation method" was initiated by Poincare [107], Bernstein [12]. Certainly, Leray and Schauder [76] in 1934 gave the first abstract formulation of "continuation principle" using the topological degree theory (see also Granas and Dugundji [53], Isac [60], Rothe [115, 116], Zeidler [146]). But in this paper, we will see how the best approximation method could be used for the study of fixed point theorems in p-vector space ($0 < p \leq 1$), which as a basic tool, will help us to develop the principle of nonlinear alterative, Leray–Schauder alternative, fixed point theorems of Rothe, Petryshyn, Atlman type for set-valued nonself mappings, and nonlinear alternative with different boundary conditions. Moreover, the new results given in this paper are highly expected to become useful tools for the study on optimization, nonlinear programming, variational inequality, complementarity, game theory, mathematical economics, and related other social science area.

It is well known that Fan's best approximation theorem given by Fan [42] in 1969 acts as a very powerful tool in nonlinear analysis, as discussed by the book of Singh et al. [125] for the study on the fixed point theory and best approximation with the KKM-map principle, among them, the related tools are Rothe type and the principle of Leray–Schauder alterative in topological vector spaces (in short, TVS) and local convex spaces (in short, LCS), which are also comprehensively studied by Chang et al. [27–30], Carbone and Conti [21], Ennassik and Taoudi [38], Ennassik et al. [37], Guo [54], Guo et al. [55], Granas and Dugundji [53], Isac [60], Kirk and Shahzad [68], Liu [81], Park [101], Rothe [115, 116], Shahzad [120–122], Xu [136], Yuan [142–145], Zeidler [146], and the references therein.

Moreover, since the celebrated so-called KKM principle established in 1929 in [70] (see also Mauldin [84]) was based on the celebrated Sperner combinatorial lemma and first applied to a simple proof of the Brouwer fixed point theorem, later it became clear that these three theorems are mutually equivalent and they were regarded as a sort of mathematical trinity (Park [101]). In particular, since Fan extended the classical KKM theorem to infinite-dimensional spaces in 1961 (see Fan [41–43]), there have been a number of generalizations and applications in numerous areas of nonlinear analysis and fixed points in TVS and LCS as developed by Browder [15–20] and the related references therein. Among them, Schauder's fixed point theorem [118] in normed spaces is one of the powerful tools in dealing with nonlinear problems in analysis. Most notably, it has played a major role in the development of fixed point theory and related nonlinear analysis and mathematical theory of partial and differential equations and others.

A generalization of Schauder's theorem from a normed space to general topological vector spaces is an old conjecture in fixed point theory, which is explained by Problem 54 of the book "The Scottish Book" by Mauldin [84] and stated as Schauder's conjecture: "Every nonempty compact convex set in a topological vector space has the fixed point property, or in its analytic statement, does a continuous function defined on a compact convex subset of a topological vector space to itself have a fixed point?"

Based on the discussion by Ennassik and Taoudi [38], Cauty [22, 23] tried to solve the Schauder conjecture, and Ennassik and Taoudi [38] gave the positive answer to the Schauder conjecture for single-valued continuous mappings under the framework of p-vector spaces, where $p \in (0,1]$. Indeed, from the respective of development on the study of fixed point theory and related topics in nonlinear analysis, a number of works have been contributed by Górniewicz [51], Górniewicz et al. [52], Ennassik et al. [37] by using the p-seminorm method under p-vector spaces; plus corresponding contributions by Askoura and Godet-Thobie [6], Chang [25], Chang et al. [27], Chen [32], Dobrowolski [35], Gholizadeh et al. [46], Huang et al. [57], Isac [60], Li [79], Li et al. [78], Liu [81], Mańka [83], Nhu [87], Okon [89], Park [100–102], Reich [110], Smart [126], Weber [132, 133], Xiao and Lu [134], Xiao and Zhu [135], Xu [139], Xu et al. [140], Yuan [142–145], and the related references therein under the general framework of p-vector spaces for even nonself set-valued mappings ($0 < p \leq 1$).

The goal of this paper is to develop new fixed points for quasi upper semicontinuous set-valued mappings, and related some useful tools for nonlinear analysis by applying the best approximation approach for classes of semiclosed 1-set contractive set-valued mappings in locally p-convex or p-vector spaces for $p \in (0,1]$. In particular, we first develop general fixed point theorems for quasi upper semicontinuous set-valued and single-valued condensing mappings, which provide answers to the Schauder conjecture in the affirmative way under the setting of locally p-convex (and p-vector spaces). Then the best approximation results for quasi upper semicontinuous and 1-set contractive set-valued are established, which are used as tools to establish some new fixed points for nonself quasi upper semicontinuous set-valued mappings with either inward or outward set conditions under various situations. These results unify or improve corresponding results in the existing literature for nonlinear analysis. We do wish that these new results such as the best approximation, Birkhoff–Kellogg type, nonlinear alternative, fixed point theorems for nonself set-valued mappings with boundary conditions, Rothe, Petryshyn type, Altman type, Leray–Schedule type, other related nonlinear problems would play important roles for the development of nonlinear analysis of p-seminorm spaces for $0 < p \leq 1$. The results discussed in this paper do not only unify or improve corresponding results in the existing literature for nonlinear analysis, but they can also be regarded as the continuation of (or) related work established by Yuan [144, 145] recently.

The paper consists of eight sections. Section 1 is the introduction. Section 2 describes general concepts for p-vector spaces, locally p-convex spaces, and p-convexity for $p \in (0,1]$. In Sect. 3, some basic results of the KKM principle related to abstract convex spaces are given. In Sect. 4, as an application of the KKM principle in abstract convex spaces, which include p-convex vector spaces as a special class for $p \in (0,1]$, by combining the graph approximation lemma for quasi upper semicontinuous set-valued mappings in locally p-convex spaces, we provide general fixed point theorems for upper semicontinuous self-mappings defined on locally p-convex compact and 1-set contractive upper semicontinuous set-valued mappings defined on noncompact p-convex subsets in locally p-convex spaces. In Sect. 5, the general best approximation result for 1-set contractive upper semicontinuous mappings is first given under the framework of locally p-convex spaces, which is used as a tool to establish the general existence theorems for fixed points and the principle of nonlinear alternative and solutions for Birkhoff–Kellogg problem, including Leray–Schauder alternative, Rothe type, Altman type associated with various boundary conditions. In Sect. 7, we focus on the study of the general principle for nonlinear alternative for semiclosed contractive set-valued mappings under various boundary conditions. In Sect. 8, we develop fixed points and a related principle of nonlinear alterative for the classes of semiclosed 1-set mappings including nonexpansive set-valued mappings as a special class under uniformly convex Banach spaces or locally convex spaces with the Opial condition.

For convenience of our discussion, throughout this paper, all p-vector spaces, locally p-convex spaces are assumed to be Hausdorff and p satisfies the condition for $p \in (0,1]$ unless specified otherwise. We also denote by $\mathbb{N}$ the set of all positive integers, i.e., $\mathbb{N} := \{1, 2, \dots, \}$. For a set X, the 2^X denotes the family of all subsets of X.

2 The basic results of p-vector spaces

We now recall some notion and definitions of p-convexities, p-vector spaces for Hausdorff topological vector spaces, and locally p-convex spaces, which will be used in what follows (see Jarchow [61], Kalton [62], Rolewicz [114], Bayoumi [8], Gholizadeh et al. [46], or Ennassik and Taoudi [37]).

Definition 2.1 Let $p \in (0,1]$. A set A in a vector space X is said to be p-convex if for any $x, y \in A$ we have $sx + ty \in A$, whenever $0 \leq s, t \leq 1$ with $s^p + t^p = 1$; the set A is said to be absolutely p-convex if for any $x, y \in A$ we have $sx + ty \in A$, whenever $|s|^p + |t|^p \leq 1$. In the case $p = 1$, the concept of the (absolutely) 1-convexity is simply the usually (absolutely) convex defined in vector spaces.

Definition 2.2 Let $p \in (0,1]$. If A is a subset of a topological vector space X, the closure of A is denoted by $\overline{A}$, then the p-convex hull of A and its closed p-convex hull are denoted by $C_p(A)$ and $\overline{C}_p(A)$, respectively, which is the smallest p-convex set containing A and the smallest closed p-convex set containing A, respectively.

Definition 2.3 Let $p \in (0,1]$, A be p-convex and $x_1, \dots, x_n \in A$, and $t_i \geq 0$, $\sum_1^n t_i^p = 1$. Then $\sum_1^n t_i x_i$ is called a p-convex combination of $\{x_i\}$ for $i = 1, 2, \dots, n$. If $\sum_1^n |t_i|^p \leq 1$, then $\sum_1^n t_i x_i$ is called an absolutely p-convex combination. It is easy to see that $\sum_1^n t_i x_i \in A$ for a p-convex set A.

Definition 2.4 A subset A of a vector space X is called balanced (or circled) if $\lambda A \subset A$ holds for all scalars λ satisfying $|\lambda| \leq 1$. We say that A is absorbing if for each $x \in X$ there is a real number $\rho_x > 0$ such that $\lambda x \in A$ for all $\lambda > 0$ with $|\lambda| \leq \rho_x$.

By Definition 2.4, it is easy to see that the system of all balanced (circled) subsets of X is easily seen to be closed under the formation of linear combinations, arbitrary unions, and arbitrary intersections. A balanced set A is symmetric, and thus $A = -A$. In particular, every set $A \subset X$ determines the smallest circled subset $\hat{A}$ of X in which it is contained: $\hat{A}$ is called the circled hull of A. It is clear that $\hat{A} = \bigcup_{|\lambda| \leq 1} \lambda A$ holds so that A is circled if and only if (in short, iff) $\hat{A} = A$. We use $\overline{\hat{A}}$ to denote the closed circled hull of $A \subset X$. In addition, if X is a topological vector space, then we use the $\mathrm{int}(A)$ to denote the interior of set $A \subset X$ and if $0 \in \mathrm{int}(A)$, then $\mathrm{int}(A)$ is also circled; and we use ∂A to denote the boundary of A in X.

Definition 2.5 Let X be a vector space and $\mathbb{R}^+$ be a nonnegative part of a real line $\mathbb{R}$. Then a mapping $P : X \longrightarrow \mathbb{R}^+$ is said to be a p-seminorm if it satisfies the requirements for $(0 < p \leq 1)$:

(i) $P(x) \geq 0$ for all $x \in X$;
(ii) $P(\lambda x) = |\lambda|^p P(x)$ for all $x \in X$ and $\lambda \in R$;
(iii) $P(x + y) \leq P(x) + P(y)$ for all $x, y \in X$.

An p-seminorm P is called a p-norm if $x = 0$ whenever $P(x) = 0$. A topological vector space with a specific p-norm is called a p-normed space. Of course if $p = 1$, then X is the usual normed space. By Lemma 3.2.5 of Balachandra [7], the following proposition gives a necessary and sufficient condition for a p-seminorm to be continuous.

Proposition 2.1 *Let X be a topological vector space, P be a p-seminorm on X and $V := \{x \in X : P(x) < 1\}$. Then P is continuous if and only if $0 \in \mathrm{int}(V)$, where $\mathrm{int}(V)$ is the interior of V.*

Now, given an p-seminorm P, the p-seminorm topology determined by P (in short, the p-topology) is the class of unions of open balls $B(x, \epsilon) := \{y \in X : P(y - x) < \epsilon\}$ for $x \in X$ and $\epsilon > 0$.

We also need the following notion for the so-called p-gauge (see Balachandra [7]).

Definition 2.6 Let A be an absorbing subset of a vector space X. For $x \in X$ and $0 < p \leq 1$, set $P_A = \inf\{\alpha > 0 : x \in \alpha^{\frac{1}{p}} A\}$, then the nonnegative real-valued function P_A is called p-gauge (gauge if $p = 1$). The p-gauge of A is also known as the Minkowski p-functional.

By Proposition 4.1.10 of Balachandra [7], we have the following proposition.

Proposition 2.2 *Let A be an absorbing subset of X. Then a p-gauge P_A has the following properties*:

(i) $P_A(0) = 0$;
(ii) $P_A(\lambda x) = |\lambda|^p P_A(x)$ *if* $\lambda \geq 0$;
(iii) $P_A(\lambda x) = |\lambda|^p P_A(x)$ *for all* $\lambda \in R$ *provided* A *is circled*;
(iv) $P_A(x + y) \leq P_A(x) + P_A(y)$ *for all* $x, y \in A$ *provided* A *is* p*-convex.*

In particular, P_A is a p-seminorm if A is absolutely p-convex (and also absorbing).

Remark 2.1 It is worthwhile to note that a 0-neighborhood in a topological vector space is absolutely 0-neighborhoods, which are also absorbing (see Lemma 2.1.16 of Balachandran [7] or Proposition 2.2.3 of Jarchow [61]), thus it makes sense for us to define a topological vector space E to be a topological p-vector space (in short, p-vector space) for $p \in (0, 1]$ by using the concept of the Minkowski p-functional, as given below.

Definition 2.7 A topological vector space X is said to be a topological p-vector space (in short, p-vector space) if the base of the origin in X is generated by a family of Minkowski p-functionals (p-gauges) (defined by Definition 2.6), where $p \in (0, 1]$.

By incorporating Proposition 2.2, it seems that the following is a natural way to lead us to have the definition for a p-vector space being locally p-convex, where $p \in (0, 1]$.

Definition 2.8 A topological vector space X is said to be locally p-convex if the origin in X has a fundamental set of absolutely p-convex 0-neighborhoods. This topology can be determined by p-seminorms which are defined in the obvious way (see p. 52 of Bayoumi [8], Jarchow [61], or Rolewicz [114]). When $p = 1$, a locally p-convex space X is reduced to being a usual locally convex space.

By Proposition 4.1.12 of Balachandra [7], we also have the following proposition.

Proposition 2.3 *Let* A *be a subset of a vector space* X*, which is absolutely* p*-convex* ($0 < p \leq 1$) *and absorbing. Then, we have that*

(i) *The* p*-gauge* P_A *is a* p*-seminorm such that if* $B_1 := \{x \in X : P_A(x) < 1\}$ *and* $\overline{B_1} = \{x \in X : P_A(x) \leq 1\}$*, then* $B_1 \subset A \subset \overline{B_1}$*; in particular,* $\ker P_A \subset A$*, where* $\ker P_A := \{x \in X : P_A(x) = 0\}$.
(ii) $A = B_1$ *or* $\overline{B_1}$*, according to whether* A *is open or closed in the* P_A*-topology.*

Remark 2.2 Let X be a topological vector space, and let U be an open absolutely p-convex neighborhood of the origin, and let ϵ be given. If $y \in \epsilon^{\frac{1}{p}} U$, then $y = \epsilon^{\frac{1}{p}} u$ for some $u \in U$ and $P_U(y) = P_U(\epsilon^{\frac{1}{p}} u) = \epsilon P_U(u) \leq \epsilon$ (as $u \in U$ implies that $P_U(u) \leq 1$). Thus, P_U is continuous at *zero*, and therefore P_U is continuous everywhere. Moreover, we have $U = \{x \in X : P_U(x) < 1\}$.

Indeed, since U is open and the scalar multiplication is continuous, we have that for any $x \in U$ there exists $0 < t < 1$ such that $x \in t^{\frac{1}{p}} U$, and so $P_U(x) \leq t < 1$. This shows that $U \subset \{x \in X : P_U(x) < 1\}$. The conclusion follows by Proposition 2.3.

The following result is a very important and useful result which allows us to make the approximation for convex subsets in topological vector spaces by p-convex subsets in p-convex vector spaces (see Lemma 2.1 of Ennassik and Taoudi [37], Remark 2.1 of Qiu and Rolewicz [109], or Lemma 2.1 of Yuan [144, 145]), thus we omit their proof.

Lemma 2.1 *Let A be a subset of a vector space X, then we have*:

(i) *If A is r-convex with $0 < r < 1$, then $\alpha x \in A$ for any $x \in A$ and any $0 < \alpha \leq 1$.*
(ii) *If A is convex and $0 \in A$, then A is s-convex for any $s \in (0, 1]$.*
(iii) *If A is r-convex for some $r \in (0, 1)$, then A is s-convex for any $s \in (0, r]$.*

Remark 2.3 We would like to point out that results (i) and (iii) of Lemma 2.1 do not hold for $p = 1$. Indeed, any singleton $\{x\} \subset X$ is convex in topological vector spaces; but if $x \neq 0$, then it is not p-convex for any $p \in (0, 1)$.

We also need the following proposition, which is Proposition 6.7.2 of Jarchow [61].

Proposition 2.4 *Let K be compact in a topological vector X and $(1 < p \leq 1)$. Then the closure $\overline{C}_p(K)$ of the p-convex hull and the closure $\overline{AC}_p(K)$ of absolutely p-convex hull of K are compact if and only if $\overline{C}_p(K)$ and $\overline{AC}_p(K)$ are complete, respectively.*

We also need the following fact, which is a special case of Lemma 2.4 of Xiao and Zhu [135].

Lemma 2.2 *Let C be a (bounded) closed p-convex subset of a topological vector space X and $0 \in \operatorname{int} C$, where $(0 < p \leq 1)$. For each $x \in X$, we define an operator by $r(x) := \frac{x}{\max\{1,(P_C(x))^{\frac{1}{p}}\}}$, where P_C is the Minkowski p-functional of C. Then C is a retract of X and $r : X \to C$ is continuous such that*:

(1) *if $x \in C$, then $r(x) = x$;*
(2) *if $x \notin C$, then $r(x) \in \partial C$;*
(3) *if $x \notin C$, then the Minkowski p-functional $P_C(x) > 1$.*

Proof Taking $s = p$ in Lemma 2.4 of Xiao and Zhu [135], Proposition 2.3, and Remark 2.2, the proof is complete. □

Remark 2.4 As discussed in Remark 2.2, Lemma 2.2 still holds if "the bounded closed p-convex subset C of the p-normed space $(X, \|\cdot\|_p)$" is replaced by "X is a p-seminorm vector space and C is a bounded closed absorbing p-convex subset with $0 \in \operatorname{int} C$ of X".

For a given p-convex subset C in a given p-vector space E with the origin (zero element) $0 \in \operatorname{int}(C)$ with the p-seminorm p (for example, thinking of the p-seminorm P_U, which is the Minkowski p-functional of U), we also denote by $d_P(x, C) := \inf\{P_U(x - y) : y \in C\}$ the distance of $\{x\}$ with the set C in space E for $p \in (0, 1]$.

For the convenience of our discussion, throughout this paper, we also assume all topological vector spaces and locally p-convex spaces are Hausdorff unless specified for $p \in (0, 1]$.

3 The KKM principle in abstract convex spaces

As mentioned above, Knaster, Kuratowski, and Mazurkiewicz (in short, KKM) [70] in 1929 obtained the so-called KKM principle (theorem) to give a new proof for the Brouwer fixed point theorem in finite dimensional spaces; and later in 1961, Fan [41] (see also Fan [43]) extended the KKM principle (theorem) to any topological vector spaces and applied it to various results including the Schauder fixed point theorem. Since then there have appeared a large number of works devoted to applications of the KKM principle (theorem). In 1992, such a research field was called the KKM theory for the first time by Park [95]. Then the KKM theory was extended to general abstract convex spaces by Park [99] (see also Park [100] and [101], Mauldin [84], Granas and Dugundji [53], Yuan [143], and the related references therein), which actually include locally p-convex spaces ($0 < p \leq 1$) as a special class.

Here we first give some notion and a brief introduction on the abstract convex spaces, which play an important role in the development of the KKM principle and related applications. Once again, for the corresponding comprehensive discussion on the KKM theory and its various applications to nonlinear analysis and related topics, we refer to Agarwal et al. [1], Granas and Dugundji [53], Mauldin [84], Park [101] and [102], Yuan [143], and the related comprehensive references therein.

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a given nonempty set D, and let 2^D denote the family of all subsets of D. We have the following definition for abstract convex spaces essentially introduced by Park [99].

Definition 3.1 An abstract convex space $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a set-valued mapping $\Gamma : \langle D\rangle \to 2^E$ with nonempty values $\Gamma_A := \Gamma(A)$ for each $A \in \langle D\rangle$, we have Γ-convex hull of any $D' \subset D$ is denoted and defined by $\mathrm{co}_\Gamma D' := \cup\{\Gamma_A | A \in \langle D'\rangle\} \subset E$.

A subset X of E is said to be a Γ-convex subset of $(E, D; \Gamma)$ relative to D' if, for any $N \in \langle D'\rangle$, we have $\Gamma_N \subseteq X$, that is, $\mathrm{co}_\Gamma D' \subset X$. For the convenience of our discussion, in the case $E = D$, the space $(E, E; \Gamma)$ is simply denoted by $(E; \Gamma)$ unless specified otherwise.

Definition 3.2 Let $(E, D; \Gamma)$ be an abstract convex space and Z be a topological space. For a set-valued mapping (or, say, multivalued mapping) $F : E \to 2^Z$ with nonempty values, if a set-value mapping $G : D \to 2^Z$ satisfies $F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y)$ for all $A \in \langle D\rangle$, then G is called a KKM mapping with respect to F. Clearly, a classical KKM mapping (see Mauldin [84]) $G : D \to 2^E$ is just a KKM mapping with respect to the identity map 1_E defined above.

Definition 3.3 The partial KKM principle for an abstract convex space $(E, D; \Gamma)$ is that, for any closed-valued KKM mapping $G : D \to 2^E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The KKM principle is that the same property also holds for any open-valued KKM mapping.

An abstract convex space is called a (partial) KKM space if it satisfies the (partial) KKM principle (resp.). We now give some known examples of (partial) KKM spaces (see Park [99] and also [100]) as follows.

Definition 3.4 A ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \{D\}$ with $|A| = n + 1$. By putting $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D\rangle$, the triple $(X, D; \Gamma)$ becomes an abstract convex space.

Remark 3.1 For a ϕ_A-space $(X, D; \{\phi_A\})$, we see that any set-valued mapping $G : D \to X$ satisfying $\phi_A(\Delta_J) \subset G(J)$ for each $A \in \langle D\rangle$ and $J \in \langle A\rangle$ is a KKM mapping.

By the definition given above, it is clear that every ϕ_A-space is a KKM space, thus we have the following fact (see Lemma 1 of Park [100]).

Lemma 3.1 *Let $(X,D;\Gamma)$ be a ϕ_A-space and $G:D\to 2^X$ be a set-valued (multimap) with nonempty closed [resp. open] values. Suppose that G is a KKM mapping, then $\{G(a)\}_{a\in D}$ has the finite intersection property.*

By Definition 2.7, we recall that a topological vector space is said to be locally p-convex if the origin has a fundamental set of absolutely p-convex 0-neighborhoods. This topology can be determined by p-seminorms, which are defined in the obvious way (see Jarchow [61] or p. 52 of Bayoumi [8]).

Now we have a new KKM space as follows inducted by the concept of p-convexity (see Lemma 2 of Park [100]).

Lemma 3.2 *Suppose that X is a subset of the topological vector space E and $p\in(0,1]$, and D is a nonempty subset of X such that $C_p(D)\subset X$. Let $\Gamma_N:=C_p(N)$ for each $N\in\langle D\rangle$ for each $pin(0,1]$. Then $(X,D;\Gamma)$ is clearly a ϕ_A-space.*

Proof Since $C_p(D)\subset X$, Γ_N is well defined. For each $N=\{x_0,x_1,\dots,x_n\}\subset D$, we define $\phi_N:\Delta_n\to\Gamma_N$ by $\sum_{i=0}^n t_ie_i\mapsto\sum_{i=0}^n(t_i)^{\frac{1}{p}}x_i$ for $p\in(0,1]$. Then, clearly, $(X,D;\Gamma)$ is a ϕ_A-space. This completes the proof. □

4 Fixed point theorems for set-valued and single-valued mappings in locally *p*-convex and *p*-vector spaces

In this section, we mainly give fixed point theorems for quasi upper semicontinuous set-valued mappings in locally p-convex spaces and compact continuous single-valued mappings in p-vector spaces. These fixed points will allow us to establish Rothe's principle, Leray–Schauder alternative in the next section, which would be useful tools in nonlinear analysis for the study of nonlinear problems arising from theory to practice. Here, we first gather together necessary definitions, notations, and known facts needed in this section.

Definition 4.1 Let X and Y be two topological spaces. A set-valued mapping (also called multifunction) $T:X\longrightarrow 2^Y$ is a point to set function such that for each $x\in X$, $T(x)$ is a subset of Y. The mapping T is said to be upper semicontinuous (USC) if the subset $T^{-1}(B):=\{x\in X:T(x)\cap B\neq\emptyset\}$ (equivalently, the set $\{x\in X:T(x)\subset B\}$) is closed (equivalently, open) for any closed (resp., open) subset B in Y. The function $T:X\to 2^Y$ is said to be lower semicontinuous (LSC) if the set $T^{-1}(A)$ is open for any open subset A in Y.

As an application of the KKM principle for general abstract convex spaces, we have the following general existence result for the "approximation" of fixed points for upper and lower semicontinuous set-valued mappings in locally p-convex spaces for $0<p\leq 1$ (see also the corresponding results given by Theorem 2.7 of Gholizadeh et al. [46], Theorem 5 of Park [100], and related discussion therein).

Theorem 4.1 *Let A be a p-convex compact subset of a locally p-convex space X, where $0<p\leq 1$. Suppose that $T:A\to 2^A$ is lower (resp. upper) semicontinuous with nonempty p-convex values. Then, for any given U, which is a p-convex neighborhood of zero in X, there exists $x_U\in A$ such that $T(x_U)\cap(x_U+U)\neq\emptyset$.*

Proof Suppose that U is any given p-convex element of $\mathfrak{U}$, there is a symmetric open p-convex neighborhood V of zero for which $\overline{V} + \overline{V} \subset U$ in p-convex neighborhood of zero, we prove the results by two cases: T is lower semicontinuous (LSC) and upper semicontinuous (USC).

Case 1, by assuming that T is lower semicontinuous: As X is a locally p-convex vector space, suppose that $\mathfrak{U}$ is a family of neighborhoods of 0 in X. For any element U of $\mathfrak{U}$, there is a symmetric open p-convex neighborhood V of zero for which $\overline{V} + \overline{V} \subset U$. Since A is compact, there exist $x_0, x_1, \dots, x_n$ in A such that $A \subset \bigcup_{i=0}^n (x_i + V)$. By using the fact that A is p-convex, we find $D := \{b_0, b_2, \dots, b_n\} \subset A$ for which $b_i - x_i \in V$ for all $i \in \{0, 1, \dots, n\}$, and we define C by $C := C_p(D) \subset A$. By the fact that T is LSC, it follows that the subset $F(b_i) := \{c \in C : T(c) \cap (x_i + V) = \emptyset\}$ is closed in C (as the set $x_i + V$ is open) for each $i \in \{0, 1, \dots, n\}$. For any $c \in C$, we have $\emptyset \neq T(c) \cap A \subset T(c) \cap \bigcup_{i=0}^n (x_i + V)$, it follows that $\bigcap_{i=0}^n F(b_i) = \emptyset$. Now, we apply Lemma 3.1 and Lemma 3.2, which implies that there is $N := \{b_{i_0}, b_{i_1}, \dots, b_{i_k}\} \in \langle D\rangle$ and $x_U \in C_p(N) \subset A$ for which $x_U \notin F(N)$, and so $T(x_u) \cap (x_{i_j} + V) \neq \emptyset$ for all $j \in \{0, 1, \dots, k\}$. As $b_i - x_i \in V$ and $\overline{V} + \overline{V} \subset U$, which imply that $x_{i_j} + \overline{V} \subset b_{i_j} + U$, which means that $T(x_U) \cap ((b_{i_j} + U) \neq \emptyset$, it follows that $N \subset \{c \in C : T(x_U) \cap (c + U) \neq \emptyset\}$. By the fact that the subsets C, $T(x_U)$, and U are p-convex, we have that $x_U \in \{c \in C : T(x_U) \cap (c + U) \neq \emptyset\}$, which means that $T(x_U) \cap (x_U + U) \neq \emptyset$.

Case 2, by assuming T is upper semicontinuous: We define $F(b_i) := \{c \in C : T(c) \cap (x_i + \overline{V}) = \emptyset\}$, which is then open in C (as the subset $x_i + \overline{V}$ is closed) for each $i = 0, 1, \dots, n$. Then the argument is similar to the proof for the case T is USC, and by applying Lemma 3.1 and Lemma 3.2 again, it follows that there exists $x_U \in A$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$. This completes the proof. □

By Theorem 4.1, we have the following Fan–Glicksberg fixed point theorems (Fan [40]) in locally p-convex vector spaces for $(0 < p \leq 1)$, which also improve or generalize the corresponding results given by Yuan [143], Xiao, and Lu [134], Xiao and Zhu [135] into locally p-convex vector spaces.

Theorem 4.2 *Let A be a p-convex compact subset of a locally p-convex vector space X, where $0 < p \leq 1$. Suppose that $T : A \to 2^A$ is upper semicontinuous with nonempty p-convex closed values. Then T has one fixed point.*

Proof We denote by $\mathfrak{U}$ the family of neighborhoods of 0 in X, and $U \in \mathfrak{U}$, by Theorem 4.1, there exists $x_U \in A$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$. Then there exist $a_U, b_U \in A$ for which $b_U \in T(a_U)$ and $b_U \in a_U + U$. Now, two nets $\{a_U\}$ and $\{b_U\}$ in Graph(T), which is a compact graph of mapping T as A is compact and T is semicontinuous, we may assume that a_U has a subnet converging to a and $\{b_U\}$ has a subnet converging to b. As $\mathfrak{U}$ is the family of neighborhoods for 0, we should have $a = b$ (e.g., by the Hausdorff separation property) and $a = b \in T(b)$ due to the fact that Graph(T) is close (e.g., see Lemma 3.1.1 in p. 40 of Yuan [142]). The proof is complete. □

In the next part of this section, we are going to establish fixed point theorems for quasi upper semicontinuous set-valued mappings in topological vector spaces, which include upper semicontinuous set-valued mappings as a special class. Now we recall the following definitions.

By following Repovš et al. [111] (see also Ewert and Neubrunn [39] and Neubrunn [86]), we recall the following definition for quasi upper semicontinuous (QUSC) mappings, which are a generalization of upper semicontinuous (USC) mappings.

Definition 4.2 Let X and Y be two topological spaces and $T : X \longrightarrow 2^Y$ be a set-valued mapping. The mapping T is said to be quasi upper semicontinuous (QUSC) at $x \in X$ if, for each of its (x') neighborhood $W(x)$ and for each neighborhood V of the origin in Y, there exists a point $q(x) \in W(x)$ such that $x \in \operatorname{Int} T_{-1}(T(q(x)) + V))$, where $T_{-1}(T(q(x)) + V)) = \{z \in X : T(z) \subset T(q(x)) + V\}$, and the notation $\operatorname{Int} T_{-1}(T(q(x)) + V))$ denotes the (topological) interior of the set $T_{-1}(T(q(x)) + V))$ in X. The mapping T is said to be quasi supper semicontinuous if it is quasi upper semicontinuous at each point of its domain.

Remark 4.1 It is clear that in Definition 4.2 for QUSC mappings, for each $x \in X$, by taking $q(x)$ just being x itself, then it is just the definition for upper semicontinuous mappings given by Definition 3.1. Therefore, a USC mapping is a QUSC one, but a QUSC mapping may not be a USC mapping as shown by the example in p. 1094 due to Repovš et al. [111]. In addition, interested readers can see Ewert and Neubrunn [39] and Neubrunn [86] and the related references therein for the comprehensive study on the quasicontinuity for both single and set-valued mappings and related applications.

For a given set A in a vector space X, we denote by "lin(A)" the "linear hull" of A in X, then we also recall the following definition.

Definition 4.3 Let A be a subset of a topological vector space X, and let Y be another topological vector space. We shall say that A can be linearly embedded in Y if there is a linear map $L : \operatorname{lin}(A) \rightarrow Y$ (not necessarily continuous) whose restriction to A is a homeomorphism.

The following Lemma 4.1 is a significant embedded result for compact convex subsets in topological vector spaces, which is Theorem 1 of Kalton [62], which says that though not every compact convex set in TVS can be linearly imbedded in a locally convex space (e.g., see Roberts [112] and Kalton et al. [64]), but for p-convex sets when $0 < p < 1$, every compact p-convex set in topological vector spaces is considered as a subset of a locally p-convex vector space, hence every such set has sufficiently many p-extreme points.

Secondly, by property (ii) of Lemma 2.1, each convex subset containing zero in a topological vector space is always p-convex for $0 < p \leq 1$. Thus it is possible for us to transfer the problem involving p-convex subsets from topological vector spaces into the locally p-convex vector spaces, which indeed allows us to establish the existence of fixed points for compact single-valued mappings for noncompact p-convex subsets in locally p-convex spaces and p-vector spaces ($0 < p \leq 1$) to cover the case when the underlying is just a topological vector space, which provides the answer for Schauder's conjecture in the affirmative for the general version of compact continuous (single-valued) mappings in topological vector spaces (following the idea due to Ennassik and Taoudi [38]).

Lemma 4.1 *Let K be a compact p-convex subset* ($0 < p < 1$) *of a topological vector space X. Then K can be linearly embedded in a locally p-convex topological vector space.*

Proof It is Theorem 1 of Kalton [62], which completes the proof. □

Remark 4.2 At this point, it is important to note that Lemma 4.1 does not hold for $p = 1$. By Theorem 9.6 of Kalton et al. [64], it was shown that the spaces $L_p = L_p(0,1)$, where $0 < p < 1$, contain compact convex sets with no extreme points, which thus cannot be linearly embedded in a locally convex space, see also Roberts [112].

Definition 4.4 We recall that for two given topological spaces X and Y, a set-valued mapping $T : X \to 2^Y$ is said to be compact if there is a compact subset C in Y such that $F(X)(= \{y \in F(x), x \in X\})$ is contained in C, i.e., $F(X) \subset C$. Now we have the following non-compact versions of fixed point theorems for compact single-valued mappings defined in locally p-convex and topological vector spaces for $0 < p \leq 1$.

We now have the following result for a continuous single-valued mapping in locally p-convex spaces or topological vector spaces.

Theorem 4.3 *If K is a nonempty closed p-convex subset of either a Hausdorff locally p-convex space or a Hausdorff topological vector space X for $p \in (0,1]$, then the compact single-valued continuous mapping $T : K \to K$ has at least a fixed point.*

Proof As T is compact, there exists a compact subset A in K such that $T(K) \subset A$. Let $K_0 := \overline{C}_p(A)$ be the closure of the p-convex hull of the subset A in K. Then K_0 is compact p-convex by Proposition 2.4, and the mapping $T : K_0 \to K_0$ is continuous.

First, if K is a nonempty closed p-convex subset of a locally p-convex space, where $p \in (0,1]$, the conclusion is obtained by considering the self-mapping T on K_0 as an application of Theorem 3.1 by Ennassik and Taoudi [38].

Second, if K is a nonempty closed p-convex subset of a Hausdorff topological vector space X, we prove the conclusion by applying Lemma 4.1 in the following two cases.

Case 1: For $0 < p < 1$, K_0 is a nonempty compact p-convex subset of a topological vector space E for $p \in (0,1)$, by Lemma 4.1, it follows that K_0 can be linearly embedded in a locally p-convex space E, which means that there exists a linear mapping $L : \mathrm{lin}(K_0) \to E$ whose restriction to K_0 is a homeomorphism. Define the mapping $S : L(K_0) \to L(K_0)$ by $S(Lx) := L(Tx)$ for each $x \in K_0$, then this mapping is easily checked to be well defined. The mapping S is continuous since L is a (continuous) homeomorphism and T is continuous on K_0. Furthermore, the set $L(K_0)$ is compact, being the image of a compact set under a continuous mapping L, and $L(K_0)$ is also p-convex since it is the image of a p-convex set under a linear mapping. Then, by the conclusion given in the first part above, T has a fixed point $x \in K_0$. Thus there exists $x \in K_0$ such that $Lx = S(Lx) = L(Tx)$, thus it implies that $x = T(x)$ since L is a homeomorphism, which is the fixed point of T.

Case 2: For $p = 1$, taking any point $x_0 \in K_0$, let $K_0' := K_0 - \{x_0\}$. Now define a new mapping $T_0 : K_0' \to K_0'$ by $T_0(x - x_0) := T(x) - x_0$ for each $x - x_0 \in K_0'$. By the fact that now K_0' is compact and s-convex by Lemma 2.1(ii) for some $s \in (0,1)$, and T_0 is also continuous and has a fixed point in K_0' by the proof in Case 1, so T has a fixed point in K_0. The proof is complete. □

Before we establish the main results for the existence of fixed point theorem for quasi upper semicontinuous set-valued mappings in locally p-convex spaces, by following the idea for the proof of Theorem 1.10 by Repovš et al. [111] for the graph approximation of quasi upper semicontinuous set-valued mappings, using the concept of the "p-convexity" in locally p-convex spaces to replace the usual concept of "convexity" in LCS and TVS (see also related discussions by Ben-El-Mechaiekh [10], Ben-El-Mechaiekh and Saidi [11], Cellina [24], Kryszewsky [73], Repovš et al. [111], and related applications), we have the following Lemma 4.2, which is then used to establish a general fixed point theorem for upper semicontinuous set-valued mappings in locally p-convex spaces for $p \in (0,1]$, which is actually an extension of Theorems 4.2 and 4.3.

We recall that if X and Y are two topological spaces and $F : X \to 2^Y$ is a set-valued mapping, and we denote by either $\operatorname{Graph} F$ or Γ_F the graph of F in $X \times Y$, and α is a given open cover of Γ_F in $X \times Y$, then a (single- or set-valued) mapping $G : X \to Y$ is said to be an α-approximation (also called α-graph approximation) of F if for each point $p \in \Gamma_G$ there exists a point $q \in \Gamma_F$ such that p and q lie in some common element of the over α; and when G is a single-valued (continuous), G is also called a selection (continuous) mapping. In the case Y is a topological vector space, if Ω is the open cover of X and V is an open neighborhood of their origin in Y, then $\Omega \times \{y + V\}_{y\in Y}$ is one open cover of $X \times Y$, which is denoted by $\Omega \times V$ as used below. The following result was first given by Chang et al. [26], we provide the proof in detail here for the convenience of self-contained reading.

Lemma 4.2 *Let X be a paracompact space and Y be a topological vector space and $p \in (0,1]$. If $F : X \to 2^Y$ is an upper semicontinuous mapping with p-convex values, then for each open cover Ω of X, and each p-convex open neighborhood V of the origin in Y, there exists a continuous single-valued $(\Omega \times V)$-approximation for the set-valued mapping F. In particular, the conclusion holds if V is any convex open neighborhood of the origin in Y.*

Proof Let Ω be an open covering of X, and let V be a p-convex open neighborhood of the origin in Y. For each $x \in X$, fix an arbitrary element $W(x) \in \Omega$ such that $x \in W(x)$, then we first claim the following statements:

(1) By the upper semicontinuity (USC) of the mapping F, for each $x \in X$, there exists an open neighborhood $U(x) \subset W(x)$ such that $F(z) \subset F(x) + V$ for all $z \in U(x)$;

(2) As X is paracompact, by Theorem 3.5 of Dugundji [36] (see also Theorem 28 in Chap. 5 of Kelly [66]), without loss of generality, let the family $\{G(x)\}_{x\in X}$ be a covering, which is a star refinement of the covering $\{U(x)\}_{x\in X}$ of X (and see also the discussion on pp. 167–168 by Dugundji [36] for the concept of the star refinement for a given covering);

(3) Using the upper semicontinuity property again for the mapping F, for each $x \in X$, there exists an open neighborhood $U'(x) \subset G(x)$ such that $F(z) \subset F(x) + V$ for all $z \in U'(x)$;

(4) Let $\{e_\alpha\}_{\alpha\in A}$ be a locally finite continuous partition of unity inscribed into the covering $\{U'(x)\}_{x\in X}$ of X, where A is the index set, with $\Sigma_{\alpha\in A} e_{\alpha(x)} = 1$ for each $x \in X$; and for each $\alpha \in A$, we can choose $x_\alpha \in X$ such that $\operatorname{supp} e_\alpha \subset U'(x_\alpha)$ and one point $y_\alpha \in F(x_\alpha)$, where $\operatorname{supp} e_\alpha$ is the support of e_α (defined by $\operatorname{supp} e_\alpha := \overline{\{x \in X : e_\alpha(x) \neq 0\}}$); and

(5) Finally, define a mapping $f : X \to Y$ by $f(x) := \Sigma_{\alpha\in A} e_\alpha^{\frac{1}{p}}(x) y_\alpha$ for each $x \in X$, where $y_\alpha \in F(x_\alpha)$ as given by (4) above, then f is well defined, where the sum is taken over all $\alpha \in A$ with $e_\alpha(x) > 0$. By (3), it follows that $\Sigma_{\alpha\in A}(e_\alpha^{\frac{1}{p}}(x))^p = \Sigma_{\alpha\in A} e_\alpha(x) = 1$.

Now we show that f is indeed the desired single-valued continuous mapping, which is the $(\Omega \times V)$-approximation for the mapping F. Indeed, for any given $x_0 \in X$, we have that

$$x_0 \in St\bigl\{x_0, \{\operatorname{supp} e_\alpha\}_{\alpha\in A}\bigr\} \subset St\bigl\{x_0, \bigl\{U'(x)\bigr\}_{x\in X}\bigr\} \subset St\bigl\{x_0, \bigl\{G(x)\bigr\}_{x\in X}\bigr\} \subset U\bigl(x'\bigr) \subset W\bigl(x'\bigr)$$

for some $x' \in X$, where $St\{x_0, \{\operatorname{supp} e_\alpha\}_{\alpha\in A}\}$ denotes the star of the point $\{x_0\}$ with respect to the family $\{\operatorname{supp} e_\alpha\}_{\alpha\in A}$ and defined by $St\{x_0, \{\operatorname{supp} e_\alpha\}_{\alpha\in A}\} := \cup\{U : x_0 \in U, U \in \{\operatorname{supp} e_\alpha\}_{\alpha\in A}\}$ (see also the corresponding discussion for the notation and concept on p. 349 given by Ageev and Repovš [4]).

By the definition of upper semicontinuity, we have that $x' \in W(x')$. Hence the points x_0 and x' are Ω-close.

Secondly, if $e_\alpha(x_0) > 0$ for $\alpha \in A$, then $x_0 \in G(x_\alpha)$ and $x_\alpha \in G(x_\alpha)$ by (3) above. Thus $x_\alpha \in St\{x_0, \{G(x)\}_{x\in X}\} \subset U(x')$. Therefore, $y_\alpha \in F(x_\alpha) \subset F(x') + V$, i.e., $y_\alpha - v_\alpha \in V$ for some $v_\alpha \in F(x')$ for $\alpha \in A$. But then, for $v := \Sigma_\alpha e_\alpha^{\frac{1}{p}}(x_0)v_\alpha \in F(x')$ as F is p-convex-valued and we know that $\Sigma_{\alpha\in A}(e_\alpha^{\frac{1}{p}}(x))^p = \Sigma_{\alpha\in A} e_\alpha(x) = 1$ as shown by (5) above, and $y_\alpha - v_\alpha \in V$, too, for $\alpha \in A$, thus we have that $f(x_0) - v = \Sigma e_\alpha^{\frac{1}{p}}(x_0)(y_\alpha - v_\alpha) \in V$ as V is p-convex. Hence, the point $(x_0, f(x_0)) \in \operatorname{Graph}(f)$ is $(\Omega \times V)$-close to the point $(x', v) \in \operatorname{Graph}(F)$.

In particular, as each convex neighborhood of the origin in Y is also p-convex for each $p \in (0, 1]$, the conclusion holds. The proof is complete. □

As an application of Lemma 4.2, we now have the following fixed point theorem for quasi upper semicontinuous set-valued mappings in locally p-convex spaces for $p \in (0, 1]$, which was first initially discussed by Chang et al. [26].

Theorem 4.4 *Let K be a compact s-convex subset of a Hausdorff locally p-convex space X, where $p, s \in (0, 1]$. If $T : K \to 2^K$ is a quasi upper continuous set-valued mapping with nonempty closed p-convex values and its graph is closed, then T has a fixed point in K.*

Proof We give the proof by using the graph approximation approach for upper semicontinuous set-valued mappings established in this section above. Let $\mathfrak{U}$ be the family of absolutely p-convex open neighborhoods of the origin in X. By the fact the family $\{x + u\}_{x\in K}$ is an open covering of K, we denote the family $\{x + u\}_{x\in K}$ by Ω. Now, by Lemma 4.2, it follows that there exists one (single-valued) continuous mapping $f_u : K \to K$, which is $(\Omega \times u)$-approximation of the mapping T. By Theorem 4.3, f_u has a fixed point $x_u = f_u(x_u)$ in K for each $u \in \mathfrak{U}$. Note that $(x_u, f_u(x_u)) = (x_u, x_u) \in \operatorname{Graph}(f_u)$, which is $(\Omega \times u)$-approximation of the Graph(T), and the graph of T is closed due to the assumption, we will prove T has a fixed point x^*, which is indeed the limit of some subnet of the family $\{x_u\}_{u\in\mathfrak{U}}$ in K, i.e., $x^* \in T(x^*)$, by using notations of language in general topology (for related references on the discussion for normed spaces or topological (vector) spaces, see Cellina [24], Ben-El-Mechaiekh [10], and Fan [40]).

Indeed, for any given open p-convex member u in $\mathfrak{U}$, as the set $\{x + u\}_{x\in K} \times \{y + u\}_{y\in K}$ is an open cover of $K \times K$, by Lemma 4.2, there exists a single-valued continuous mapping $f_u : K \to K$, which is $(\Omega \times u)$-approximation of the Graph(T), where $\Omega := \{x + u\}_{x\in K}$ as mentioned above. By Theorem 4.3, f_u has a fixed point $x_u = f_u(x_u)$ in K for each $u \in \mathfrak{U}$. Now, for $x_u \in K$, by following the proof of Lemma 4.2, we observe that, firstly, there exists $x'_u \in K$ such that $x_u \in x'_u + u$; and secondly, there also exists $v_u \in F(x'_u)$ such that $f_u(x_u) - v_u \in u$, which means that $f_u(x_u) \in v_u + u$.

In summary, for any given $u \in \mathfrak{U}$, there exists a continuous mapping $f_u : K \to K$, which has at least one fixed point $x_u \in K$ such that $x_u = f_u(x_u)$ with $(x_u, x_u) = (x_u, f_u(x_u)) \in \operatorname{Graph}(f_u)$, and we also have the following statements:

(1) There exists $x'_u \in K$ such that $x_u \in x'_u + u$; and

(2) There exists $v_u \in F(x'_u)$ such that $f_u(x_u) - v_u \in u$, which means $f_u(x_u) \in v_u + u$.

Since K is compact, without loss of generality, we may assume that there exists a subnet $(x_{u_i})_{u_i\in\mathfrak{U}}$ converging to x^* in K. Now we will show that x^* is the fixed point of T, i.e., $x^* \in T(x^*)$.

As K is compact, without loss of generality, we may assume that two nets $\{x_u\}_{u\in\mathfrak{U}}$ and $\{x'_u\}_{u\in\mathfrak{U}}$ in K have the subnet $\{x_{u_i}\}_{u_i\in\mathfrak{U}}$ converging to x^*, and the subnet $\{x'_{u_i}\}_{u_i\in\mathfrak{U}}$ converges to x'^* respectively in K. By the statement of (1) above, it is clear that we must have $x^* = x'^*$; otherwise, as the family $\mathfrak{U}$ is the base of absolutely p-convex open neighborhoods of the origin in X, by (1) we will have the contradiction, and thus our claim that $x^* = x'^*$ is true in a locally p-convex space X.

Now we prove that x^* is a fixed point of T by using the statement of (2) for all $u \in \mathfrak{U}$. As the net $\{v_u\}_{u\in\mathfrak{U}} \subset K$, we may assume its subnet $\{v_{u_i}\}_{u_i\in\mathfrak{U}}$ converges to v^*. Then, by the statement given by (2), it is clear that we have that $\lim_{u_i\in\mathfrak{U}} v_{u_i} = v^* = \lim_{u_i\in\mathfrak{U}} f_{u_i}(x_{u_i}) = \lim_{u_i\in\mathfrak{U}} x_{u_i} = x^*$. By the fact that $(v_{u_i}, x'_{u_i}) \in \mathrm{Graph}(T)$ and the graph of T is closed, it follows that $x^* = v^* \in T(x^*)$, which means that x^* is a fixed point of T. The proof is complete. □

We note that Theorem 4.4 improves or unifies corresponding results given by Cauty [22], Cauty [23], Chang et al. [27], Dobrowolski [35], Nhu [87], Park [101], Reich [110], Smart [126], Xiao and Lu [134], Xiao and Zhu [135], Yuan [143–145] under the framework of compact single-valued or upper semicontinuous set-valued mappings.

Remark 4.3 Theorem 4.3 says that each compact single-valued mapping defined on a closed p-convex subsets ($0 < p \leq 1$) in topological vector spaces has the fixed point property, which does not only include or improve most available results for fixed point theorems in the existing literature as special cases (just to mention a few, Ben-El-Mechaiekh [10], Ben-El-Mechaiekh and Saidi [11], Ennassik and Taoudi [38], Mauldin [84], Granas and Dugundji [53], O'Regan and Precup [93], Reich [110], Park [101], and the references therein), but also provides an answer to Schauder conjecture in topological vector spaces in the affirmative for compact single-valued mappings defined on noncompact convex p-convex subsets in locally p-convex spaces for $0 < p \leq 1$ or topological vector spaces. In particular, we note that the answer to Schauder conjecture in the affirmative for a single-valued continuous mapping recently was obtained by Ennaassik and Taoudi [38] defined on a nonempty compact p-convex subset in TVS. Actually, we will show that Schauder conjecture is also true for quasi upper semicontinuous set-valued mappings in locally p-convex spaces as discussed by Theorems 4.4 and 4.7.

In addition. we we would like to point out that it is not clear if the assumption "$T(x)$ is with nonempty closed p-convex values" could be replaced with the condition "$T(x)$ is with nonempty closed s-convex values" in Theorem 4.4. In fact, it seems that the proof of Theorem 4.3 given by Ennassik et al. [37] only goes through for the case $s \leq p$, not for the general case when both $s, p \in (0, 1]$ (please note that the letter p is denoted as the letter r by Ennassik et al. [37]). Thus, we are still looking for a proper way to prove if the conclusion of Theorem 4.4 is true under Hausdorff topological vector spaces instead of locally p-convex spaces for $p \in (0, 1]$.

Now, as an immediate consequence of Theorem 4.4, we have the following fixed point result for QUSC mappings in a locally p-convex space X.

Corollary 4.1 *If K is a nonempty compact s-convex subset of a locally convex space X, where $s \in (0,1]$, then any quasi upper semicontinuous set-valued mapping $T : K \to 2^K$ with nonempty closed convex values and its graph being closed has at least one fixed point.*

Proof Apply Theorem 4.4 with $p = 1$, this completes the proof. □

Corollary 4.1 indeed improves or unifies the corresponding results given by Askoura and Godet-Thobie [6], Cauty [22], Cauty [23], Chang et al. [27], Chen [32], Theorem 3.1 and Theorem 3.3 of Ennssik and Taoudi [38], Theorem 3.14 of Gholizadeh et al. [46], Isac [60], Li [79], Nhu [87], Okon [89], Park [102], Reich [110], Smart [126], Xiao and Lu [134], Yuan [143] under the framework of locally p-convex spaces for set-valued (instead of single-valued) mappings.

As an application of Theorem 4.4, we have the following fixed point theorem for quasi upper semicontinuous set-valued mappings in locally p-convex spaces, which could be regarded as the extension or a set-valued version of Theorem 3.1 and Theorem 3.3 of Ennassik and Taoudi [38].

Theorem 4.5 *If K is a nonempty compact p-convex subset of a Hausdorff locally p-convex space X, where $p \in (0,1]$, then any quasi upper semicontinuous set-valued mapping $T : K \to 2^K$ with nonempty p-convex values and with a closed graph, has at least one fixed point.*

Proof By taking $s = p$ in Theorem 4.4, the conclusion follows. This completes the proof. □

By following the same idea used in the proof of Theorem 4.3, the conclusion of Theorem 4.4 still holds for compact quasi upper semicontinuous set-valued mappings as stated by Theorem 4.6 (and thus we omit its proof here).

Theorem 4.6 *If K is a nonempty closed s-convex subset of a Hausdorff locally p-convex space X, where $s, p \in (0,1]$, then any compact quasi upper semicontinuous set-valued mapping $T : K \to 2^K$ with nonempty p-convex values and with a closed graph has at least one fixed point.*

Now, as a special case in Theorem 4.6 with $p = 1$, we have the following results for compact QUSC mappings defined on s-convex subsets in locally convex spaces, where $s \in (0,1]$.

Corollary 4.2 *If K is a nonempty closed s-convex subset of a Hausdorff locally convex space X, then any compact quasi upper semicontinuous set-valued mapping $T : K \to 2^K$ with nonempty convex values and with a closed graph has at least one fixed point.*

Corollary 4.3 *Let K be a closed convex compact subset of a Hausdorff locally convex space X. If $T : K \to 2^K$ is a quasi upper continuous set-valued mapping with nonempty closed convex values and its graph is closed, then T has a fixed point in K.*

Corollary 4.4 (Schauder fixed point theorem for USC mappings in LCS) *Let K be a closed convex compact subset of a Hausdorff locally convex space X. If $T : K \to 2^K$ is an upper* *continuous set-valued mapping with nonempty closed convex values, then T has a fixed point in K.*

So far in this section, as the application of graph approximation for quasi upper semicontinuous mappings, which is Lemma 4.2, we have established general fixed point theorems for general (compact) quasi upper semicontinuous set-valued mappings in locally p-convex spaces, which allows us not only to answer Schauder's conjecture in the affirmative under the general framework of locally p-convex spaces, but also to unify or improve the corresponding results in the existing literature for nonlinear analysis, where $pin(0, 1]$.

We would like to mention that by comparing with topological degree approach or other related methods used or developed by Cauty [22, 23], Nhu [87], and others, the arguments used in this section actually provide an accessible way for the study of nonlinear analysis for p-convex vector spaces for $p \in (0, 1]$. The results given in this paper are new and may be easily understood and used by general readers in the mathematical community. In addition, the general fixed point theorems established for quasi upper semicontinuous set-valued mappings in locally p-convex spaces for $p \in (0, 1]$ or in topological vector spaces would play important roles for the study in functional analysis as those by Agarwal et al. [1], Ben-El-Mechaiekh [10], Ben-El-Mechaiekh and Saidi [11], Browder [17], Cellina [24], Chang [25], Chang et al. [27], Ennassik et al. [37], Fan [40, 41], Górniewicz [51], Granas and Dugundji [53], Guo et al. [55], Nhu [87], Park [101], Reich [110], Smart [126], Tychonoff [130], Weber [132, 133], Xiao and Lu [134], Xiao and Zhu [135], Xu [137], Yuan [142–145], Zeidler [146], and the related references therein. We would also like to point out that the results given in this part are new, which is the continuation of the related work given by Yuan [144, 145] recently.

In order to establish fixed point theorems for the classes of USC 1-set contractive and condensing mappings in locally p-convex spaces by using the concept of the measure of noncompactness (or saying, the noncompactness measures) that were introduced and widely accepted in mathematical community by Kuratowski [74], Darbo [33], and the related references therein, by following recent work due to Yuan [144, 145], we first need to have a brief introduction for the concept of noncompactness measures for the so-called Kuratowski or Hausdorff measures of noncompactness in normed spaces (see Alghamdi et al. [5], Machrafi and Oubbi [82], Nussbaum [88], Sadovskii [117], Silva et al. [123], Xiao and Lu [134] for the general concepts under the framework of p-seminorm or locally convex p-convex settings for $p \in (0, 1]$, which will be discussed below, too).

The same as those given by Yuan [144, 145], for a given metric space (X, d) (or a p-normed space $(X, \|\cdot\|_p)$), here we recall some notions and concepts for the completeness, boundedness, relative compactness, and compactness, which will be used in what follows. Let (X, d) and (Y, d) be two metric spaces and $T : X \to Y$ be a mapping (or, say, operator). Then: 1) T is said to be bounded if for each bounded set $A \subset X$, $T(A)$ is a bounded set of Y; 2) T is said to be continuous if for every $x \in X$, $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} T(x_n) = T$; and 3) T is said to be completely continuous if T is continuous and $T(A)$ is relatively compact for each bounded subset A of X.

Let $A_1, A_2 \subset X$ be bounded of a metric space (X, d), we also recall that the Hausdorff metric $d_H(A_1, A_2)$ between A_1 and A_2 is defined by

$$d_H(A_1, A_2) := \max\Big\{\sup_{x\in A_1}\inf_{y\in A_2} d(x, y), \sup_{y\in A_2}\inf_{x\in A_1} d(x, y)\Big\}.$$

The Hausdorff and Kurotowskii measures of noncompactness (denoted by β_H and β_K, respectively) for a nonempty bounded subset D in X are the nonnegative real numbers $\beta_H(D)$ and $\beta_K(D)$ defined by

$$\beta_H(D) := \inf\{\epsilon > 0 : D \text{ has a finite } \epsilon\text{-net}\}$$

and

$$\beta_K(D) := \inf\left\{\epsilon > 0 : D \subset \bigcup_{i=1}^{n} D_i, \text{where } D_i \text{ is bounded and } \operatorname{diam} D_i \leq \epsilon, n \text{ is an integer}\right\},$$

here $\operatorname{diam} D_i$ means the diameter of the set D_i, and it is well known that $\beta_H \leq \beta_K \leq 2\beta_H$. We also point out that the notions above can be well defined under the framework of p-seminorm spaces $(E, \|\cdot\|_p)_{p\in\mathfrak{P}}$ by following a similar idea and method used by Chen and Singh [31], Ko and Tasi [71], and Kozlov et al. [72]; see the references therein for more details.

Let T be a mapping from $D \subset X$ to X. Then we have that: 1) T is said to be a k-set contraction with respect to β_K (or β_H) if there is a number $k \in [0, 1)$ such that $\beta_K(T(A)) \leq k\beta_K(A)$ (or $\beta_H(T(A)) \leq k\beta_H(A)$) for all bounded sets A in D; and 2) T is said to be β_K-condensing (or β_H-condensing) if $(\beta_K(T(A)) < \beta_K(A))$ (or $\beta_H(T(A)) < \beta_H(A))$ for all bounded sets A in D with $\beta_K(A) > 0$ (or $\beta_H(A) > 0$).

For the convenience of our discussion, throughout the rest part of this paper, if a mapping "is β_K-condensing (or β_H-condensing)", we simply say it is "a condensing mapping" unless specified otherwise.

Moreover, it is easy to see that: (1) if T is a compact operator, then T is a k-set contraction; and (2) if T is a k-set contraction for $k \in (0, 1)$, then T is condensing.

To establish the fixed points of set-valued condensing mappings in locally p-convex spaces (and also p-vector spaces) for $p \in (0, 1]$, we need to recall some notions introduced by Machrafi and Oubbi [82] for the measure of noncompactness in locally p-convex vector spaces, which also satisfies some necessary (common) properties of the classical measures of noncompactness such as β_K and β_H mentioned above introduced by Kuratowski [74], Sadovskii [117](see also related discussion by Alghamdi et al. [5], Nussbaum [88], Silva et al. [123], Xiao and Lu [134], and the references therein). In particular, the measures of noncompactness in locally p-vector spaces (for $0 < p \leq 1$) should have the stable property, which means the measure of noncompactness A is the same by transition to the (closure) for the p-convex hull of subset A.

For the convenience of discussion, we follow up to use α and β to denote the Kuratowski and the Hausdorff measures of noncompactness in topological vector spaces, respectively (see the same way used by Machrafi and Oubbi [82]), unless otherwise stated. The E is used to denote a Hausdorff topological vector space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{Q}\}$, here $\mathbb{R}$ denotes all real numbers and $\mathbb{Q}$ all complex numbers, and $p \in (0, 1]$. Here, the base set of family of all balanced zero neighborhoods in E is denoted by $\mathfrak{V}_0$.

We recall that $U \in \mathfrak{V}_0$ is said to be shrinkable if it is absorbing, balanced, and $rU \subset U$ for all $r \in (0, 1)$, and we know that any topological vector space admits a local base at zero consisting of shrinkable sets (see Klee [69], or Jarchow [61] for details).

Recall that a topological vector space E is said to be a locally p-convex space if E has a local base at zero consisting of p-convex sets. The topology of a locally p-convex space is always given by an upward directed family P of p-seminorms, where a p-seminorm on E is any nonnegative real-valued and subadditive functional $\|\cdot\|_p$ on E such that $\|\lambda x\|_p = |\lambda|^p\|x\|_p$ for each $x \in E$ and $\lambda \in \mathbb{R}$ (i.e., the real number line). When E is Hausdorff, then for every $x \neq 0$, there is some $p \in P$ such that $P(x) \neq 0$. Whenever the family P is reduced to a singleton, one says that $(E, \|\cdot\|)$ is a p-seminorm space. A p-normed space is a Hausdorff p-seminorm space when $p = 1$, which is the usual locally convex case. Furthermore, a p-normed space is a metric vector space with the translation invariant metric $d_p(x, y) := \|x - y\|_p$ for all $x, y \in E$, which is the same notation as that used above.

By Remark 2.2, if P is a continuous p-seminorm on E, then the ball $B_p(0, s) := \{x \in E : P(x) < s\}$ is shrinkable for each $s > 0$. Indeed, if $r \in (0, 1)$ and $x \in \overline{rB_p(0, s)}$, then there exists a net $(x_i)_{i\in I} \subset B_p(0, s)$ such that rx_i converges to x. By the continuity of P, we get $P(x) \leq r^p s < s$, which means that $\overline{rB_p(0, s)} \subset B_P(0, s)$. In general, it can be shown that every p-convex $U \in \mathfrak{V}_0$ is shrinkable.

We recall that given such a neighborhood U, a subset $A \subset E$ is said to be U-small if $A - A \subset U$ (or, say, small of order U by Robertson [113]). Now, by following the idea of Kaniok [65] in the setting of a topological vector space E, we use zero neighborhoods in E instead of seminorms to define the measure of noncompactness in (local convex) p-vector spaces ($0 < p \leq 1$) as follows: For each $A \subset E$, the U-measures of noncompactness $\alpha_U(A)$ and $\beta_U(A)$ for A are defined by

$$\begin{aligned}\alpha_U(A) := {} & \inf\{: r > 0 : A \text{ is covered by a finite number of } rU\text{-small sets } A_i \\ & \text{for } i = 1, 2, \dots, n\}\end{aligned}$$

and

$$\beta_U(A) := \inf\left\{ r > 0 : \text{there exists } x_1, \dots, x_n \in E \text{ such that } A \subset \bigcup_{i=1}^{n}(x_i + rU) \right\},$$

here we set $\inf \emptyset := \infty$.

By the definition above, it is clear that when E is a normed space and U is the closed unit ball of E, α_U and β_U are nothing else but the Kuratowski measure β_K and Hausdorff measure β_H of noncompactness, respectively. Thus, if $\mathfrak{U}$ denotes a fundamental system of balanced and closed zero neighborhoods in E and $\mathfrak{F}_{\mathfrak{U}}$ is the space of all functions $\phi : \mathfrak{U} \to R$, endowed with the pointwise ordering, then the α_U (resp., β_U) measures for noncompactness introduced by Kaniok [65] can be expressed by the Kuratowski (resp., the Hausdorf) measure of noncompact $\alpha(A)$(resp., $\beta(A)$) for a subset A of E as the function defined from $\mathfrak{U}$ into $[0, \infty)$ by

$$\alpha(A)(U) := \alpha_U(A) \quad \bigl(\text{resp.}, \beta(A)(U) := \beta_U(A)\bigr).$$

By following Machrafi and Oubbi [82], to define the measure of noncompactness in (locally convex) p-vector space E, we need the following notions of basic and sufficient collections for zero neighborhoods in a topological vector space. To do this, let us introduce an equivalence relation on V_0 by saying that U is related to V, written $U\mathfrak{R}V$, if and only if there exist $r, s > 0$ such that $rU \subset V \subset sU$. We now have the following definition.

Definition 4.5 (BCZN) We say that $\mathfrak{B} \subset \mathfrak{V}_0$ is a basic collection of zero neighborhoods (in short, BCZN) if it contains at most one representative member from each equivalence class with respect to $\mathfrak{R}$. It is said to be sufficient (in short, SCZN) if it is basic and, for every $V \in \mathfrak{V}_0$, there exist some $U \in \mathfrak{B}$ and some $r > 0$ such that $rU \subset V$.

Remark 4.4 By Remark 2.2, it follows that for a locally p-convex space E, its base set $\mathfrak{U}$, the family of all open p-convex subsets for 0 is BCZB. We also note that: 1) In the case when E is a normed space, if f is a continuous functional on E, $U := \{x \in E : |f(x)| < 1\}$ and V is the open unit ball of E, then $\{U\}$ is basic but not sufficient, but $\{V\}$ is sufficient; 2) Secondly, if (E, τ) is a locally convex space, whose topology is given by an upward directed family P of seminorms so that no two of them are equivalent, then the collection $(B_p)_{p \in \mathbb{P}}$ is SCZN, where B_p is the open unit ball of p. Further, if $\mathfrak{W}$ is a fundamental system of zero neighborhoods in a topological vector space E, then there exists SCZN consisting of $\mathfrak{W}$ members; and 3) By following Oubbi [94], we recall that a subset A of E is called uniformly bounded with respect to a sufficient collection $\mathfrak{B}$ of zero neighborhoods if there exists $r > 0$ such that $A \subset rV$ for all $V \in \mathfrak{B}$. Note that in the locally convex space $C_c(X) := C_c(X, \mathbb{K})$, the set $B_\infty := \{f \in C(X) : \|f\|_\infty \leq 1\}$ is uniformly bounded with respect to the SCZN $\{B_k, k \in \mathbb{K}\}$, where B_k is the (closed or) open unit ball of the seminorm P_k, where $k \in \mathbb{K}$.

Now we are ready to give the definition for the measure of noncompactness in (locally p-convex) topological vector space E as follows.

Definition 4.6 Let $\mathfrak{B}$ be SCZN in E. For each $A \subset E$, we define the measure of noncompactness of A with respect to $\mathfrak{B}$ by $\alpha_{\mathfrak{B}}(A) := \sup_{U \in \mathfrak{B}} \alpha_U(A)$.

By the definition above, it is clear that: 1) The measure of noncompactness α_B holds the semiadditivity, i.e., $\alpha_B(A \cup B) = \max\{\alpha_B(A), \alpha_B(B)\}$; and 2) $\alpha_B(A) = 0$ if and only if A is a precompact subset of E (for more properties in detail, see Proposition 1 and related discussion by Machraf and Oubbi [94]).

As we know, under the normed spaces (and even seminormed spaces), Kuratowski [74], Darbo [33], and Sadovskii [117] introduced the notions of k-set-contractions for $k \in (0, 1)$ and condensing mappings to establish fixed point theorems in the setting of Banach spaces, normed or seminorm spaces. By following the same idea, if E is a Hausdorff locally p-convex space, we have the following definition for general (nonlinear) mappings.

Definition 4.7 A mapping $T : C \to 2^C$ is said to be a k-set contraction (resp., condensing) if there is some SCZN $\mathfrak{B}$ in E consisting of p-convex sets, such that (resp., condensing) for any $U \in \mathfrak{B}$, there exists $k \in (0, 1)$ (resp., condensing) such that $\alpha_U(T(A)) \leq k\alpha_U(A)$ for $A \subset C$ (resp., $\alpha_U(T(A)) < \alpha_U(A)$ for each $A \subset C$ with $\alpha_U(A) > 0$).

It is clear that a contraction mapping on C is a k-set contraction mapping (where we always mean $k \in (0, 1)$), and a k-set contraction mapping on C is condensing; and they all reduce to the usual cases by the definitions for β_K and β_H, which are the Kuratowski measure and the Hausdorff measure of noncompactness, respectively, in normed spaces (see Kuratowski [74]).

From now on, denote by $\mathfrak{V}_0$ the set of all shrinkable zero neighborhoods in E, we then have the following result, which is Theorem 1 of Machrafi and Oubbi [82], saying that in the general setting of locally p-convex spaces, the measure of noncompactness α for U given by Definition 4.3 is stable from U to its p-convex hull $C_p(A)$ of the subset A in E, which is key for us to establish fixed points for condensing mappings in locally p-convex spaces for $0 < p \leq 1$. This also means that the key property for the measures due to the Kurotowski and Hausdorff measures of noncompactness in normed (or p-seminorm) spaces also holds for the measure of noncompactness by Definition 4.3 in the setting of locally p-convex spaces with $(0 < p \leq 1)$ (for more details, see similar and related discussion by Alghamdi et al. [5] and Silva et al. [123]).

Lemma 4.3 *If $U \in \mathfrak{V}_0$ is p-convex for some $0 < p \leq 1$, then $\alpha(C_p(A)) = \alpha(A)$ for every $A \subset E$.*

Proof It is Theorem 1 of Machrafi and Oubbi [82]. The proof is complete. □

Now, based on the definition for the measure of noncompactness given by Definition 4.3 (originally from Machrafi and Oubbi [82]), we have the following general extended version of Schauder, Darbo, and Sadovskii type fixed point theorems in the context of locally p-convex vector spaces for condensing mappings.

Theorem 4.7 *Let $C \subset E$ be a complete s-convex subset of a locally p-convex space E with $s, p \in (0, 1]$. If $T : C \to 2^C$ is quasi upper semicontinuous and (α) condensing set-valued mappings with nonempty p-convex values and with a closed graph, then T has a fixed point in C.*

Proof Let $\mathfrak{B}$ be a sufficient collection of p-convex zero neighborhoods in E with respect to which T is condensing for any given $U \in \mathfrak{B}$. We choose some $x_0 \in C$ and let $\mathfrak{F}$ be the family of all closed p-convex subsets A of C with $x_0 \in A$ and $T(A) \subset A$. Note that $\mathfrak{F}$ is not empty since $C \in \mathfrak{F}$. Let $A_0 = \bigcap_{A \in \mathfrak{F}} A$. Then A_0 is a nonempty closed p-convex subset of C such that $T(A_0) \subset A_0$. We shall show that A_0 is compact. Let $A_1 = \overline{C_p(T(A_0) \cup \{x_0\})}$. Since $T(A_0) \subset A_0$ and A_0 is closed and p-convex, $A_1 \subset A_0$. Hence, $T(A_1) \subset T(A_0) \subset A_1$. It follows that $A_1 \in \mathfrak{F}$, and therefore $A_1 = A_0$. Now, by Proposition 1 of Machrafi and Oubbi [82] and Lemma 4.3 above (i.e., Theorem 1 and Theorem 2 in [82]), we get $\alpha_U(T(A_0)) = \alpha_U(A)$. Our assumption on T shows that $\alpha_U(A_0) = 0$ since T is condensing. As U is arbitrary from the family $\mathfrak{B}$, thus A_0 is p-convex and compact (see Proposition 4 in [82]). Now, the conclusion follows by Theorem 4.4 (or Theorem 4.6) above. The proof is complete. □

As an application of Theorem 4.7, we have the following general result, which answers Schauder conjecture for quasi upper semicontinuous set-valued mappings defined on s-convex subsets in locally convex spaces, where $p \in (0, 1]$.

Theorem 4.8 (Schauder fixed point theorem for QUSC condensing mappings in LCS) *Let K be a nonempty closed p-convex subset of a locally p convex space, where $p \in (0, 1]$, then any quasi upper semicontinuous set-valued (α) condensing mapping $T : K \to 2^K$ with nonempty convex values and with a closed graph has at least a fixed point.*

Proof By letting $s = p$ in Theorem 4.7, the conclusion follows by Theorem 4.7. Thus we complete the proof. □

As a special case of Theorem 4.8, we have the following result.

Theorem 4.9 *Let K be a closed p-convex subset of a Hausdorff locally p-convex space X, where $p \in (0,1]$. If $T : K \to 2^K$ is an upper continuous condensing set-valued mapping with nonempty closed p-convex values, then T has a fixed point in K.*

Proof By the fact that each upper semicontinuous (USC) set-valued mapping is quasi upper semicontinuous and each USC with closed value has a closed graph, the conclusion follows by Theorem 4.7. This completes the proof. □

As applications of Theorem 4.9, we have a few theorems of fixed points for condensing mappings in locally p-convex spaces for $p \in (0,1]$ as follows.

Corollary 4.5 (Darbo type fixed point theorem) *Let C be a complete p-convex subset of a Hausdorff locally p-convex space E with $0 < p \leq 1$. If $T : C \to 2^C$ is a (k)-set-contraction (where $k \in (0,1)$) with closed and p-convex values, then T has a fixed point.*

Corollary 4.6 (Sadovskii type fixed point theorem) *Let $(E, \|\cdot\|)$ be a complete p-normed space and C be a bounded, closed, and p-convex subset of E, where $0 < p \leq 1$. Then every USC and condensing mapping $T : C \to 2^C$ with closed and p-convex values has a fixed point.*

Proof In Theorem 4.7, let $\mathfrak{B} := \{B_p(0,1)\}$, where $B_p(0,1)$ stands for the closed unit ball of E, and by the fact that it is clear that $\alpha(A) = (\alpha_{\mathfrak{B}}(A))^p$ for each $A \subset E$. Then that T satisfies all conditions of Theorem 4.7. This completes the proof. □

Corollary 4.7 (Darbo type) *Let $(E, \|\cdot\|)$ be a complete p-normed space and C be a bounded, closed, and p-convex subset of E, where $0 < p \leq 1$. Then each mapping $T : C \to C$ that is continuous and a set-contraction has a fixed point.*

Theorem 4.7 and also Theorem 4.8 improve Theorem 5 of Machrafi and Oubbi [82] for general condensing mappings that are general upper semicontinuous mappings with closed p-convex values and also unify the corresponding results in the existing literature, e.g., see Alghamdi et al. [5], Górniewicz [51], Górniewicz et al. [52], Nussbaum [88], Silva et al. [123], Xiao and Lu [134], Xiao and Zhu [135], and the references therein.

Secondly, as an application of the KKM principle for abstract convex spaces with graph approximation Lemma 4.2 for quasi upper semicontinuous set-valued mappings in locally p-convex spaces, we establish general fixed point theorems for quasi upper semicontinuous set-valued mappings, which allow us to answer Schauder's conjecture in the affirmative way under the framework of locally p-convex spaces for $p \in (0,1]$.

Before the ending of this section, we would also like to remark that by comparing with topological method or related arguments used by Askoura et al. [6], Cauty [22, 23], Dobrowolski [35], Nhu [87], Reich [110], the fixed points given in this section improve or unify the corresponding ones given by Alghamdi et al. [5], Darbo [33], Liu [81], Machrafi and Oubbi [82], Sadovskii [117], Silva et al. [123], Xiao and Lu [134], Yuan [144, 145], and those from the references therein.

5 Best approximation for the class of 1-set contractive mappings in locally *p*-convex spaces

The goal of this section is first to establish one general best approximation result for 1-set upper semicontinuous and hemicompact (see its definition below) nonself set-valued mappings, which in turn is used as a tool to derive the general principle for the existence of solutions for Birkhoff–Kellogg problems (see Birkhoff and Kellogg [14]) and fixed points for nonself 1-set contractive set-valued mappings.

Here, we recall that since the Birkhoff–Kellogg theorem was first introduced and proved by Birkhoff and Kellogg [14] in 1922 in discussing the existence of solutions for the equation $x = \lambda F(x)$, where λ is a real parameter and F is a general nonlinear nonself mapping defined on an open convex subset U of a topological vector space E, now the general form of the Birkhoff–Kellogg problem is to find the so-called invariant direction for nonlinear set-valued mappings F, i.e., to find $x_0 \in \overline{U}$ (or $x_0 \in \partial\overline{U}$) and $\lambda > 0$ such that $\lambda x_0 \in F(x_0)$.

Since the Birkhoff and Kellogg theorem given by Birkhoff and Kellogg in 1920s, the study on Birkhoff–Kellogg problem has been received a lot of scholars' attention. For example, one of the fundamental results in nonlinear functional analysis, called the Leray–Schauder alternative, was established via topological degree by Leray and Schauder [76] in 1934. Thereafter, certain other types of Leray–Schauder alternatives were proved using different techniques other than topological degree, see the work by Granas and Dugundji [53], Furi and Pera [44] in the Banach space setting and applications to the boundary value problems for ordinary differential equations, and a general class of mappings for nonlinear alternative of Leray–Schauder type in normal topological spaces, and also Birkhoff–Kellogg type theorems for general class mappings in TVS by Agarwal et al. [1], Agarwal and O'Regan [2, 3], Park [98]. In particular, recently O'Regan [91] used the Leray–Schauder type coincidence theory to establish some Birkhoff–Kellogg problem and Furi–Pera type results for a general class of set-valued mappings.

In this section, one best approximation result for 1-set contractive mappings in p-seminorm spaces is first established, which is then used to the general principle for solutions of Birkhoff–Kellogg problems and related nonlinear alternatives, then it allows us to give general existence results for the Leray–Schauder type and related fixed point theorems of nonself mappings in p-seminorm spaces for $p \in (0, 1]$. The new results given in this part not only include the corresponding results in the existing literature as special cases, but are also expected to be useful tools for the study of nonlinear problems arising from theory to practice for 1-set contractive mappings.

We also note that the general nonlinear alternative related to Leray–Schauder alternative under the framework of p-seminorm spaces for $p \in (0, 1]$ given in this section would be a useful tool for the study of nonlinear problems. In addition, we also note that corresponding results in the existing literature for Birkhoff–Kellogg problems and the Leray–Schauder alternative have been studied comprehensively by Granas and Dugundji [53], Isac [60], Park [99–101], Carbone and Conti [21], Chang and Yen [30], Chang et al. [28, 29], Kim et al. [67], Shahzad [120–122], Singh [125]; and in particular, many general forms have been recently obtained by O'Regan [92] (see also the references therein).

To study the existence of fixed points for nonself mappings in p-vector spaces, we need the following definitions.

Definition 5.1 (Inward and outward sets in p-vector spaces) Let C be a subset of a p-vector space E and $x \in E$ for $0 < p \leq 1$. Then the p-inward set $I_C^p(x)$ and p-outward set $O_C^p(x)$ are defined by

$I_C^p(x) := \{x + r(y - x) : y \in C$ for any $r \geq 0$ (1) if $0 \leq r \leq 1$ with $(1-r)^p + r^p = 1$; or (2) if $r \geq 1$ with $(\frac{1}{r})^p + (1 - \frac{1}{r})^p = 1\}$; and

$O_C^p(x) := \{x + r(y - x) : y \in C$ for any $r \leq 0$ (1) if $0 \leq |r| \leq 1$ with $(1-|r|)^p + |r|^p = 1$; or (2) if $|r| \geq 1$ with $(\frac{1}{|r|})^p + (1 - \frac{1}{|r|})^p = 1\}$.

From the definition, it is obvious that when $p = 1$, both the inward and outward sets $I_C^p(x)$, $O_C^p(x)$ are reduced to the definition for the inward set $I_C(x)$ and the outward set $O_C(x)$, respectively, in topological vector spaces introduced by Halpern and Bergman [56] and used for the study of nonself mappings related to nonlinear functional analysis in the literature. In this paper, we mainly focus on the study of the p-inward set $I_U^p(x)$ for the best approximation related to the boundary condition for the existence of fixed points in p-vector spaces. By the special property of p-convex concept when $p \in (0,1)$ and $p = 1$, we have the following fact.

Lemma 5.1 *Let C be a subset of a p-vector space E and $x \in E$, where $0 < p \leq 1$. Then for both p-inward and outward sets $I_C^p(x)$ and $O_C^p(x)$ defined above, we have*

(I) *when $p \in (0,1)$, $I_C^p(x) = [\{x\} \cup C]$ and $O_C^p(x) = [\{x\} \cup \{2x\} \cup -C]$,*

(II) *when $p = 1$, in general $[\{x\} \cup C] \subset I_C^p(x)$ and $[\{x\} \cup \{2x\} \cup -C] \subset O_C^p(x)$.*

Proof First, when $p \in (0,1)$, by the definitions of $I_C^p(x)$, the only real number $r \geq 0$ satisfying the equation $(1-r)^p + r^p = 1$ for $r \in [0,1]$ is $r = 0$ or $r = 1$, and when $r \geq 1$, the equation $(\frac{1}{r})^p + (1 - \frac{1}{r})^p = 1$ implies that $r = 1$. The same reason for $O_C^p(x)$, it follows that $r = 0$ and $r = -1$.

Secondly when $p = 1$, all $r \geq 0$ and all $r \leq 0$ satisfy the requirement of definition for $I_C^p(x)$ and $O_C^p(x)$, respectively, thus the proof is complete. □

By following the original idea by Tan and Yuan [129] for hemicompact mappings in metric spaces, we introduce the following definition for a mapping being hemicompact in p-seminorm spaces for $p \in (0,1]$, which is indeed the "*(H) condition*" used in Theorem 5.1 to prove the existence of best approximation results for 1-set contractive set-valued mappings in p-seminorm vector spaces for $p \in (0,1]$.

Definition 5.2 (Hemicompact mapping) Let E be a p-vector space with p-seminorm for $1 < p \leq 1$. For a given bonded (closed) subset D in E, a mapping $F : D \to 2^E$ is said to be hemicompact if each sequence $\{x_n\}_{n \in N}$ in D has a convergent subsequence with limit x_0 such that $x_0 \in F(x_0)$, whenever $\lim_{n\to\infty} d_{P_U}(x_n, F(x_n)) = 0$ for each $U \in \mathfrak{U}$, where $d_{P_U}(x, C) := \inf\{P_U(x - y) : y \in C\}$ is the distance of a single point x with the subset C in E based on P_U, P_U is the Minkowski p-functional in E for $U \in \mathfrak{U}$, which is the base of the family consisting of all subsets of 0-neighborhoods in E.

Remark 5.1 We would like to point out that Definition 5.2 is indeed an extension for a "hemicompact mapping" defined from a metric space to a p-vector space with the p-seminorm, where $p \in (0,1]$ (see Tan and Yuan [129]). By the monotonicity of Minkowski

p-functionals, i.e., the bigger 0-neighborhoods, the smaller Minkowski p-functionals' values (see also p. 178 of Balachandran [7]), Definition 5.2 describes the convergence for the distance between x_n and $F(x_n)$ by using the language of seminorms in terms of Minkowski p-functionals for each 0-neighborhood in $\mathfrak{U}$ (the base), which is the family consisting of its 0-neighborhoods in p-vector space E.

Now we have the following Schauder fixed point theorem for 1-set contractive mappings in locally p-convex spaces for $p \in (0, 1]$.

Theorem 5.1 (Schauder fixed point theorem for 1-set contractive mappings) *Let U be a nonempty bounded open p-convex subset of a (Hausdorff) locally p-convex space E and its zero $0 \in U$, and let $C \subset E$ be a closed p-convex subset of E such that $0 \in C$ with $0 < p \leq 1$. If $F : C \cap \overline{U} \to 2^{C \cap \overline{U}}$ is a quasi upper semicontinuous and 1-set contractive set-valued mapping with nonempty p-convex values and with a closed graph and satisfying the following (H) or (H1) condition:*

(H) Condition: The sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\overline{U}$ has a convergent subsequence with limit $x_0 \in \overline{U}$ such that $x_0 \in F(x_0)$, whenever $\lim_{n\to\infty} d_{P_U}(x_n, F(x_n)) = 0$, where $d_{P_U}(x_n, F(x_n)) := \inf\{P_U(x_n - z) : z \in F(x_n)\}$, where P_U is the Minkowski p-functional for any $U \in \mathfrak{U}$, which is the family of all nonempty open p-convex subset containing zero in E.

(H1) Condition: There exists x_0 in $\overline{U}$ with $x_0 \in F(x_0)$ if there exists $\{x_n\}_{n\in\mathbb{N}}$ in $\overline{U}$ such that $\lim_{n\to\infty} d_{P_U}(x_n, F(x_n)) = 0$, where P_U is the Minkowski p-functional for any $U \in \mathfrak{U}$, which is the family of all nonempty open p-convex subsets containing zero in E.

Then F has at least one fixed point in $C \cap \overline{U}$.

Proof Let $\mathfrak{U}$ be a family of all nonempty open p-convex subset containing zero in E, and let U be any element in $\mathfrak{U}$. As the mapping T is 1-set contractive, take an increasing sequence $\{\lambda_n\}$ such that $0 < \lambda_n < 1$ and $\lim_{n\to\infty} \lambda_n = 1$, where $n \in \mathbb{N}$. Now we define a mapping $F_n : C \to 2^C$ by $F_n(x) := \lambda_n F(x)$ for each $x \in C$ and $n \in \mathbb{N}$. Then it follows that F_n is a λ_n-set-contractive mapping with $0 < \lambda_n < 1$, and it is also quasi upper semicontinuous with p-convex values, and its graph is also closed. Now, by Theorem 4.8 on the condensing mapping F_n in locally p-convex spaces with p-seminorm P_U (which is the Minkowski p-functional for $U \in \mathfrak{U}$), for each $n \in \mathbb{N}$, there exists $x_n \in C$ such that $x_n \in F_n(x_n) = \lambda_n F(x_n)$. Thus there exists $y_n \in F(x_n)$ such that $x_n = \lambda_n y_n$. As P_U is the Minkowski p-functional of U in E, it follows that P_U is continuous as $0 \in \operatorname{int}(U) = U$. Note that for each $n \in \mathbb{N}$, $\lambda_n x_n \in \overline{U} \cap C$, which implies that $x_n = r(\lambda_n y_n) = \lambda_n y_n$, thus $P_U(\lambda_n y_n) \leq 1$ by Lemma 2.2. Note that

$$\begin{aligned} P_U(y_n - x_n) &= P_U(y_n - x_n) \\ &= P_U(y_n - \lambda_n y_n) \\ &= P_U\left(\frac{(1-\lambda_n)\lambda_n y_n}{\lambda_n}\right) \leq \left(\frac{1-\lambda_n}{\lambda_n}\right)^p P_U(\lambda_n y_n) \leq \left(\frac{1-\lambda_n}{\lambda_n}\right)^p, \end{aligned}$$

which implies that $\lim_{n\to\infty} P_U(y_n - x_n) = 0$ for all $U \in \mathfrak{U}$.

Now (1) if F satisfies the (H) condition, it implies that the consequence $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence that converges to x_0 such that $x_0 \in F(x_0)$. Without loss of generality, we assume that $\lim_{n\to\infty} x_n = x_0$, here $y_n \in F(x_n)$ is with $x_n = \lambda_n y_n$, and $\lim_{n\to\infty} \lambda_n = 1$, it implies that $x_0 = \lim_{n\to\infty}(\lambda_n y_n)$, which means $y_0 := \lim_{n\to\infty} y_n = x_0$. There exists $y_0(= x_0) \in F(x_0)$.

(ii) If F satisfies the (H1) condition, then by the (H1) condition, it follows that there exists x_0 in $\overline{U}$ such that $x_0 \in F(x_0)$, which is a fixed point of F. We complete the proof. □

Theorem 5.2 (Best approximation for 1-set-contractive mappings) *Let U be a bounded open p-convex subset of a locally p-convex space E $(0 < p \leq 1)$, zero $0 \in U$, and C be a (bounded) closed convex subset of E with also zero $0 \in C$. Assume that $F : \overline{U} \cap C \to 2^C$ is a 1-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph, and for each $x \in \partial_C U$ with $y \in F(x) \cap (C \setminus \overline{U}))$, $(P_U^{\frac{1}{p}}(y) - 1)^p \leq P_U(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$). In addition, if F satisfies the following (H) or (H1) condition:*

(H) Condition: The sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\overline{U}$ has a convergent subsequence with limit $x_0 \in \overline{U}$ such that $x_0 \in F(x_0)$, whenever $\lim_{n\to\infty} d_{P_U}(x_n, F(x_n)) = 0$, where $d_{P_U}(x_n, F(x_n)) := \inf\{P_U(x_n - z) : z \in F(x_n)\}$, where P_U is the Minkowski p-functional for any $U \in \mathfrak{U}$, which is the family of all nonempty open p-convex subsets containing zero in E.

(H1) Condition: There exists x_0 in $\overline{U}$ with $x_0 \in F(x_0)$ if there exists $\{x_n\}_{n\in\mathbb{N}}$ in $\overline{U}$ such that $\lim_{n\to\infty} d_{P_U}(x_n, F(x_n)) = 0$, where P_U is the Minkowski p-functional for any $U \in \mathfrak{U}$, which is the family of all nonempty open p-convex subsets containing zero in E.

Then we have that there exist $x_0 \in C \cap \overline{U}$ and $y_0 \in F(x_0)$ such that

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\big(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C\big),$$

where P_U is the Minkowski p-functional of U. More precisely, we have that either (I) or (II) holds:

(I) F has a fixed point $x_0 \in \overline{U} \cap C$, i.e.,
$0 = P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C)$;

(II) There exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \overline{U}$ with

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\big(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C\big) = \big(P_U^{\frac{1}{p}}(y_0) - 1\big)^p > 0.$$

Proof As E is a p-convex space and U is a bounded open p-convex subset of E, it suffices to prove that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\overline{U}$ and $y_n \in F(x_n)$ such that $\lim_{n\to\infty} P_U(y_n - x_n) = 0$, and the conclusion follows by applying the (H) condition.

Let $r : E \to U$ be a retraction mapping defined by $r(x) := \frac{x}{\max\{1,(P_U(x))^{\frac{1}{p}}\}}$ for each $x \in E$, where P_U is the Minkowski p-functional of U. Since the space E's zero $0 \in U(= \operatorname{int} U$ as U is open), it follows that r is continuous by Lemma 2.2. As the mapping F is 1-set contractive, take an increasing sequence $\{\lambda_n\}$ such that $0 < \lambda_n < 1$ and $\lim_{n\to\infty} \lambda_n = 1$, where $n \in \mathbb{N}$. Now, for each $n \in \mathbb{N}$, we define a mapping $F_n : C \cap \overline{U} \to 2^C$ by $F_n(x) := \lambda_n F \circ r(x)$ for each $x \in C \cap \overline{U}$. By the fact that C and $\overline{U}$ are p-convex, it follows that $r(C) \subset C$ and $r(\overline{U}) \subset \overline{U}$, thus $r(C \cap \overline{U}) \subset C \cap \overline{U}$. Therefore F_n is a mapping from $\overline{U} \cap C$ to itself. For each $n \in \mathbb{N}$, by the fact that F_n is a λ_n-set-contractive mapping with $0 < \lambda_n < 1$, it is also QUSC with nonempty p-convex and its graph is also closed. Then it follows by Theorem 4.8 for the condensing mapping that there exists $z_n \in C \cap \overline{U}$ such that $z_n \in F_n(z_n) = \lambda_n F \circ r(z_n)$. As $r(C \cap \overline{U}) \subset C \cap \overline{U}$, let $x_n = r(z_n)$. Then we have that $x_n \in C \cap \overline{U}$, and there exists $y_n \in F(x_n)$

with $x_n = r(\lambda_n y_n)$ such that the following (1) or (2) holds for each $n \in \mathbb{N}$: (1) $\lambda_n y_n \in C \cap \overline{U}$; or (2) $\lambda_n y_n \in C \setminus \overline{U}$.

Now we prove the conclusion by considering the following two cases under (H) condition and (H1) condition.

Case (I) For each $n \in N$, $\lambda_n y_n \in C \cap \overline{U}$; or

Case (II) There exists a positive integer n such that $\lambda_n y_n \in C \setminus \overline{U}$.

First, by case (I), for each $n \in \mathbb{N}$, $\lambda_n y_n \in \overline{U} \cap C$, which implies that $x_n = r(\lambda_n y_n) = \lambda_n y_n$, thus $P_U(\lambda_n y_n) \leq 1$ by Lemma 2.2. Note that

$$\begin{aligned} P_U(y_n - x_n) &= P_U(y_n - x_n) \\ &= P_U(y_n - \lambda_n y_n) \\ &= P_U\left(\frac{(1-\lambda_n)\lambda_n y_n}{\lambda_n}\right) \\ &\leq \left(\frac{1-\lambda_n}{\lambda_n}\right)^p P_U(\lambda_n y_n) \\ &\leq \left(\frac{1-\lambda_n}{\lambda_n}\right)^p, \end{aligned}$$

which implies that $\lim_{n\to\infty} P_U(y_n - x_n) = 0$. Now, for any $V \in \mathbb{U}$, without loss of generality, let $U_0 = V \cap U$. Then we have the following conclusion:

$$\begin{aligned} P_{U_0}(y_n - x_n) &= P_{U_0}(y_n - x_n) \\ &= P_{U_0}(y_n - \lambda_n y_n) \\ &= P_{U_0}\left(\frac{(1-\lambda_n)\lambda_n y_n}{\lambda_n}\right) \\ &\leq \left(\frac{1-\lambda_n}{\lambda_n}\right)^p P_{U_0}(\lambda_n y_n) \\ &\leq \left(\frac{1-\lambda_n}{\lambda_n}\right)^p, \end{aligned}$$

which implies that $\lim_{n\to\infty} P_{U_0}(y_n - x_n) = 0$, where P_{U_0} is the Minkowski p-functional of U_0 in E.

Now, if F satisfies the (H) condition, then it follows that the consequence $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence that converges to x_0 such that $x_0 \in F(x_0)$. Without loss of generality, we assume that $\lim_{n\to\infty} x_n = x_0$, where $y_n \in F(x_n)$ is with $x_n = \lambda_n y_n$ and $\lim_{n\to\infty} \lambda_n = 1$, and as $x_0 = \lim_{n\to\infty}(\lambda_n y_n)$, which implies that $y_0 = \lim_{n\to\infty} y_n = x_0$. Thus there exists $y_0 (= x_0) \in F(x_0)$, we have $0 = d_p(x_0, F(x_0)) = d(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C))$ as indeed $x_0 = y_0 \in F(x_0) \in \overline{U} \cap C \subset \overline{I_{\overline{U}}^p(x_0)} \cap C)$.

If F satisfies the (H1) condition, then it follows that there exists $x_0 \in \overline{U} \cap C$ with $x_0 \in F(x_0)$. Then we have $0 = P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C)$.

Second, by case (II) there exists a positive integer n such that $\lambda_n y_n \in C \setminus \overline{U}$. Then we have that $P_U(\lambda_n y_n) > 1$, and also $P_U(y_n) > 1$ as $\lambda_n < 1$. As $x_n = r(\lambda_n y_n) = \frac{\lambda_n y_n}{(P_U(\lambda_n y_n))^{\frac{1}{p}}}$, which implies that $P_U(x_n) = 1$, thus $x_n \in \partial_C(U)$. Note that

$$P_U(y_n - x_n) = P_U\left(\frac{(P_U(y_n)^{\frac{1}{p}} - 1)y_n}{P_U(y_n)^{\frac{1}{p}}}\right) = \left(P_U^{\frac{1}{p}}(y_n) - 1\right)^p.$$

By the assumption, we have $(P_U^{\frac{1}{p}}(y_n) - 1)^p \leq P_U(y_n - x)$ for $x \in C \cap \partial\overline{U}$, it follows that

$$\begin{aligned} P_U(y_n) - 1 &\leq P_U(y_n) - \sup\{P_U(z) : z \in C \cap \overline{U}\} \\ &\leq \inf\{P_U(y_n - z) : z \in C \cap \overline{U}\} = d_p(y_n, C \cap \overline{U}). \end{aligned}$$

Thus we have the best approximation: $P_U(y_n - x_n) = d_P(y_n, \overline{U} \cap C) = (P_U^{\frac{1}{p}}(y_n) - 1)^p > 0$.

Now we want to show that $P_U(y_n - x_n) = d_P(y_n, \overline{U} \cap C) = d_p(y_n, \overline{I_{\overline{U}}^p(x_0)} \cap C) > 0$.

By the fact that $(\overline{U} \cap C) \subset I_{\overline{U}}^p(x_n) \cap C$, let $z \in I_{\overline{U}}^p(x_n) \cap C \setminus (\overline{U} \cap C)$, we first claim that $P_U(y_n - x_n) \leq P_U(y_n - z)$. If not, we have $P_U(y_n - x_n) > P_U(y_n - z)$. As $z \in I_{\overline{U}}^p(x_n) \cap C \setminus (\overline{U} \cap C)$, there exist $y \in \overline{U}$ and a nonnegative number c (actually $c \geq 1$ as shown soon below) with $z = x_n + c(y - x_n)$. Since $z \in C$, but $z \notin \overline{U} \cap C$, it implies that $z \notin \overline{U}$. By the fact that $x_n \in \overline{U}$ and $y \in \overline{U}$, we must have the constant $c \geq 1$; otherwise, it implies that $z(= (1 - c)x_n + cy) \in \overline{U}$, this is impossible by our assumption, i.e., $z \notin \overline{U}$. Thus we have that $c \geq 1$, which implies that $y = \frac{1}{c}z + (1 - \frac{1}{c})x_n \in C$ (as both $x_n \in C$ and $z \in C$). On the other hand, as $z \in I_{\overline{U}}^p(x_n) \cap C \setminus (\overline{U} \cap C)$, and $c \geq 1$ with $(\frac{1}{c})^p + (1 - \frac{1}{c})^p = 1$, combining with our assumption that for each $x \in \partial_C\overline{U}$ and $y \in F(x_n) \setminus \overline{U}$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$, it then follows that

$$\begin{aligned} P_U(y_n - y) &= P_U\left[\frac{1}{c}(y_n - z) + \left(1 - \frac{1}{c}\right)(y_n - x_n)\right] \\ &\leq \left[\left(\frac{1}{c}\right)^p P_U(y_n - z) + \left(1 - \frac{1}{c}\right)^p P_U(y_n - x_n)\right] \\ &< P_U(y_n - x_n), \end{aligned}$$

which contradicts that $P_U(y_n - x_n) = d_P(y_n, \overline{U} \cap C)$ as shown above, we know that $y \in \overline{U} \cap C$, we should have $P_U(y_n - x_n) \leq P_U(y_n - y)$! This helps us to complete the claim: $P_U(y_n - x_n) \leq P_U(y_n - z)$ for any $z \in I_{\overline{U}}^p(x_n) \cap C \setminus (\overline{U} \cap C)$, which means that the following best approximation of Fan type (see [42, 43]) holds:

$$0 < d_P(y_n, \overline{U} \cap C) = P_U(y_n - x_n) = d_p\left(y_n, I_{\overline{U}}^p(x_n) \cap C\right).$$

Now, by the continuity of P_U, it follows that the following best approximation of Fan type is also true:

$$0 < P_U(y_n - x_n) = d_P(y_n, \overline{U} \cap C) = d_p\left(y_n, I_{\overline{U}}^p(x_n) \cap C\right) = d_p\left(y_n, \overline{I_{\overline{U}}^p(x_n)} \cap C\right).$$

The proof is complete. □

Remark 5.2 Based on the proof of Theorem 5.2, we have that (1): For the condition "$x \in \partial_C U$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$", indeed we only need that for "$x \in$

$\partial_C U$ with $y \in F(x) \cap (C \setminus \overline{U})$), $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$"; (2): Theorem 5.2 also improves the corresponding best approximation for 1-set contractive mappings given by Li et al. [78], Liu [81], Xu [139], Xu et al. [140], and the results from the references therein; and (3): When $p = 1$, we have a similar best approximation result for the mapping F in the locally convex spaces with outward set boundary condition below (see Theorem 3 of Park [97] and related discussion by the references therein).

For the p-vector space with $p = 1$ being a topological vector space E, we have the following best approximation for the outward set $\overline{O_{\overline{U}}(x_0)}$ based on the point $\{x_0\}$ with respect to the convex subset U in E.

Theorem 5.3 (Best approximation for outward sets) *Let U be a bounded open convex subset of a locally convex space E (i.e., $p = 1$) with zero $0 \in \operatorname{int} U = U$ (the interior $\operatorname{int} U = U$ as U is open), and let C be a closed p-convex subset of E with also zero $0 \in C$. Assume that $F : \overline{U} \cap C \to 2^C$ is a 1-set-contractive quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph that satisfies condition* (*H*) *or* (*H*1) *above. Then there exist $x_0 \in \overline{U} \cap X$ and $y_0 \in F(x_0)$ such that $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{O_{\overline{U}}(x_0)} \cap C)$, where P_U is the Minkowski p-functional of U. More precisely, we have that either* (I) *or* (II) *holds*:

(I) *F has a fixed point $x_0 \in U \cap C$, i.e., $x_0 \in F(x_0)$ such that* $P_U(y_0 - x_0) = P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{O_{\overline{U}}(x_0)} \cap C)) = 0$;

(II) *There exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \overline{U}$ with*

$$\begin{aligned} P_U(y_0 - x_0) &= d_P(y_0, \overline{U} \cap C) \\ &= d_p\big(y_0, O_{\overline{U}}(x_0) \cap C\big) = d_p\big(y_0, \overline{O_{\overline{U}}(x_0)} \cap C\big) > 0. \end{aligned}$$

Proof We define a new mapping $F_1 : \overline{U} \cap C \to 2^C$ by $F_1(x) := \{2x\} - F(x)$ for each $x \in \overline{U} \cap C$, then F_1 is also compact and upper semicontinuous mapping with nonempty closed convex values, and F_1 satisfies all hypotheses of Theorem 5.2 with $p = 1$. If follows by Theorem 5.2 that there exist $x_0 \in \overline{U} \cap X$ and $y_1 \in F_1(x_0)$ such that $P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p(y_1, \overline{I_{\overline{U}}(x_0)} \cap C)$. More precisely, we have the following either (I) or (II) holding:

(I) F_1 has a fixed point $x_0 \in U \cap C$ (so $0 = P_U(y_1 - x_0) = P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p(y_1, \overline{I_{\overline{U}}(x_0)} \cap C)$);

(II) There exist $x_0 \in \partial_C(U)$ and $y_1 \in F_1(x_0) \setminus \overline{U}$ with

$$P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p\big(y_1, \overline{O_{\overline{U}}(x_0)} \cap C\big) > 0.$$

Now, for any $x \in O_{\overline{U}}(x_0)$, there exist $r < 0$, $u \in \overline{U}$ such that $x = x_0 + r(u - x_0)$. Let $x_1 = 2x_0 - x$, then $x_1 = 2x_0 - x_0 - r(u - x_0) = x_0 + (-r)(u - x_0) \in I_{\overline{U}}(x_0)$. Let $y_1 = 2x_0 - y_0$ for some $y_0 \in F(x_0)$. As we have $P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p(y_1, \overline{I_{\overline{U}}(x_0)} \cap C)$, it follows that $P_U(y_1 - x_0) \leq P_U(y_1 - x_1)$, which implies that

$$\begin{aligned} P_U(x_0 - y_0) &= P_U(y_1 - x_0) \\ &\leq P_U(y_1 - x_1) = P_U\big(2x_0 - y_0 - (2x_0 - x)\big) = P_U(y_0 - x) \end{aligned}$$

for all $x \in O_{\overline{U}}(x_0)$. Thus we have $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, O_{\overline{U}}(x_0) \cap C)$, and by the continuity of P_U, it follows that

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\big(y_0, \overline{O_{\overline{U}}(x_0)} \cap C\big)\big(P_U^{\frac{1}{p}}(y_0) - 1\big)^p > 0.$$

This completes the proof. □

Now, by the application of Theorem 5.2, Theorem 5.3, Remark 5.2, and the argument used in Theorem 5.2, we have the following general principle for the existence of solutions for Birkhoff–Kellogg problems in p-seminorm spaces, where $(0 < p \leq 1)$.

Theorem 5.4 (Principle of Birkhoff–Kellogg alternative) *Let U be a bounded open p-convex subset of a locally p-convex space E $(0 < p \leq 1)$ with zero $0 \in \operatorname{int} U = U$, and let C be a closed p-convex subset of E with also zero $0 \in C$. Assume that $F : \overline{U} \cap C \to 2^C$ is a 1-set-contractive quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph, satisfying the (H) or $(H1)$ condition above. Then F has at least one of the following two properties*:

(I) *F has a fixed point $x_0 \in U \cap C$ such that $x_0 \in F(x_0)$*;

(II) *There exist $x_0 \in \partial_C(U)$, $y_0 \in F(x_0)\setminus\overline{U}$, and $\lambda = \frac{1}{(P_U(y_0))^{\frac{1}{p}}} \in (0,1)$ such that $x_0 = \lambda y_0 \in \lambda F(x_0)$; In addition if for each $x \in \partial_C U$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$), then the best approximation between x_0 and y_0 is given by*

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\big(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C\big) = \big(P_U^{\frac{1}{p}}(y_0) - 1\big)^p > 0.$$

Proof If (I) is not the case, then (II) is proved by Remark 5.2 and by following the proof in Theorem 5.2 for case (ii): $y_0 \in C\setminus\overline{U}$ with $y_0 := f(x_0) \in F(x_0)$. Indeed, as $y_0 \notin \overline{U}$, it follows that $P_U(y_0) > 1$ and $x_0 = f(y_0) = y_0\frac{1}{(P_U(y_0))^{\frac{1}{p}}}$. Now let $\lambda = \frac{1}{(P_U(y_0))^{\frac{1}{p}}}$, we have $\lambda < 1$ and $x_0 = \lambda y_0$ with $y_0 \in F(x_0)$. Finally, the additionally assumption in (II) allows us to have the best approximation between x_0 and y_0 obtained by following the proof of Theorem 5.2 as $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C) > 0$. This completes the proof. □

As an application of Theorem 5.2 for the nonself set-valued mappings discussed in Theorem 5.3 with outward set condition, we have the following general principle of Birkhoff–Kellogg alternative in topological vector spaces.

Theorem 5.5 (Principle of Birkhoff–Kellogg alternative in TVS) *Let U be a bounded open p-convex subset of a locally p-convex space E with zero $0 \in U$, and let C be a closed convex subset of E with also zero $0 \in C$. Assume that $F : \overline{U} \cap C \to 2^C$ is a 1-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph, satisfying the (H) or $(H1)$ condition (H) above. Then it has at least one of the following two properties*:

(I) *F has a fixed point $x_0 \in U \cap C$ such that $x_0 \in F(x_0)$*;

(II) *There exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)\setminus\overline{U}$ and $\lambda \in (0,1)$ such that $x_0 = \lambda y_0$, and the best approximation between x_0 and y_0 is given by $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C) > 0$.*

On the other hand, by the proof of Theorem 5.2, we note that for case (II) of Theorem 5.2, the assumption "each $x \in \partial_C U$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$" is only used to guarantee the best approximation "$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C) > 0$", thus we have the following Leray–Schauder alternative in p-vector spaces, which, of course, includes the corresponding results in locally convex spaces as special cases.

Theorem 5.6 (The Leray–Schauder nonlinear alternative) *Let C be a closed p-convex subset of p-seminorm space E with $0 \leq p \leq 1$ and zero $0 \in C$. Assume that $F : C \to 2^C$ is a 1-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph, satisfying the (H) or $(H1)$ condition above. Let $\varepsilon(F) := \{x \in C : x \in \lambda F(x)$ for some $0 < \lambda < 1\}$. Then either F has a fixed point in C or the set $\varepsilon(F)$ is unbounded.*

Proof We prove the conclusion by assuming that F has no fixed point, then we claim that the set $\varepsilon(F)$ is unbounded. Otherwise, assume the set $\varepsilon(F)$ is bounded, and assume that P is the continuous p-seminorm for E, then there exists $r > 0$ such that the set $B(0, r) := \{x \in E : P(x) < r\}$, which contains the set $\varepsilon(F)$, i.e., $\varepsilon(F) \subset B(0, r)$, which means for any $x \in \varepsilon(F)$, $P(x) < r$. Then $B(0.r)$ is an open p-convex subset of E and zero $0 \in B(0, r)$ by Lemma 2.2 and Remark 2.4. Now let $U := B(0, r)$ in Theorem 5.4, it follows that the mapping $F : B(0, r) \cap C \to 2^C$ satisfies all general conditions of Theorem 5.4, and we have that any $x_0 \in \partial_C B(0, r)$, no any $\lambda \in (0, 1)$ such that $x_0 = \lambda y_0$, where $y_0 \in F(x_0)$. Indeed, for any $x \in \varepsilon(F)$, it follows that $P(x) < r$ as $\varepsilon(F) \subset B(0, r)$, but for any $x_0 \in \partial_C B(0, r)$, we have $P(x_0) = r$, thus conclusion (II) of Theorem 5.4 does not hold. By Theorem 5.4 again, F must have a fixed point, but this contradicts our assumption that F is fixed point free. This completes the proof. □

Now assume a given p-vector space E equipped with the P-seminorm (by assuming it is continuous at zero) for $0 < p \leq 1$, then we know that $P : E \to \mathbb{R}^+$, $P^{-1}(0) = 0$, $P(\lambda x) = |\lambda|^p P(x)$ for any $x \in E$ and $\lambda \in \mathbb{R}$. Then we have the following useful result for fixed points due to Rothe and Altman types in locally p-convex spaces, which plays important roles for optimization problem, variational inequality, complementarity problems (see Isac [60] or Yuan [143] and the references therein for related study in detail).

Corollary 5.1 *Let U be a bounded open p-convex subset of a locally p-convex space E and zero $0 \in U$, plus C is a closed p-convex subset of E with $U \subset C$, where $0 < p \leq 1$. Assume that $F : \overline{U} \to 2^C$ is a 1-set contractive quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph, satisfying the (H) or $(H1)$ condition above. If one of the following is satisfied*:

(1) (*Rothe type condition*): *$P_U(y) \leq P_U(x)$ for $y \in F(x)$, where $x \in \partial U$*;

(2) (*Petryshyn type condition*): *$P_U(y) \leq P_U(y - x)$ for $y \in F(x)$, where $x \in \partial U$*;

(3) (*Altman type condition*): *$|P_U(y)|^{\frac{2}{p}} \leq [P_U(y) - x)]^{\frac{2}{p}} + [P_U(x)]^{\frac{2}{p}}$ for $y \in F(x)$, where $x \in \partial U$*,

then F has at least one fixed point.

Proof By conditions (1), (2), and (3), it follows that the conclusion of (II) in Theorem 5.4 "there exist $x_0 \in \partial_C(U)$ and $\lambda \in (0, 1)$ such that $x_0 \notin \lambda F(x_0)$" does not hold, thus by the alternative of Theorem 5.4, F has a fixed point. This completes the proof. □

By the fact that when $p = 1$, each locally p-convex space is a locally convex space, we have the following classical Fan's best approximation (see [42]) as a powerful tool for the study in the optimization, mathematical programming, games theory, and mathematical economics, and other related topics in applied mathematics.

Corollary 5.2 (Fan's best approximation) *Let U be a bounded open convex subset of a locally convex space E with zero $0 \in U$, let C be a closed convex subset of E with also zero $0 \in C$, and assume that $F : \overline{U} \cap C \to 2^C$ is a 1-set contractive and quasi upper semicontinuous mapping with nonempty closed convex values satisfying the (H) or $(H1)$ condition above. Assume that P_U is the Minkowski p-functional of U in E. Then there exist $x_0 \in \overline{U} \cap X$ and $y_0 \in T(x_0)$ such that $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C)$. More precisely, we have the following either* (I) *or* (II) *holding, where $W_{\overline{U}}(x_0)$ is either the inward set $I_{\overline{U}}(x_0)$ or the outward set $O_{\overline{U}}(x_0)$*:

(I) *F has a fixed point $x_0 \in U \cap C$,*
$0 = P_U(y_0 - x_0) = P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{W_{\overline{U}}(x_0)} \cap C))$;

(II) *There exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \overline{U}$ with*

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\big(y_0, \overline{W_{\overline{U}}(x_0)} \cap C\big) = P_U(y_0) - 1 > 0.$$

Proof When $p = 1$, it automatically satisfies the inequality $P_U^{\frac{1}{p}}(y) - 1 \le P_U^{\frac{1}{p}}(y - x)$, and indeed we have that for $x_0 \in \partial_C(U)$, with $y_0 \in F(x_0)$, we have $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{W_{\overline{U}}(x_0)} \cap C) = P_U(y_0) - 1$. The conclusions are given by Theorem 5.2 (or Theorem 5.3). The proof is complete. □

We would like to point out that similar results on Rothe and Leray–Schauder alternative have been developed by Isac [60], Park [96], Potter [108], Shahzad [120–122], Xiao and Zhu [135], and the related references therein as tools of nonlinear analysis in locally p-convex spaces. As mentioned above, when $p = 1$ and take F as a continuous mapping, then we obtain the version of Leray–Schauder in locally convex spaces, and thus we omit its statement in detail.

6 Fixed points for the class of nonself semiclosed 1-set contractive mappings

In this section, based on the best approximation Theorem 5.2 for classes of semiclosed 1-set contractive mappings developed in Sect. 5, we show how it can be used as a useful tool to establish fixed point theorems for nonself upper semicontinuous mappings in locally p-convex spaces for $p \in (0, 1]$, including norm spaces and uniformly convex Banach spaces as special classes.

By following Browder [18], Li [77], Goebel and Kirk [48], Petryshyn [104, 105], Tan and Yuan [129], Xu [139], and the references therein, we recall some definitions as follows for p-seminorm spaces, where $p \in (0, 1]$.

Definition 6.1 Let D be a nonempty (bounded) closed subset of p-vector spaces $(E, \|\cdot\|_p)$ with p-seminorm, where $p \in (0, 1]$. Suppose that $f : D \to X$ is a (single-valued) mapping, then: (1) f is said to be nonexpansive if for each $x, y \in D$, we have $\|f(x) - f(y)\|_p \le \|x - y\|_p$; (2) f (actually, $(I - f)$) is said to be demiclosed (see Borwder [18]) at $y \in X$ if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in D, the conditions $x_n \to x_0 \in D$ weakly, and $(I - f)(x_n) \to y_0$ strongly imply that $(I-f)(x_0) = y_0$, where I is the identity mapping; (3) f is said to be hemicompact (see p. 379 of Tan and Yuan [129]) if each sequence $\{x_n\}_{n\in\mathbb{N}}$ in D has a convergent subsequence with the limit x_0 such that $x_0 = f(x_0)$, whenever $\lim_{n\to\infty} d_p(x_n, f(x_n)) = 0$, here $d_P(x_n, f(x_n)) := \inf\{P_U(x_n - z) : z \in f(x_n)\}$, and P_U is the Minkowski p-functional for any $U \in \mathfrak{U}$, which is the family of all nonempty open p-convex subset containing zero in E; (4) f is said to be demicompact (by Petryshyn [104]) if each sequence $\{x_n\}_{n\in\mathbb{N}}$ in D has a convergent subsequence whenever $\{x_n - f(x_n)\}_{n\in\mathbb{N}}$ is a convergent sequence in X; (5) f is said to be a semiclosed 1-set contractive mapping if f is 1-set contractive mapping and $(I-f)$ is closed, where I is an identity mapping (by Li [77]); and (6) f is said to be semicontractive (see Petryshyn [105] and Browder [18]) if there exists a mapping $V : D \times D \to 2^X$ such that $f(x) = V(x,x)$ for each $x \in D$, with (a) for each fixed $x \in D$, $V(\cdot, x)$ is nonexpansive from D to X; and (b) for each fixed $x \in D$, $V(x, \cdot)$ is completely continuous from D to X, uniformly for u in a bounded subset of D (which means if v_j converges weakly to v in D and u_j is a bounded sequence in D, then $V(u_j, v_j) - V(u_j, v) \to 0$, strongly in D).

From the definition above, we first observe that definitions (1) to (6) for set-valued mappings can be given in a similar way with the Hausdorff metric H (we omit their definitions here in detail to save space); Secondly, if f is a continuous demicompact mapping, then $(I-f)$ is closed, where I is the identity mapping on X. It is also clear from the definitions that every demicompact map is hemicompact in seminorm spaces, but the converse is not true by the example on p. 380 by Tan and Yuan [129]. It is evident that if f is demicompact, then $I-f$ is demiclosed. It is known that for each condensing mapping f, when D or $f(D)$ is bounded, then f is hemicompact; and also f is demicompact in metric spaces by Lemma 2.1 and Lemma 2.2 of Tan and Yuan [129], respectively. In addition, it is known that every nonexpansive map is a 1-set-contractive mapping; and also if f is a hemicompact 1-set-contractive mapping, then f is a 1-set-contractive mapping satisfying the following *(H1) condition* (which is the same as "condition (H1)" in Sect. 5, but slightly different from condition (H) used there in Sect. 5):

(H1) condition: Let D be a nonempty bounded subset of a space E, and assume that $F : \overline{D} \to 2^E$ a set-valued mapping. If $\{x_n\}_{n\in\mathbb{N}}$ is any sequence in D such that for each x_n, there exists $y_n \in F(x_n)$ with $\lim_{n\to\infty}(x_n - y_n) = 0$, then there exists a point $x \in \overline{D}$ such that $x \in F(x)$.

We first note that the "(H1) condition" above is actually the same one as the "condition (C)" used in Theorem 1 by Petryshyn [105]. Secondly, it was shown by Browder [18] that indeed the nonexpansive mapping in a uniformly convex Banach X enjoys condition (H1) as shown below.

Lemma 6.1 *Let D be a nonempty bonded convex subset of a uniformly convex Banach space E. Assume that $F : \overline{D} \to E$ is a nonexpansive (single-valued) mapping, then the mapping $P := I - F$ defined by $P(x) := (x - F(x))$ for each $x \in \overline{D}$ is demiclosed, and in particular, the "(H1) condition" holds.*

Proof By following the argument given on p. 329 (see the proof of Theorem 2.2 and Corollary 2.1) by Petryshyn [105], the mapping F is demiclosed (which actually is called Browder's demiclosedness principle), which says that by the assumption of (H1) condition, if $\{x_n\}_{n\in\mathbb{N}}$ is any sequence in D such that for each x_n there exists $y_n \in F(x_n)$ with

$\lim_{n\to\infty}(x_n - y_n) = 0$, then we have $0 \in (I - F)(\overline{D})$, which means that there exists $x_0 \in \overline{D}$ with $0 \in (I - F)(x_0)$, this implies that $x_0 \in F(x_0)$. The proof is complete. □

Remark 6.1 When a p-vector space E is with a p-norm, then "(H) condition" satisfies the "(H1) condition". The (H1) condition is mainly supported by the so-called demiclosedness principle after the work by Browder [18].

Lemma 6.1 above shows that s single-valued nonexpansive mapping defined in a uniformly convex Banach space satisfied the (H1) condition. Actually, the nonexpansive set-valued mappings defined on a special class of Banach spaces with the so-called the "Opial's condition" do not only satisfy condition (H1), but also belong to the classes of semiclosed 1-set contractive mappings, as shown below.

The notion of the so-called "Opial's condition" first given by Opial [90] says that a Banach space X is said to satisfy Opial's condition if $\liminf_{n\to\infty} \|w_n - w\| < \liminf_{n\to\infty} \|w_n - p\|$ whenever (w_n) is a sequence in X weakly convergent to w and $p \neq w$. We know that Opial's condition plays an important role in the fixed point theory, e.g., see Lami Dozo [75], Goebel and Kirk [49], Xu [137], and the references therein. The following result shows that there are nonexpansive set-valued mappings in Banach spaces with Opial's condition (see Lami Dozo [75] satisfying the condition (H1).

Lemma 6.2 *Let C be convex weakly compact of a Banach space X that satisfies Opial's condition. Let $T : C \to K(C)$ be a nonexpansive set-valued mapping with nonempty compact values. Then the graph of $(I - T)$ is closed in $(X, \sigma(X, X^*) \times (X, \|\cdot\|))$, thus T satisfies the "(H1) condition", where I denotes the identity on X, $\sigma(X, X^*)$ is the weak topology, and $\|\cdot\|$ is the norm (or strong) topology.*

Proof By following Theorem 3.1 of Lami Dozo [75], it follows that the mapping T is demiclosed, thus T satisfies the "(H1) condition". The proof is complete. □

For the convenience of our study, for the fixed point theory for a class of semiclosed 1-set contractive mappings in p-seminorm spaces, we also need to introduce the following definition, which is a set-valued generalization of single-value semiclosed 1-set mappings first discussed by Li [77], Xu [139] (see also Li et al. [78], Xu et al. [140], and the references therein).

Definition 6.2 Let D be a nonempty (bounded) closed subset of p-vector spaces $(E, \|\cdot\|_p)$ with p-seminorm, where $p \in (0, 1]$ (which includes norm spaces or Banach spaces as special classes), and suppose that $T : D \to X$ is a set-valued mapping. Then F is said to be a semiclosed 1-set contraction mapping if T is 1-set contraction, and $(I - T)$ is closed, which means that for a given net $\{x_n\}_{i\in I}$, for each $i \in I$, there exists $y_i \in T(x_i)$ with $\lim_{i\in I}(x_i - y_i) = 0$, then $0 \in (I - T)(\overline{D})$, i.e., there exists $x_0 \in \overline{D}$ such that $x_0 \in T(x_0)$.

By Lemmas 6.1 and 6.2, it follows that each nonexpansive (single-valued) mapping defined on a subset of uniformly convex Banach spaces and each nonexpansive set-valued mapping defined on a subset of Banach spaces satisfying Opial's condition is a semiclosed 1-set contractive mapping (see also Goebel [47], Goebel and Kirk [48], Petrusel et al. [103], Xu [137], Yangai [141], and the references therein for related discussion). In particular, under the setting of metric spaces or Banach spaces with certain property, it is clear that each semiclosed 1-set contractive mapping satisfies condition (H1) above.

We know that compared to the single-valued case, based on the study in the literature about the approximation of fixed points for multivalued mappings, a well-known counterexample due to Pietramala [106] (see also Muglia and Marino [85]) proved in 1991 that Browder approximation Theorem 1 given by Browder [16] cannot be extended to the genuine multivalued case even on a finite dimensional space $\mathbb{R}^2$. Moreover, if a Banach space X satisfies Opial's property (see Opial [90]) that is, if x_n weakly converges to x, then we have that $\limsup \|x_n - x\| < \limsup \|x_n - y\|$ for all $x \in X$ and $y \neq x$), then $I - f$ is demiclosed at 0 (see Lami Dozo [75], Yanagi [141], and the related references therein) provided $f : C :\to K(C)$ is nonexpansive (here $K(C)$ denotes a family of nonempty compact subsets of C). We know that all Hilbert spaces and L^p spaces $p \in (1, \infty)$ have Opial's property, but it seems that whether $I - f$ is demiclosed at zero 0 if f is a nonexpansive set-valued mapping defined on the space X which is uniformly convex (e.g., $L[0, 1]$, $1 < p < \infty$, $\neq 2$) and $f : C \to K(C)$ is nonexpansive. Here we remark that for a single-valued nonexpansive mapping f is yes, which is the famous theorem of Browder [15]. A remarkable fixed point theorem for multivalued mappings is Lim's result in [80], which says that: If C is a nonempty closed bounded convex subset of a uniformly convex Banach space X and $f : C \to K(C)$ is nonexpansive, then f has a fixed point.

Now, based on the concept for the semiclosed 1-set contractive mappings, we give the existence results for their best approximation, fixed points, and related nonlinear alterative under the framework of p-seminorm spaces for $p \in (0, 1]$.

Theorem 6.1 (Schauder fixed point theorem for semiclosed 1-set contractive mappings) *Let U be a nonempty bounded open p-subset of a (Hausdorff) locally p-convex space E and its zero $0 \in U$, and let $C \subset E$ be a closed p-convex subset of E such that $0 \in C$ with $0 < p \leq 1$. If $F : C \cap \overline{U} \to 2^{C\cap\overline{U}}$ is a quasi upper semicontinuous and semiclosed 1-set contractive set-valued mappings with nonempty convex p-convex values and with a closed graph, then T has at least one fixed point in $C \cap \overline{U}$.*

Proof As the mapping T is 1-set contractive, take an increasing sequence $\{\lambda_n\}$ such that $0 < \lambda_n < 1$ and $\lim_{n\to\infty} \lambda_n = 1$, where $n \in \mathbb{N}$. Now we define a mapping $F_n : C \to 2^C$ by $F_n(x) := \lambda_n F(x)$ for each $x \in C$ and $n \in \mathbb{N}$. Then it follows that F_n is a λ_n-set-contractive mapping with $0 < \lambda_n < 1$, quasi upper semicontinuous with nonempty p-convex, and its graph is closed. Now, by Theorem 4.8 on the condensing mapping F_n in p-vector space with p-seminorm P_U for each $n \in \mathbb{N}$, there exists $x_n \in C$ such that $x_n \in F_n(x_n) = \lambda_n F(x_n)$. Thus there exists $y_n \in F(x_n)$ such that $x_n = \lambda_n y_n$. Let P_U be the Minkowski p-functional of U in E, it follows that P_U is continuous as $0 \in \text{int}(U) = U$. Note that for each $n \in \mathbb{N}$, $\lambda_n x_n \in \overline{U} \cap C$, which implies that $x_n = r(\lambda_n y_n) = \lambda_n y_n$, thus $P_U(\lambda_n y_n) \leq 1$ by Lemma 2.2. Note that

$$
\begin{aligned}
P_U(y_n - x_n) &= P_U(y_n - x_n) \\
&= P_U(y_n - \lambda_n y_n) \\
&= P_U\left(\frac{(1-\lambda_n)\lambda_n y_n}{\lambda_n}\right) \leq \left(\frac{1-\lambda_n}{\lambda_n}\right)^p P_U(\lambda_n y_n) \leq \left(\frac{1-\lambda_n}{\lambda_n}\right)^p,
\end{aligned}
$$

which implies that $\lim_{n\to\infty} P_U(y_n - x_n) = 0$. Now, by the assumption that F is semiclosed, which means that $(I - F)$ is closed at zero, there exists one point $x_0 \in \overline{C}$ such that $0 \in (I - F)(\overline{C})$, thus we have $x_0 \in F(x_0)$.

Indeed, without loss of generality, we assume that $\lim_{n\to\infty} x_n = x_0$, here $y_n \in F(x_n)$ is with $x_n = \lambda_n y_n$, and $\lim_{n\to\infty} \lambda_n = 1$, it implies that $x_0 = \lim_{n\to\infty}(\lambda_n y_n)$, which means $y_0 := \lim_{n\to\infty} y_n = x_0$. There exists $y_0(= x_0) \in F(x_0)$. We complete the proof. □

Theorem 6.2 (Best approximation for semiclosed 1-set contractive mappings) *Let U be a bounded open p-convex subset of a locally p-convex space E $(0 < p \le 1)$ zero $0 \in U$, and let C be a (bounded) closed p-convex subset of E with also zero $0 \in C$. Assume that $F : \overline{U} \cap C \to 2^C$ is a semiclosed 1-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph, and for each $x \in \partial_C U$ with $y \in F(x) \cap (C \setminus \overline{U})$, $(P_U^{\frac{1}{p}}(y) - 1)^p \le P_U(y - x)$ for $0 < p \le 1$ (this is trivial when $p = 1$). Then we have that there exist $x_0 \in C \cap \overline{U}$ and $y_0 \in F(x_0)$ such that $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C)$, where P_U is the Minkowski p-functional of U. More precisely, we have that either* (I) *or* (II) *holds*:

(I) *F has a fixed point $x_0 \in U \cap C$, i.e.,*
$0 = P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C)$;

(II) *There exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \overline{U}$ with*

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\left(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C\right) = \left(P_U^{\frac{1}{p}}(y_0) - 1\right)^p > 0.$$

Proof Let $r : E \to U$ be a retraction mapping defined by $r(x) := \frac{x}{\max\{1,(P_U(x))^{\frac{1}{p}}\}}$ for each $x \in E$, where P_U is the Minkowski p-functional of U. Since the space E's zero $0 \in U(= \operatorname{int} U$ as U is open), it follows that r is continuous by Lemma 2.2. As the mapping F is 1-set contractive, take an increasing sequence $\{\lambda_n\}$ such that $0 < \lambda_n < 1$ and $\lim_{n\to\infty} \lambda_n = 1$, where $n \in \mathbb{N}$. Now we define a mapping $F_n : C \cap \overline{U} \to 2^C$ by $F_n(x) := \lambda_n F \circ r(x)$ for each $x \in C \cap \overline{U}$ and $n \in \mathbb{N}$. Then it follows that F_n is a λ_n-set-contractive mapping with $0 < \lambda_n < 1$ for each $n \in \mathbb{N}$. As C and $\overline{U}$ are p-convex, we have $r(C) \subset C$ and $r(\overline{U}) \subset \overline{U}$, so $r(C \cap \overline{U}) \subset C \cap \overline{U}$. Thus F_n is a self-mapping defined on $C \cap \overline{U}$, and we can also show that F_n satisfies all conditions of Theorem 4.8. By Theorem 4.8 for condensing mapping F_n, for each $n \in \mathbb{N}$, there exists $z_n \in C \cap \overline{U}$ such that $z_n \in F_n(z_n) = \lambda_n F \circ r(z_n)$. Let $x_n = r(z_n)$, then we have $x_n \in C \cap \overline{U}$, and there exists $y_n \in F(x_n)$ with $x_n = r(\lambda_n y_n)$ such that the following (1) or (2) holds for each $n \in \mathbb{N}$:

(1): $\lambda_n y_n \in C \cap \overline{U}$; or (2): $\lambda_n y_n \in C \setminus \overline{U}$.

Now we prove the conclusion by considering the following two cases:

Case (I): For each $n \in N$, $\lambda_n y_n \in C \cap \overline{U}$; or

Case (II): There exists a positive integer n such that $\lambda_n y_n \in C \setminus \overline{U}$.

First, by case (I), for each $n \in \mathbb{N}$, $\lambda_n y_n \in \overline{U} \cap C$, which implies that $x_n = r(\lambda_n y_n) = \lambda_n y_n$, thus $P_U(\lambda_n y_n) \le 1$ by Lemma 2.2. Note that

$$\begin{aligned} P_U(y_n - x_n) &= P_U(y_n - x_n) \\ &= P_U(y_n - \lambda_n y_n) \\ &= P_U\left(\frac{(1-\lambda_n)\lambda_n y_n}{\lambda_n}\right) \le \left(\frac{1-\lambda_n}{\lambda_n}\right)^p P_U(\lambda_n y_n) \le \left(\frac{1-\lambda_n}{\lambda_n}\right)^p, \end{aligned}$$

which implies that $\lim_{n\to\infty} P_U(y_n - x_n) = 0$. Now by the fact that F is semiclosed, it implies that there exists a point $x_0 \in \overline{U}$ (i.e., the consequence $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence with the limit x_0) such that $x_0 \in F(x_0)$. Indeed, without loss of generality, we assume that $\lim_{n\to\infty} x_n = x_0$, where $y_n \in F(x_n)$ is with $x_n = \lambda_n y_n$ and $\lim_{n\to\infty} \lambda_n = 1$, and as $x_0 = \lim_{n\to\infty}(\lambda_n y_n)$, it implies that $y_0 = \lim_{n\to\infty} y_n = x_0$. Thus there exists $y_0(= x_0) \in F(x_0)$, we have $0 = d_p(x_0, F(x_0)) = d(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I^p_{\overline{U}}(x_0)} \cap C))$ as indeed $x_0 = y_0 \in F(x_0) \in \overline{U} \cap C \subset \overline{I^p_{\overline{U}}(x_0)} \cap C)$.

Second, by case (II), there exists a positive integer n such that $\lambda_n y_n \in C \setminus \overline{U}$. Then we have that $P_U(\lambda_n y_n) > 1$, and also $P_U(y_n) > 1$ as $\lambda_n < 1$. As $x_n = r(\lambda_n y_n) = \frac{\lambda_n y_n}{(P_U(\lambda_n y_n))^{\frac{1}{p}}}$, it implies that $P_U(x_n) = 1$, thus $x_n \in \partial_C(U)$. Note that

$$P_U(y_n - x_n) = P_U\left(\frac{(P_U(y_n)^{\frac{1}{p}} - 1)y_n}{P_U(y_n)^{\frac{1}{p}}}\right) = \left(P_U^{\frac{1}{p}}(y_n) - 1\right)^p.$$

By the assumption, we have $(P_U^{\frac{1}{p}}(y_n) - 1)^p \le P_U(y_n - x)$ for $x \in C \cap \partial\overline{U}$, it follows that

$$\begin{aligned} P_U(y_n) - 1 &\le P_U(y_n) - \sup\{P_U(z) : z \in C \cap \overline{U}\} \\ &\le \inf\{P_U(y_n - z) : z \in C \cap \overline{U}\} = d_p(y_n, C \cap \overline{U}). \end{aligned}$$

Thus we have the best approximation: $P_U(y_n - x_n) = d_P(y_n, \overline{U} \cap C) = (P_U^{\frac{1}{p}}(y_n) - 1)^p > 0$.

Now we want to show that $P_U(y_n - x_n) = d_P(y_n, \overline{U} \cap C) = d_p(y_n, \overline{I^p_{\overline{U}}(x_0)} \cap C) > 0$.

By the fact that $(\overline{U} \cap C) \subset I^p_{\overline{U}}(x_n) \cap C$, let $z \in I^p_{\overline{U}}(x_n) \cap C \setminus (\overline{U} \cap C)$, we first claim that $P_U(y_n - x_n) \le P_U(y_n - z)$. If not, we have $P_U(y_n - x_n) > P_U(y_n - z)$. As $z \in I^p_{\overline{U}}(x_n) \cap C \setminus (\overline{U} \cap C)$, there exist $y \in \overline{U}$ and a nonnegative number c (actually $c \ge 1$ as shown soon below) with $z = x_n + c(y - x_n)$. Since $z \in C$, but $z \notin \overline{U} \cap C$, it implies that $z \notin \overline{U}$. By the fact that $x_n \in \overline{U}$ and $y \in \overline{U}$, we must have the constant $c \ge 1$; otherwise, it implies that $z(= (1 - c)x_n + cy) \in \overline{U}$, this is impossible by our assumption, i.e., $z \notin \overline{U}$. Thus we have that $c \ge 1$, which implies that $y = \frac{1}{c}z + (1 - \frac{1}{c})x_n \in C$ (as both $x_n \in C$ and $z \in C$). On the other hand, as $z \in I^p_{\overline{U}}(x_n) \cap C \setminus (\overline{U} \cap C)$ and $c \ge 1$ with $(\frac{1}{c})^p + (1 - \frac{1}{c})^p = 1$, combining with our assumption that for each $x \in \partial_C \overline{U}$ and $y \in F(x_n) \setminus \overline{U}$, $P_U^{\frac{1}{p}}(y) - 1 \le P_U^{\frac{1}{p}}(y - x)$ for $0 < p \le 1$, it then follows that

$$\begin{aligned} P_U(y_n - y) &= P_U\left[\frac{1}{c}(y_n - z) + \left(1 - \frac{1}{c}\right)(y_n - x_n)\right] \\ &\le \left[\left(\frac{1}{c}\right)^p P_U(y_n - z) + \left(1 - \frac{1}{c}\right)^p P_U(y_n - x_n)\right] < P_U(y_n - x_n), \end{aligned}$$

which contradicts that $P_U(y_n - x_n) = d_P(y_n, \overline{U} \cap C)$ as shown above. We know that $y \in \overline{U} \cap C$, and we should have $P_U(y_n - x_n) \le P_U(y_n - y)$! This helps us to complete the claim: $P_U(y_n - x_n) \le P_U(y_n - z)$ for any $z \in I^p_{\overline{U}}(x_n) \cap C \setminus (\overline{U} \cap C)$, which means that the following best approximation of Fan's type (see [42, 43]) holds:

$$0 < d_P(y_n, \overline{U} \cap C) = P_U(y_n - x_n) = d_p\left(y_n, I^p_{\overline{U}}(x_n) \cap C\right).$$

Now, by the continuity of P_U, it follows that the following best approximation of Fan type is also true:

$$0 < P_U(y_n - x_n) = d_P(y_n, \overline{U} \cap C) = d_p\big(y_n, I^p_{\overline{U}}(x_n) \cap C\big) = d_p\big(y_n, \overline{I^p_{\overline{U}}(x_n)} \cap C\big).$$

The proof is complete. □

For a p-vector space when $p = 1$, it is a (Hausdorff) topological vector space E, we have the following best approximation for the outward set $\overline{O_{\overline{U}}(x_0)}$ based on the point $\{x_0\}$ with respect to the convex subset U in E.

Theorem 6.3 (Best approximation for outward sets) *Let U be a bounded open convex subset of a locally convex space E (i.e., $p = 1$) with zero $0 \in \operatorname{int} U = U$ (the interior $\operatorname{int} U = U$ as U is open), and C be a closed convex subset of E with also zero $0 \in C$. Assume that $F : \overline{U} \cap C \to 2^C$ is a semiclosed 1-set-contractive quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph, then there exist $x_0 \in \overline{U} \cap X$ and $y_0 \in F(x_0)$ such that $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{O_{\overline{U}}(x_0)} \cap C)$, where P_U is the Minkowski p-functional of U. More precisely, we have that either* (I) *or* (II) *holds*:

(I) *F has a fixed point $x_0 \in U \cap C$, i.e.,*
$P_U(y_0 - x_0) = P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{O_{\overline{U}}(x_0)} \cap C)) = 0$;

(II) *There exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \overline{U}$ with*

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\big(y_0, O_{\overline{U}}(x_0) \cap C\big) = d_p\big(y_0, \overline{O_{\overline{U}}(x_0)} \cap C\big) > 0.$$

Proof We define a new mapping $F_1 : \overline{U} \cap C \to 2^C$ by $F_1(x) := \{2x\} - F(x)$ for each $x \in \overline{U} \cap C$, then F_1 is also compact and upper semicontinuous mapping with nonempty closed convex values, and F_1 satisfies all hypotheses of Theorem 5.2 with $p = 1$. It follows by Theorem 5.2 that there exist $x_0 \in \overline{U} \cap X$ and $y_1 \in F_1(x_0)$ such that $P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p(y_1, \overline{I_{\overline{U}}(x_0)} \cap C)$. More precisely, we have that either (I) or (II) holds:

(I) F_1 has a fixed point $x_0 \in U \cap C$ (so
$0 = P_U(y_1 - x_0) = P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p(y_1, \overline{I_{\overline{U}}(x_0)} \cap C))$;

(II) There exist $x_0 \in \partial_C(U)$ and $y_1 \in F_1(x_0) \setminus \overline{U}$ with

$$P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p\big(y_1, \overline{O_{\overline{U}}(x_0)} \cap C\big) > 0.$$

Now, for any $x \in O_{\overline{U}}(x_0)$, there exist $r < 0$, $u \in \overline{U}$ such that $x = x_0 + r(u - x_0)$. Let $x_1 = 2x_0 - x$, then $x_1 = 2x_0 - x_0 - r(u - x_0) = x_0 + (-r)(u - x_0) \in I_{\overline{U}}(x_0)$. Let $y_1 = 2x_0 - y_0$ for some $y_0 \in F(x_0)$. As we have $P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p(y_1, \overline{I_{\overline{U}}(x_0)} \cap C)$, it follows that $P_U(y_1 - x_0) \leq P_U(y_1 - x_1)$, which implies that

$$P_U(x_0 - y_0) = P_U(y_1 - x_0) \leq P_U(y_1 - x_1) = P_U\big(2x_0 - y_0 - (2x_0 - x)\big) = P_U(y_0 - x)$$

for all $x \in O_{\overline{U}}(x_0)$. Thus we have $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, O_{\overline{U}}(x_0) \cap C)$ and by the continuity of P_U, it follows that

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\big(y_0, \overline{O_{\overline{U}}(x_0)} \cap C\big)\big(P_U^{\frac{1}{p}}(y_0) - 1\big)^p > 0.$$

This completes the proof. □

Now, by the application of Theorems 6.2 and 6.3, we have the following general principle for the existence of solutions for Birkhoff–Kellogg problems in p-seminorm spaces, where $(0 < p \leq 1)$.

Theorem 6.4 (Principle of Birkhoff–Kellogg alternative) *Let U be a bounded open p-convex subset of a locally p-convex space E $(0 < p \leq 1)$ with zero $0 \in \operatorname{int} U = (U)$ (the interior $\operatorname{int} U$ as U is open), and let C be a closed p-convex subset of E with also zero $0 \in C$. Assume that $F : \overline{U} \cap C \to 2^C$ is a semiclosed 1-set-contractive quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph, then F has at least one of the following two properties*:

(I) *F has a fixed point $x_0 \in U \cap C$ such that $x_0 \in F(x_0)$*;

(II) *There exist $x_0 \in \partial_C(U)$, $y_0 \in F(x_0) \setminus \overline{U}$, and $\lambda = \frac{1}{(P_U(y_0))^{\frac{1}{p}}} \in (0,1)$ such that $x_0 = \lambda y_0 \in \lambda F(x_0)$; In addition, if for each $x \in \partial_C U$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$), then the best approximation between x_0 and y_0 is given by*

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\bigl(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C\bigr) = \bigl(P_U^{\frac{1}{p}}(y_0) - 1\bigr)^p > 0.$$

Proof If (I) is not the case, then (II) is proved by Remark 5.2 and by following the proof in Theorem 6.2 for case (ii): $y_0 \in C \setminus \overline{U}$ with $y_0 := f(x_0) \in F(x_0)$. Indeed, as $y_0 \notin \overline{U}$, it follows that $P_U(y_0) > 1$ and $x_0 = f(y_0) = y_0 \frac{1}{(P_U(y_0))^{\frac{1}{p}}}$. Now let $\lambda = \frac{1}{(P_U(y_0))^{\frac{1}{p}}}$, we have $\lambda < 1$ and $x_0 = \lambda y_0$ with $y_0 \in F(x_0)$. Finally, the additional assumption in (II) allows us to have the best approximation between x_0 and y_0 obtained by following the proof of Theorem 6.2 as $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C) > 0$. This completes the proof. □

As an application of Theorem 6.2 for the nonself set-valued mappings discussed in Theorem 6.3 with the outward set condition, we have the following general principle of Birkhoff–Kellogg alternative in locally p-convex spaces.

Theorem 6.5 (Principle of Birkhoff–Kellogg alternative in LCS) *Let U be a bounded open p-convex subset of a locally p-convex space E $(0 < p \leq 1)$ with zero $0 \in U$, and let C be a closed convex subset of E with also zero $0 \in C$. Assume that $F : \overline{U} \cap C \to 2^C$ is a semiclosed 1-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph, then it has at least one of the following two properties*:

(I) *F has a fixed point $x_0 \in U \cap C$ such that $x_0 \in F(x_0)$*;

(II) *There exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0) \setminus \overline{U}$ and $\lambda \in (0,1)$ such that $x_0 = \lambda y_0$, and the best approximation between x_0 and y_0 is given by $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C) > 0$.*

On the other hand, by the proof of Theorem 6.2, we note that for case (II) of Theorem 6.2, the assumption "each $x \in \partial_C U$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$" is only used to guarantee the best approximation "$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C) > 0$", thus we have the following Leray–Schauder alternative in p-vector spaces, which, of course, includes the corresponding results in locally convex spaces as special cases.

Theorem 6.6 (Leray–Schauder nonlinear alternative) *Let C be a closed p-convex subset of p-seminorm space E with $0 < p \leq 1$ and zero $0 \in C$. Assume that $F : C \to 2^C$ is a semiclosed*

1*-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph. Let* $\varepsilon(F) := \{x \in C : x \in \lambda F(x)$ *for some* $0 < \lambda < 1\}$. *Then either F has a fixed point in C or the set* $\varepsilon(F)$ *is unbounded.*

Proof By assuming that case (I) is not true, i.e., F has no fixed point, we claim that the set $\varepsilon(F)$ is unbounded. Otherwise, assume that the set $\varepsilon(F)$ is bounded, and assume that P is the continuous p-seminorm for E, then there exists $r > 0$ such that the set $B(0,r) := \{x \in E : P(x) < r\}$, which contains the set $\varepsilon(F)$, i.e., $\varepsilon(F) \subset B(0,r)$, which means for any $x \in \varepsilon(F)$, $P(x) < r$. Then $B(0.r)$ is an open p-convex subset of E and zero $0 \in B(0,r)$ by Lemma 2.2 and Remark 2.4. Now, let $U := B(0,r)$ in Theorem 6.4, it follows that the mapping $F : B(0,r) \cap C \to 2^C$ satisfies all general conditions of Theorem 6.4, and we have that any $x_0 \in \partial_C B(0,r)$, no any $\lambda \in (0,1)$ such that $x_0 = \lambda y_0$, where $y_0 \in F(x_0)$. Indeed, for any $x \in \varepsilon(F)$, it follows that $P(x) < r$ as $\varepsilon(F) \subset B(0,r)$, but for any $x_0 \in \partial_C B(0,r)$, we have $P(x_0) = r$, thus conclusion (II) of Theorem 6.4 does not hold. By Theorem 6.4 again, F must have a fixed point, but this contradicts with our assumption that F is fixed point free. This completes the proof. □

Now assume a given p-vector space E equipped with the P-seminorm (by assuming it is continuous at zero) for $0 < p \leq 1$, then we know that $P : E \to \mathbb{R}^+$, $P^{-1}(0) = 0$, $P(\lambda x) = |\lambda|^p P(x)$ for any $x \in E$ and $\lambda \in \mathbb{R}$. Then we have the following useful result for fixed points due to Rothe and Altman types in p-vector spaces, which plays important roles for optimization problems, variational inequalities, complementarity problems.

Corollary 6.1 *Let U be a bounded open p-convex subset of a locally p-convex space E and zero* $0 \in U$, *plus C is a closed p-convex subset of E with* $U \subset C$, *where* $0 < p \leq 1$. *Assume that* $F : \overline{U} \to 2^C$ *is a semiclosed* 1*-set contractive quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph, and one of the following conditions is satisfied*:

(1) (*Rothe type condition*): $P_U(y) \leq P_U(x)$ *for* $y \in F(x)$, *where* $x \in \partial U$;
(2) (*Petryshyn type condition*): $P_U(y) \leq P_U(y - x)$ *for* $y \in F(x)$, *where* $x \in \partial U$;
(3) (*Altman type condition*): $|P_U(y)|^{\frac{2}{p}} \leq [P_U(y) - x)]^{\frac{2}{p}} + [P_U(x)]^{\frac{2}{p}}$ *for* $y \in F(x)$, *where* $x \in \partial U$,

then F has at least one fixed point.

Proof By conditions (1), (2), and (3), it follows that the conclusion of (II) in Theorem 6.4 "there exist $x_0 \in \partial_C(U)$ and $\lambda \in (0,1)$ such that $x_0 \notin \lambda F(x_0)$" does not hold, thus by the alternative of Theorem 6.4, F has a fixed point. This completes the proof. □

By the fact that when $p = 1$, each p-vector space is a topological vector space, we have the following classical Fan's best approximation (see [42]) as a powerful tool for the study in the optimization, mathematical programming, games theory, mathematical economics, and other related topics in applied mathematics.

Corollary 6.2 (Fan's best approximation in LCS) *Let U be a bounded open convex subset of a locally convex space E with zero* $0 \in U$, *let C be a closed convex subset of E with also zero* $0 \in C$, *and assume that* $F : \overline{U} \cap C \to 2^C$ *is a semiclosed* 1*-set contractive and quasi upper* *semicontinuous mapping with nonempty convex values and with a closed graph. Then there exist* $x_0 \in \overline{U} \cap X$ *and* $y_0 \in T(x_0)$ *such that* $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C)$, *where* P_U *is the Minkowski p-functional of U in E. More precisely, we have that either* (I) *or* (II) *holds, where* $W_{\overline{U}}(x_0)$ *is either the inward set* $I_{\overline{U}}(x_0)$ *or the outward set* $O_{\overline{U}}(x_0)$:

(I) *F has a fixed point* $x_0 \in U \cap C$,
$0 = P_U(y_0 - x_0) = P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{W_{\overline{U}}(x_0)} \cap C))$;

(II) *There exist* $x_0 \in \partial_C(U)$ *and* $y_0 \in F(x_0) \setminus \overline{U}$ *with*

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\bigl(y_0, \overline{W_{\overline{U}}(x_0)} \cap C\bigr) = P_U(y_0) - 1 > 0.$$

Proof When $p = 1$, it automatically satisfies that the inequality: $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$, and indeed we have that for $x_0 \in \partial_C(U)$ with $y_0 \in F(x_0)$, we have $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{W_{\overline{U}}(x_0)} \cap C) = P_U(y_0) - 1$. The conclusions are given by Theorem 6.2 (or Theorem 6.3). The proof is complete. □

We would like to point out that similar results on Rothe and Leray–Schauder alternative have been developed by Isac [60], Park [96], Potter [108], Shahzad [120–122], Xiao and Zhu [135], and the related references therein as tools of nonlinear analysis in topological vector spaces. As mentioned above, when $p = 1$ and take F as a continuous mapping, then we obtain a version of Leray–Schauder in general local convex spaces, and thus we omit its statement in detail.

7 Principle of nonlinear alternatives for nonself semiclosed 1-set contractive mappings

As applications of results in Sect. 6, we now establish general results for the existence of solutions for Birkhoff–Kellogg problem and the principle of Leray–Schauder alternatives for semiclosed 1-set contractive mappings in locally p-convex spaces for $0 < p \leq 1$.

Theorem 7.1 (Birkhoff–Kellogg alternative in p-vector spaces) *Let U be a bounded open p-convex subset of a locally p-convex space E (where,* $0 < p \leq 1$*) with zero* $0 \in U$, *let C be a closed p-convex subset of E with also zero* $0 \in C$, *and assume that* $F : \overline{U} \cap C \to 2^C$ *is a semiclosed* 1*-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph. In addition, for each* $x \in \partial_C(U)$ *with* $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ *for* $0 < p \leq 1$ (*this is trivial when* $p = 1$), *where* P_U *is the Minkowski p-functional of U. Then we have that either* (I) *or* (II) *holds*:

(I) *There exists* $x_0 \in \overline{U} \cap C$ *such that* $x_0 \in F(x_0)$;

(II) *There exists* $x_0 \in \partial_C(U)$ *with* $y_0 \in F(x_0) \setminus \overline{U}$ *and* $\lambda > 1$ *such that* $\lambda x_0 = y_0 \in F(x_0)$, *i.e.,* $F(x_0) \cap \{\lambda x_0 : \lambda > 1\} \neq \emptyset$.

Proof By following the argument and notations used in Theorem 6.2, we have that either

(1) F has a fixed point $x_0 \in U \cap C$; or

(2) there exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with $x_0 = f(y_0)$ such that

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\bigl(y_0, \overline{I_{\overline{U}}(x_0)} \cap C\bigr) = P_U(y_0) - 1 > 0,$$

where $\partial_C(U)$ denotes the boundary of U relative to C in E and f is the restriction of the continuous retraction r with respect to the set U in E.

If F has no fixed point, then (2) above holds and $x_0 \notin F(x_0)$. As given in the proof of Theorem 6.2, we have that $y_0 \in F(x_0)$ and $y_0 \notin \overline{U}$, thus $P_U(y_0) > 1$ and $x_0 = f(y_0) = \frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}$, which means $y_0 = (P_U(y_0))^{\frac{1}{p}} x_0$. Let $\lambda = (P_U(y_0))^{\frac{1}{p}}$, then $\lambda > 1$, and we have $\lambda x_0 = y_o \in F(x_0)$. This completes the proof. □

Theorem 7.2 (Birkhoff–Kellogg alternative in LCS) *Let U be a bounded open convex subset of a locally convex space E with zero $0 \in U$, let C be a closed convex subset of E with also zero $0 \in C$, and assume that $F : \overline{U} \cap C \to 2^C$ is a semiclosed 1-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph. Then we have that either* (I) *or* (II) *holds, where $W_{\overline{U}}(x_0)$ is either the inward set $I_{\overline{U}}(x_0)$ or the outward set $O_{\overline{U}}(x_0)$:*

(I) *There exists $x_0 \in \overline{U} \cap C$ such that $x_0 \in F(x_0)$;*

(II) *There exists $x_0 \in \partial_C(U)$ with $y_0 \in F(x_0) \setminus \overline{U}$ and $\lambda > 1$ such that $\lambda x_0 = y_0 \in F(x_0)$, i.e., $F(x_0) \cap \{\lambda x_0 : \lambda > 1\} \neq \emptyset$.*

Proof When $p = 1$, it automatically satisfies that the inequality $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$, and indeed we have that for $x_0 \in \partial_C(U)$ with $y_0 \in F(x_0)$, we have $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{W_{\overline{U}}(x_0)} \cap C) = P_U(y_0) - 1$. The conclusions are given by Theorems 6.3 and 6.4. The proof is complete. □

Indeed, we have the following fixed points for nonself mappings in locally p-convex spaces for $0 < p \leq 1$ under various boundary conditions.

Theorem 7.3 (Fixed points of nonself mappings) *Let U be a bounded open p-convex subset of a locally p-convex space E (where $0 < p \leq 1$) with zero $0 \in U$, let C be a closed p-convex subset of E with also zero $0 \in C$, and assume that $F : \overline{U} \cap C \to 2^C$ is a semiclosed 1-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph. In addition, for each $x \in \partial_C(U)$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$), where P_U is the Minkowski p-functional of U. If F satisfies any one of the following conditions for any $x \in \partial_C(U) \setminus F(x)$:*

(i) *For each $y \in F(x)$, $P_U(y - z) < P_U(y - x)$ for some $z \in \overline{I_{\overline{U}}(x)} \cap C$;*

(ii) *For each $y \in F(x)$, there exists λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{I_{\overline{U}}(x)} \cap C$;*

(iii) *$F(x) \subset \overline{I_{\overline{U}}(x)} \cap C$;*

(iv) *$F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;*

(v) *$F(\partial U) \subset \overline{U} \cap C$;*

(vi) *For each $y \in F(x)$, $P_U(y - x) \neq ((P_U(y))^{\frac{1}{p}} - 1)^p$;*

then F must have a fixed point.

Proof By following the argument and symbols used in the proof of Theorem 6.2 (see also Theorem 6.4), we have that either

(1) F has a fixed point $x_0 \in U \cap C$; or

(2) There exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with $x_0 = f(y_0)$ such that

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p\big(y_0, \overline{I_{\overline{U}}(x_0)} \cap C\big) = P_U(y_0) - 1 > 0,$$

where $\partial_C(U)$ denotes the boundary of U relative to C in E and f is the restriction of the continuous retraction r with respect to the set U in E.

First, suppose that F satisfies condition (i), if F has no fixed point, then (2) above holds and $x_0 \notin F(x_0)$. Then, by condition (i), it follows that $P_U(y_0 - z) < P_U(y_0 - x_0)$ for some $z \in \overline{I_{\overline{U}}(x)} \cap C$, this contradicts the best approximation equations given by (2) above, thus F mush have a fixed point.

Second, suppose that F satisfies condition (ii), if F has no fixed point, then (2) above holds and $x_0 \notin F(x_0)$. Then, by condition (ii), there exists $\lambda > 1$ such that $\lambda x_0 + (1-\lambda)y_0 \in \overline{I_{\overline{U}}(x)} \cap C$. It follows that

$$\begin{aligned} P_U(y_0 - x_0) &\leq P_U\big(y_0 - \big(\lambda x_0 + (1 - \lambda y_0)\big)\big) \\ &= P_U\big(\lambda(y_0 - x_0)\big) = |\lambda|^p P_U(y_0 - x_0) < P_U(y_0 - x_0), \end{aligned}$$

this is impossible, and thus F must have a fixed point in $\overline{U} \cap C$.

Third, suppose that F satisfies condition (iii), i.e., $F(x) \subset \overline{I_{\overline{U}}(x)} \cap C$; then (2), we have that $P_U(y_0 - x_0)$ and thus $x_0 = y_0 \in F(x_0)$, which means F has a fixed point.

Forth, suppose that F satisfies condition (iv), if F has no fixed point, then (2) above holds and $x_0 \notin F(x_0)$. As given by the proof of Theorem 6.2, we have that $y_0 \notin \overline{U}$, thus $P_U(y_0) > 1$ and $x_0 = f(y_0) = \frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}$, which means $y_0 = (P_U(y_0))^{\frac{1}{p}} x_0$, where $(P_U(y_0))^{\frac{1}{p}} > 1$, this contradicts the assumption (iv), thus F must have a fixed point in $\overline{U} \cap C$.

Fifth, suppose that F satisfies condition (v), then $x_0 \notin F(x_0)$. As $x_0 \in \partial_C U$, now by condition (v), we have that $F(\partial U) \subset \overline{U} \cap C$. It follows that for any $y_0 \in F(x_0)$, we have $y_0 \in \overline{U} \cap C$, thus $y \notin \overline{U} \setminus \cap C$, which implies that $0 < P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = 0$, this is impossible, thus F must have a fixed point. Here, as pointed out by Remark 5.2, we know that based on condition (v) the mapping F has a fixed point by applying $F(\partial U) \subset \overline{U} \cap C$ is enough, we do not need the general hypothesis: "for each $x \in \partial_C(U)$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$".

Finally, suppose that F satisfies condition (vi), if F has no fixed point, then (2) above holds and $x_0 \notin F(x_0)$. Then condition (v) implies that $P_U(y_0 - x_0) \neq ((P_U(y))^{\frac{1}{p}} - 1)^p$, but our proof in theorem shows that $P_U(y_0 - x_0) = ((P_U(y))^{\frac{1}{p}} - 1)^p$, this is impossible, thus F must have a fixed point. Then the proof is complete. □

Now by taking the set C in Theorem 7.1 as the whole p-vector space E itself, we have the following general results for nonself upper semicontinuous set-valued mappings, which include the results of Rothe, Petryshyn, Altman, and Leray–Schauder type fixed points as special cases.

Taking $p = 1$ and $C = E$ in Theorem 7.3, we have fixed points for nonself upper semicontinuous set-valued mappings associated with inward or outward sets in locally convex spaces (LCS) as follows.

Theorem 7.4 (Fixed point theorem of nonself mappings with boundary conditions) *Let U be a bounded open convex subset of a locally convex spaces E with zero $0 \in U$, and assume that $F : \overline{U} \to 2^E$ is a semiclosed 1-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph. If F satisfies any one of the following conditions for any $x \in \partial(U) \setminus F(x)$:*

(i) *For each $y \in F(x)$, $P_U(y - z) < P_U(y - x)$ for some $z \in \overline{I_{\overline{U}}(x)}$ (or $z \in \overline{O_{\overline{U}}(x)}$);*

(ii) *For each $y \in F(x)$, there exists λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{I_{\overline{U}}(x)}$ (or $\overline{O_{\overline{U}}(x)}$);*

(iii) $F(x) \subset \overline{I_{\overline{U}}(x)}$ (*or* $\overline{O_{\overline{U}}(x)}$);
(iv) $F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;
(v) $F(\partial(U) \subset \overline{U}$;
(vi) *For each* $y \in F(x)$, $P_U(y - x) \neq P_U(y) - 1$;
then F *must have a fixed point.*

In what follows, based on the best approximation theorem in a p-seminorm space, we will also give some fixed point theorems for nonself set-valued mappings with various boundary conditions, which are related to the study for the existence of solutions for PDE and differential equations with boundary problems (see Browder [18], Petryshyn [104, 105], Reich [110]), which would play roles in nonlinear analysis for p-seminorm space as shown below.

First, as discussed by Remark 5.2, the proof of Theorem 7.2 with the strongly boundary condition "$F(\partial(U)) \subset \overline{U} \cap C$" only, we can prove that F has a fixed point, thus we have the following fixed point theorem of Rothe type in p-vector spaces.

Theorem 7.5 (Rothe type) *Let* U *be a bounded open* p*-convex subset of a locally* p*-convex space* E (*where* $0 < p \leq 1$) *with zero* $0 \in U$. *Assume* $F : \overline{U} \to 2^E$ *is a semiclosed* 1*-set contractive and quasi upper semicontinuous mapping with nonempty* p*-convex values, with a closed graph, and such that* $F(\partial(U)) \subset \overline{U}$, *then* F *must have a fixed point.*

Now, as applications of Theorem 7.5, we give the following Leray–Schauder alternative in p-vector spaces for nonself set-valued mappings associated with the boundary condition, which often appears in the applications (see Isac [60] and the references therein for the study of complementary problems and related topics in optimization).

Theorem 7.6 (Leray–Schauder alternative in p-vector spaces) *Let* E *be a locally* p*-convex space* E, *where* $0 < p \leq 1$, $B \subset E$ *is bounded closed* p*-convex such that* $0 \in \operatorname{int} B$. *Let* $F : [0,1] \times B \to 2^E$ *be a semiclosed* 1*-set contractive and quasi upper semicontinuous set-valued mapping with nonempty* p*-convex values, with a closed graph, and such that the set* $F([0,1] \times B)$ *is relatively compact in* E. *If the following assumptions are satisfied*:
(1) $x \notin F(t,x)$ *for all* $x \notin \partial B$ *and* $t \in [0,1]$;
(2) $F(\{0\} \times \partial B) \subset B$,
then there is an element $x^* \in B$ *such that* $x^* \in F(1,x^*)$.

Proof For any $n \in N$, we consider the mapping

$$F_n(x) = \begin{cases} F(\frac{1-P_B(x)}{\epsilon_n}, \frac{x}{P_B(x)}) & \text{if } 1-\epsilon \leq P_B(x) \leq 1, \\ F(1, \frac{X}{1-\epsilon_n}) & \text{if } P_B(x) < 1-\epsilon_n, \end{cases} \tag{1}$$

where P_B is the Minkowski p-functional of B and $\{\epsilon_n\}_{n\in N}$ is a sequence of real numbers such that $\lim_{n\to\infty} \epsilon_n = 0$ and $0 < \epsilon_n < \frac{1}{2}$ for any $n \in N$. We observe that for each $n \in N$, the mapping F_n is 1-set contractive upper semicontinuous with nonempty closed p-convex values on B. From assumption (2), we have that $F_n(\partial B) \subset B$, and the assumptions of Theorem 7.5 are satisfied, then for each $n \in N$, there exists an element $u_n \in B$ such that $u_n \in F_n(u_n)$.

We first prove the following statement: "It is impossible to have an infinite number of the elements u_n satisfying the following inequality: $1-\epsilon_n \leq P_B(u_n) \leq 1$."

If not, we assume to have an infinite number of elements u_n satisfying the following inequality:

$$1-\epsilon_n \leq P_B(u_n) \leq 1.$$

As $F_n(B)$ is relatively compact and by the definition of mappings F_n, we have that $\{u_n\}_{n\in N}$ is contained in a compact set in E. Without loss of generality (indeed, each compact set is also countably compact), we define the sequence $\{t_n\}_{n\in N}$ by $t_n := \frac{1-P_B(u_n)}{\epsilon}$ for each $n \in N$. Then we have that $\{t_n\}_{n\in N} \subset [0,1]$, and we may assume that $\lim_{n\to\infty} t_n = t \in [0,1]$. The corresponding subsequence of $\{u_n\}_{n\in N}$ is denoted again by $\{u_n\}_{n\in N}$, and it also satisfies the inequality $1-\epsilon_n \leq P_B(u_n) \leq 1$, which implies that $\lim_{n\to\infty} P_B(u_n) = 1$.

Now let u^* be an accumulation point of $\{u_n\}_{n\in N}$, thus have $\lim_{n\to\infty}(t_n, \frac{u_n}{P_B(u_n)}, u_n) = (t, u^*, u^*)$. By the fact that F is compact, we assume that $u_n \in F(t_n, \frac{u_n}{P_B(u_n)})$ for each $n \in N$, it follows that $u^* \in F(t, u^*)$, this contradicts assumption (1) as we have $\lim_{n\to\infty} P_B(u_n) = 1$ (which means that $u^* \in \partial B$, this is impossible).

Thus it is impossible "to have an infinite number of elements u_n satisfying the inequality $1-\epsilon_n \leq P_B(u_n) \leq 1$", which means that there is only a finite number of elements of sequence $\{u_n\}_{n\in N}$ satisfying the inequality $1-\epsilon_n \leq P_B(u_n) \leq 1$. Now, without loss of generality, for $n \in N$, we have the following inequality:

$$P_B(u_n) < 1-\epsilon_n.$$

By the fact that $\lim_{n\to}(1-\epsilon_n) = 1$, $u_n \in F(1, \frac{u_n}{1-\epsilon})$ for all $n \in N$ and assuming that $\lim_{n\to} u_n = u^*$, the upper semicontinuity of F with nonempty closed values implies that the graph of F is closed, and by the fact $u_n \in F(1, \frac{u_n}{1-\epsilon})$, it implies that $u^* \in F(1, u^*)$. This completes the proof. □

As a special case of Theorem 7.6, we have the following principle for the implicit form of Leray–Schauder type alternative for set-valued mappings in p-vector spaces for $0 < p \leq 1$.

Corollary 7.1 (The implicit Leray–Schauder alternative) *Let E be a locally p-convex space E, where $0 < p \leq 1$, $B \subset E$ be bounded closed p-convex such that $0 \in \operatorname{int} B$. Let $F : [0,1] \times B \to 2^E$ be semiclosed 1-set contractive and quasi upper semicontinuous with nonempty p-convex values and with a closed graph, and let the set $F([0,1] \times B)$ be relatively compact in E. If the following assumptions are satisfied*:

(1) $F(\{0\} \times \partial B) \subset B$,

(2) $x \notin F(0,x)$ *for all* $x \in \partial B$,

then at least one of the following properties is satisfied:

(i) *there exists* $x^* \in B$ *such that* $x^* \in F(1, x^*)$; *or*

(ii) *there exists* $(\lambda^*, x^*) \in (0,1) \times \partial B$ *such that* $x^* \in F(\lambda^*, x^*)$.

Proof The result is an immediate consequence of Theorem 7.6, this completes the proof. □

We would like to point out that similar results on Rothe and Leray–Schauder alternative have been developed by Furi and Pera [44], Granas and Dugundji [53], Górniewicz [51], Górniewicz et al. [52], Isac [60], Li et al. [78], Liu [81], Park [96], Potter [108], Shahzad [120–122], Xu [139], Xu et al. [140] (see also the related references therein) as tools of nonlinear analysis in the Banach space setting and applications to the boundary value problems for ordinary differential equations in noncompact problems and a general class of mappings for nonlinear alternative of Leray–Schauder type in normal topological spaces. Some Birkhoff–Kellogg type theorems for general class mappings in topological vector spaces have also been established by Agarwal et al. [1], Agarwal and O'Regan [2, 3], and Park [98] (see the references therein for more details); and in particular, recently O'Regan [91] used the Leray–Schauder type coincidence theory to establish some Birkhoff–Kellogg problems, Furi–Pera type results for a general class of mappings.

Before closing this section, we would like to share that as the application of the best approximation result for 1-set contractive mappings we can establish fixed point theorems and the general principle of Leray–Schauder alternative for nonself mappings, which would seen to play important roles for the nonlinear analysis under the framework of p-seminorm spaces, as the achievement of nonlinear analysis for the underling being locally convex spaces, normed spaces, or in Banach spaces.

8 Fixed points for nonself semiclosed 1-set contractive mappings with various boundary conditions

In this section, based on the best approximation Theorem 6.2 established for the 1-set contractive mappings in Sect. 6, we will show how it is used as a useful tool for us to develop fixed point theorems for semiclosed 1-set contractive nonself upper semicontinuous mappings in p-seminorm spaces, where $p \in (0, 1]$, by including seminorm, norm spaces, and uniformly convex Banach spaces as special cases.

By following Definitions 6.1 and 6.2 above, we first observe that if f is a continuous demicompact mapping, then $(I - f)$ is closed, where I is the identity mapping on X. It is also clear from definitions that every demicompact map is hemicompact in seminorm spaces, but the converse is not true in general (e.g., see the example in p. 380 by Tan and Yuan [129]). It is evident that if f is demicompact, then $I - f$ is demiclosed. It is known that for each condensing mapping f, when D or $f(D)$ is bounded, then f is hemicompact; and also f is demicompact in metric spaces by Lemma 2.1 and Lemma 2.2 of Tan and Yuan [129], respectively. In addition, it is known that every nonexpansive map is a 1-set-contractive map; and also if f is a hemicompact 1-set-contractive mapping, then f is a 1-set-contractive mapping satisfying the following "*Condition (H1)*" (the same as (H1), and slightly different from condition (H) used in Sect. 5):

(H1) Condition: Let D be a nonempty bounded subset of a space E and assume that $F : \overline{D} \to 2^E$ is a set-valued mapping. If $\{x_n\}_{n\in\mathbb{N}}$ is any sequence in D such that for each x_n, there exists $y_n \in F(x_n)$ with $\lim_{n\to\infty}(x_n - y_n) = 0$, then there exists a point $x \in \overline{D}$ such that $x \in F(x)$.

We first note that the "(H1) Condition" above is actually "Condition (C)" used by Theorem 1 of Petryshyn [105]. Indeed, by following Goebel and Kirk [49] (see also Xu [137] and the references therein), Browder [18] (see also [19], p. 103) proved that if K is a closed and convex subset of a uniformly convex Banach space X, and if $T : K \to X$ is nonexpansive, then the mapping $f := I - T$ is demiclosed on X. This result, known as Browder's demiclosedness principle (Browder's proof, which was inspired by the technique of Göhde in [50]), is one of the fundamental results in the theory of nonexpansive mappings that satisfies the "(H1) condition".

The following is Browder's demiclosedness principle proved by Browder [18] that says that a nonexpansive mapping in a uniformly convex Banach X enjoys condition (H1) as shown below.

Lemma 8.1 *Let D be a nonempty bonded convex subset of a uniformly convex Banach space E. Assume that $F : \overline{D} \to E$ is a nonexpansive single-valued mapping, then the mapping $P := I - F$ defined by $P(x) := (x - F(x))$ for each $x \in \overline{D}$ is demiclosed, and in particular, the "$(H1)$ condition" holds.*

Proof By following the argument given in p. 329 (see also the proof of Theorem 2.2 and Corollary 2.1) by Petryshyn [105], by the Browder demiclosedness principle (see Goebel and Kirk [49] or Xu [137]), $P = (I - F)$ is closed at zero, thus there exists $x_0 \in \overline{U}$ such $0 \in (I - F)x_0)$, which means that $x_0 \in F(x_0)$. The proof is complete. □

On the other hand, by following the notion called "Opial's condition" given by Opial [90], which says that a Banach space X is said to satisfy Opial's condition if $\liminf_{n\to\infty} \|w_n - w\| < \liminf_{n\to\infty} \|w_n - p\|$ whenever (w_n) is a sequence in X weakly convergent to w and $p \neq w$, we know that Opial's condition plays an important role in the fixed point theory, e.g., see Lami Dozo [75], Goebel and Kirk [49], Xu [137], and the references therein. Actually, the following result shows that there exists a class of nonexpansive set-valued mappings in Banach spaces with Opial's condition (see Lami Dozo [75] satisfying the "(H1) Condition".

Lemma 8.2 *Let C be a nonempty convex weakly compact subset of a Banach space X that satisfies Opial's condition. Let $T : C \to K(C)$ be a nonexpansive set-valued mapping with nonempty compact values. Then the graph of $(I - T)$ is closed $(X, \sigma(X, X^*) \times (X, \|\cdot\|))$, thus T satisfies the "$(H1)$ condition", where I denotes the identity on X, $\sigma(X, X^*)$is the weak topology, and $\|\cdot\|$ is the norm (or strong) topology.*

Proof By following Theorem 3.1 of Lami Dozo [75], it follows that the mapping T is demiclosed, thus T satisfies the "(H1) condition". The proof is complete. □

By Theorem 3.1 of Lami Dozo [75], indeed, we have the following statement, which is another version by using the term of "distance convergence" for Lemma 8.2.

Lemma 8.3 *Let C be a nonempty closed convex subset of a Banach space (X, d) that satisfies the Opial condition. Let $T : C \to K(C)$ be a multivalued nonexpansive mapping (with fixed points). Let $(y_n)_{n\in\mathbb{N}}$ be a bounded sequence such that $_{n\to\infty} d(y, T(y_n)) = 0$, then the weak cluster points of (y_n), $n \in \mathbb{N}$ is a fixed point of T.*

Proof It is Theorem 3.1 of Lami Dozo [75] (see also Lemma 3.2 of Xu and Muglia [138]). □

We note that another class of set-valued mappings, called "$*$-nonexpansive mappings in Banach spaces (introduced by Husain and Tarafdar [59], see also Husain and Latif [58]), was proved to hold the demiclosedness principle in reflexive Banach spaces satisfying Opial's condition by Muglia and Marino (i.e., Lemma 3.4 in [85]), thus the demiclosedness principle also holds in reflexive Banach spaces with duality mapping that is weakly sequentially continuous since these satisfy Opial's condition.

Let E denote a Hausdorff locally convex topological vector space and $\mathfrak{F}$ denote the family of continuous seminorms generating the topology of E. Also $C(E)$ will denote the family of nonempty compact subsets of E. For each $p \in \mathfrak{F}$ and $A, B \in C(E)$, we can define $\delta(A,B) := \sup\{p(a-b) : a \in A, b \in B\}$ and $D_p(A,B) := \max\{\sup_{a\in A}\inf_{b\in B} P(a-b), \sup_{b\in B}\inf_{a\in A} P(a-b)\}$. Though p is only a seminorm, D_p is a Hausdorff metric on $C(E)$ (e.g., see Ko and Tsai [71]).

Definition 8.1 Let K be a nonempty subset of E. A mapping $T : K \to C(E)$ is said to be a multivalued contraction if there exists a constant $k_p \in (0,1)$ such that $D_p(T(x), T(y)) \le k_p P(x-y)$. T is said to be nonexpansive if for any $x, y \in K$, we have $P_p(T(x), T(y))) \le P(x-y)$.

By Chen and Singh [31], we now have the following definition of Opial's condition in locally convex spaces.

Definition 8.2 The locally convex space E is said to satisfy Opial's condition if for each $x \in E$ and every net (x_α) converging weakly to x, for each $P \in \mathfrak{F}$, we have $\liminf P(x_\alpha - y) > \liminf P(x_\alpha - x)$ for any $y \neq x$.

Now we have the following demiclosedness principle for nonexpansive set-valued mappings in (Hausdorff) local convex spaces E, which is indeed Theorem 1 of Chen and Singh [31].

Lemma 8.4 *Let K be a nonempty, weakly, compact, and convex subset of E. Let $T : K \to C(E)$ be nonexpansive. If E satisfies Opial's condition, then the graph $(I - G)$ is closed in $E_w \times E$, where E_w is E with its weak topology and I is the identity mapping.*

Proof The conclusion follows by Theorem 1 of Chen and Singh [31]. □

Remark 8.1 When a p-vector space E is with a p-norm, then both (H1) and (H) conditions for their convergence can be described by the convergence weakly and strongly by the weak topology and strong topology induced by p-norm for $p \in (0,1]$. Secondly, if a given p-vector space E has a nonempty open p-convex subset U containing zero, then any mapping satisfying the "(H) condition" is a hemicompact mapping (with respect P_U for a given bounded open p-convex subset U containing zero of p-vector space E), thus satisfying the "(H) condition" used in Theorem 5.1.

By the fact that each semiclosed 1-set mapping satisfies the "(H1) condition", we have the existence of fixed points for the class of semiclosed 1-set mappings. First, as an application of Theorem 8.2, we have the following result for nonself mappings in p-seminorm spaces for $p \in (0,1]$.

Theorem 8.1 *Let U be a bounded open p-convex subset of a p-seminorm space E $(0 < p \le 1)$ zero $0 \in U$. Assume that $F : \overline{U} \to 2^E$ is a semiclosed 1-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph. In addition, for any $x \in \partial\overline{U}$ and $y \in F(x)$, we have $\lambda x \neq y$ for any $\lambda > 1$ (i.e., the "Leray–Schauder boundary condition"). Then F has at least one fixed point.*

Proof By the proof of Theorem 6.2 with $C = E$, we actually have thaat either (I) or (II) holds:

(I) F has a fixed point $x_0 \in U$, i.e., $P_U(y_0 - x_0) = 0$;

(II) There exist $x_0 \in \partial(U)$ and $y_0 \in F(x_0)$ with $P_U(y_0 - x_0) = (P_U^{\frac{1}{p}}(y_0) - 1)^p > 0$.

If F has no fixed point, then (II) above holds and $x_0 \notin F(x_0)$. By the proof of Theorem 6.2, we have that $x_0 = f(y_0)$ and $y_0 \notin \overline{U}$. Thus $P_U(y_0) > 1$ and $x_0 = f(y_0) = \frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}$, which means $y_0 = (P_U(y_0))^{\frac{1}{p}} x_0$, where $(P_U(y_0))^{\frac{1}{p}} > 1$, this contradicts the assumption. Thus F must have a fixed point. The proof is complete. □

By following the idea used and developed by Browder [18], Li [77], Li et al. [78], Goebel and Kirk [48], Petryshyn [104, 105], Tan and Yuan [129], Xu [139], Xu et al. [140] and the references therein, we have the following existence theorems for the principle of Leray–Schauder type alternatives in p-seminorm spaces $(E, \|\cdot\|_p)$ for $p \in (0, 1]$.

Theorem 8.2 *Let U be a bounded open p-convex subset of a p-seminorm space $(E, \|\cdot\|_p)$ $(0 < p \leq 1)$ zero $0 \in U$. Assume that $F : \overline{U} \to 2^E$ is a semiclosed 1-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph. In addition, there exist $\alpha > 1$, $\beta \geq 0$ such that for each $x \in \partial\overline{U}$, we have that for any $y \in F(x)$, $\|y - x\|_p^{\alpha/p} \geq \|y\|_p^{(\alpha+\beta)/p} \|x\|_p^{-\beta/p} - \|x\|_p^{\alpha/p}$. Then F has at least one fixed point.*

Proof By assuming F has no fixed point, we prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 8.1 does not hold. If we assume that F has no fixed point, by the boundary condition of Theorem 8.1, there exist $x_0 \in \partial\overline{U}$, $y_0 \in F(x_0)$, and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t - 1)^\alpha - t^{\alpha+\beta} + 1$ for $t \geq 1$. We observe that f is a strictly decreasing function for $t \in [1, \infty)$ as the derivative of $f'(t) = \alpha(t-1)^{\alpha-1} - (\alpha + \beta)t^{\alpha+\beta-1} < 0$ by the differentiation, thus we have $t^{\alpha+\beta} - 1 > (t-1)^\alpha$ for $t \in (1, \infty)$. By combining the boundary condition, we have that $\|y_0 - x_0\|_p^{\alpha/p} = \|\lambda_0 x_0 - x_0\|_p^{\alpha/p} = (\lambda_0 - 1)^\alpha \|x_0\|_p^{\alpha/p} < (\lambda_0^{\alpha+\beta} - 1)\|x_0\|_p^{(\alpha+\beta)/p} \|x_0\|_p^{-\beta/p} = \|y_0\|_p^{(\alpha+\beta)/p} \|x_0\|_p^{-\beta/p} - \|x_0\|_p^{\alpha/p}$, which contradicts the boundary condition given by Theorem 8.2. Thus, the conclusion follows. □

Theorem 8.3 *Let U be a bounded open p-convex subset of a p-seminorm space $(E, \|\cdot\|_p)$ $(0 < p \leq 1)$ zero $0 \in U$. Assume that $F : \overline{U} \to 2^E$ is a semiclosed 1-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph. In addition, there exist $\alpha > 1$, $\beta \geq 0$ such that for each $x \in \partial\overline{U}$, we have that for any $y \in F(x)$, $\|y + x\|_p^{(\alpha+\beta)/p} \leq \|y\|_p^{\alpha/p} \|x\|_p^{\beta/p} + \|x\|_p^{(\alpha+\beta)/p}$. Then F has at least one fixed point.*

Proof We prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 8.1 does not hold. If we assume that F has no fixed point, by the boundary condition of Theorem 8.1, there exist $x_0 \in \partial\overline{U}$, $y_0 \in F(x_0)$, and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t + 1)^{\alpha+\beta} - t^\alpha - 1$ for $t \geq 1$. We then can show that f is a strictly increasing function for $t \in [1, \infty)$, thus we have $t^\alpha + 1 < (t + 1)^{\alpha+\beta}$ for $t \in (1, \infty)$. By the boundary condition given in Theorem 8.3, we have that

$$\begin{aligned}\|y_0 + x_0\|_p^{(\alpha+\beta)/p} &= (\lambda_0 + 1)^{\alpha+\beta} \|x_0\|_p^{(\alpha+\beta)/p} \\ &> \left(\lambda_0^\alpha + 1\right) \|x_0\|_p^{(\alpha+\beta)/p}\end{aligned}$$

$$= \|y_0\|_p^{\alpha/p}\|x_0\|_p^{\beta/p} + \|x_0\|_p^{\alpha/p},$$

which contradicts the boundary condition given by Theorem 8.3. Thus, the conclusion follows and the proof is complete. □

Theorem 8.4 *Let U be a bounded open p-convex subset of a p-seminorm space $(E, \|\cdot\|_p)$ $(0 < p \leq 1)$ zero $0 \in U$. Assume that $F : \overline{U} \to 2^E$ is a semiclosed 1-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph. In addition, there exist $\alpha > 1$, $\beta \geq 0$ (or alternatively, $\alpha > 1$, $\beta \geq 0$) such that for each $x \in \partial\overline{U}$, we have that for any $y \in F(x)$, $\|y - x\|_p^{\alpha/p}\|x\|_p^{\beta/p} \geq \|y\|_p^{\alpha/p}\|y + x\|_p^{\beta/p} - \|x\|_p^{(\alpha+\beta)/p}$. Then F has at least one fixed point.*

Proof The same as above, we prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 8.1 does not hold. If we assume that F has no fixed point, by the boundary condition of Theorem 8.1, there exist $x_0 \in \partial\overline{U}$, $y_0 \in F(x_0)$, and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t-1)^\alpha - t^\alpha(t-1)^\beta + 1$ for $t \geq 1$. We then can show that f is a strictly decreasing function for $t \in [1, \infty)$, thus we have $(t-1)^\alpha < t^\alpha(t+1)^\beta - 1$ for $t \in (1, \infty)$. By the boundary condition given in Theorem 8.4, we have that

$$\begin{aligned}\|y_0 - x_0\|_p^{\alpha/p}\|x_0\|_p^{\beta/p} &= (\lambda_0 - 1)^\alpha\|x_0\|_p^{(\alpha+\beta)/p} \\ &< \left(\lambda_0^\alpha(\lambda_0 + 1)^\beta - 1\right)\|x_0\|_p^{(\alpha+\beta)/p} \\ &= \|y_0\|_p^{\alpha/p}\|y_0 + x_0\|_p^{\beta/p} - \|x_0\|_p^{(\alpha+\beta)/p},\end{aligned}$$

which contradicts the boundary condition given by Theorem 8.4. Thus, the conclusion follows and the proof is complete. □

Theorem 8.5 *Let U be a bounded open p-convex subset of a p-seminorm space $(E, \|\cdot\|_p)$ $(0 < p \leq 1)$ zero $0 \in U$. Assume that $F : \overline{U} \to 2^E$ is a semiclosed 1-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph. In addition, there exist $\alpha > 1$, $\beta \geq 0$, we have that for any $y \in F(x)$, $\|y + x\|_p^{(\alpha+\beta)/p} \leq \|y - x\|_p^{\alpha/p}\|x\|_p^{\beta/p} + \|y\|_p^{\beta/p}\|x\|^{\alpha/p}$. Then F has at least one fixed point.*

Proof The same as above, we prove the conclusion by showing that the Leray–Schauder boundary condition in Theorem 8.1 does not hold. If we assume F has no fixed point, by the boundary condition of Theorem 8.1, there exist $x_0 \in \partial\overline{U}$, $y_0 \in F(x_0)$, and $\lambda_0 > 1$ such that $y_0 = \lambda_0 x_0$.

Now, consider the function f defined by $f(t) := (t+1)^{\alpha+\beta} - (t-1)^\alpha - t^\beta$ for $t \geq 1$. We then can show that f is a strictly increasing function for $t \in [1, \infty)$, thus we have $(t+1)^{\alpha+\beta} > (t-1)^\alpha + t^\beta$ for $t \in (1, \infty)$.

By the boundary condition given in Theorem 8.5, we have that $\|y_0 + x_0\|_p^{(\alpha+\beta)/p} = (\lambda_0 + 1)^{\alpha+\beta}\|x_0\|_p^{(\alpha+\beta)/p} > ((\lambda_0 - 1)^\alpha + \lambda_0^\beta)\|x_0\|_p^{(\alpha+\beta)/p} = \|\lambda_0 x_0 - x_0\|_p^{\alpha/p}\|x_0\|_p^{\beta/p} + \|\lambda_0 x_0\|_p^{\beta/p}\|x_0\|_p^{\alpha/p} = \|y_0 - x_0\|_p^{\beta/p}\|x_0\|_p^{\alpha/p} + \|y_0\|_p^{\beta/p}\|x_9\|^{\alpha/p}$, which implies that

$$\|y_0 + x_0\|_p^{(\alpha+\beta)/p} > \|y_0 - x_0\|_p^{\beta/p}\|x_0\|_p^{\alpha/p} + \|y_0\|_p^{\beta/p}\|x_9\|^{\alpha/p},$$

this contradicts the boundary condition given by Theorem 8.5. Thus, the conclusion follows and the proof is complete. □

As an application of Theorem 8.1, by testing the Leray–Schauder boundary condition, we have the following conclusion for each special case, and thus we omit their proofs in detail here.

Corollary 8.1 *Let U be a bounded open p-convex subset of a p-seminorm space $(E, \|\cdot\|_p)$ $(0 < p \leq 1)$ zero $0 \in U$. Assume that $F : \overline{U} \to 2^E$ is a semiclosed 1-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph. Then F has at least one fixed point if one of the following (strong) conditions holds for $x \in \partial\overline{U}$ and $y \in F(x)$:*

(i) $\|y\|_p \leq \|x\|_p$,
(ii) $\|y\|_p \leq \|y - x\|_p$,
(iii) $\|y + x||_p \leq \|y\|_p$,
(iv) $\|y + x\|_p \leq \|x\|_p$,
(v) $\|y + x\|_p \leq \|y - x\|_p$,
(vi) $\|y\|_p \cdot \|y + x\|_p \leq \|x\|_p^2$,
(vii) $\|y\|_p \cdot \|y + x\|_p \leq \|y - x\|_p \cdot \|x\|_p$.

If the p-(semi)norm space E is a uniformly convex Banach space $(E, \|\cdot\|)$ (for p-norm space with $p = 1$), then we have the following general existence result, which can apply to general nonexpansive (single-valued) mappings, too.

Theorem 8.6 *Let U be a bounded open convex subset of a uniformly convex Banach space $(E, \|\cdot\|)$ (with $p = 1$) with zero $0 \in U$. Assume that $F : \overline{U} \to E$ is a semicontractive and continuous (single-valued) mapping. In addition, for any $x \in \partial\overline{U}$, we have $\lambda x \neq F(x)$ for any $\lambda > 1$ (i.e., the "Leray–Schauder boundary condition"). Then F has at least one fixed point.*

Proof By Lemma 8.1, F is a semiclosed 1-set contractive mapping. Moreover, by the assumption that E is a uniformly convex Banach, the mapping $(I - F)$ is closed at zero, and thus F is semiclosed at zero (see Browder [18] or Goebel and Kirk [48]). Thus all assumptions of Theorem 8.2 are satisfied. The conclusion follows by Theorem 8.2. The proof is complete. □

Now we have the following results for nonexpansive set-valued mappings in a Banach space X with Opial's condition.

Theorem 8.7 *Let C be a nonempty convex weakly compact subset of a Banach space X that satisfies Opial's condition and $0 \in \operatorname{int} C$. Let $T : C \to K(X)$ be a nonexpansive set-valued mapping with nonempty compact convex values. In addition, for any $x \in \partial\overline{C}$, we have $\lambda x \neq F(x)$ for any $\lambda > 1$ (i.e., the "Leray–Schauder boundary condition"). Then F has at least one fixed point.*

Proof As T is nonexpansive, it is 1-set contractive. By Lemma 8.2, it is then semicontractive and continuous. Then all conditions of Theorem 8.1 are satisfied, the conclusion follows by Theorem 8.1, and the proof is complete. □

By using Lemma 8.4, we have the following result in local convex spaces for nonexpansive set-valued mappings.

Theorem 8.8 *Let C be a nonempty convex weakly compact subset of a local convex space X that satisfies Opial's condition and $0 \in \operatorname{int} C$. Let $T : C \to K(X)$ be a nonexpansive set-valued mapping with nonempty compact convex values. In addition, for any $x \in \partial\overline{C}$, we have $\lambda x \neq F(x)$ for any $\lambda > 1$ (i.e., the "Leray–Schauder boundary condition"). Then F has at least one fixed point.*

Proof As T is nonexpansive, it is 1-set contractive. By Lemma 8.4, it is then semicontractive and continuous. Then all conditions of Theorem 8.1 are satisfied, the conclusion follows by Theorem 10.1, and the proof is complete. □

By considering a p-seminorm space $(E, \|\cdot\|)$ with a seminorm for $p = 1$, the following corollary is a special case of the corresponding results from Theorem 8.2 to Theorem 8.5, and thus we omit its proof.

Corollary 8.2 *Let U be a bounded open convex subset of a norm space $(E, \|\cdot\|)$. Assume that $F : \overline{U} \to 2^E$ is a semiclosed 1-set contractive and quasi upper semicontinuous mapping with nonempty p-convex values and with a closed graph. Then F has at least one fixed point if there exist $\alpha > 1$, $\beta \geq 0$ such that any one of the following conditions is satisfied:*

(i) *For each $x \in \partial\overline{U}$ and any $y \in F(x)$, $\|y - x\|^{\alpha} \geq \|y\|^{(\alpha+\beta)}\|x\|^{-\beta} - \|x\|^{\alpha}$;*
(ii) *For each $x \in \partial\overline{U}$ and any $y \in F(x)$, $\|y + x\|^{(\alpha+\beta)} \leq \|y\|^{\alpha}\|x\|^{\beta} + \|x\|^{(\alpha+\beta)}$;*
(iii) *For each $x \in \partial\overline{U}$ and any $y \in F(x)$, $\|y - x\|^{\alpha}\|x\|^{\beta} \geq \|y\|^{\alpha}\|y + x\|^{\beta} - \|x\|^{(\alpha+\beta)}$;*
(iv) *For each $x \in \partial\overline{U}$ and any $y \in F(x)$, $\|y + x\|^{(\alpha+\beta)} \leq \|y - x\|^{\alpha}\|x\|^{\beta} + \|y\|^{\beta}\|x\|^{\alpha}$.*

Remark 8.2 As discussed by Lemma 8.1 and the proof of Theorem 8.6, when the p-vector space is a uniformly convex Banach space, the semicontractive or nonexpansive mappings automatically satisfy the conditions (see (H1)) required by Theorem 8.1, that is, the mappings are indeed semiclosed. Moreover, our results from Theorem 8.1 to Theorem 8.6, Corollary 8.1 and Corollary 8.2 also improve or unify corresponding results given by Browder [18], Li [77], Li et al. [78], Goebel and Kirk [48], Petryshyn [104, 105], Reich [110], Tan and Yuan [129], Xu [136], Xu [139], Xu et al. [140], and the results from the references therein by extending the nonself mappings to the classes of semiclosed 1-set contractive set-valued mappings in p-seminorm spaces with $p \in (0, 1]$, including the norm space or Banach space when $p = 1$ for p-seminorm spaces.

Before the ending of this paper, we would like to share with readers that the main goal of this paper was to develop new fixed point theorems and tools in nonlinear analysis for 1-set contractive upper semicontinuous set-valued mappings in locally p-convex spaces for $p \in (0, 1]$.

Actually, the corresponding theory in nonlinear functional analysis could be developed by applying Theorem 4.3 as a tool in locally p-convex, p-vector and topological vector spaces for singe-valued mappings for $pin \in (0, 1]$, and we do not discuss them in detail here due to the limited space.

In addition, we do expect that results established in this paper would become useful tools for the study on optimization, nonlinear programming, variational inequality, complementarity, game theory, mathematical economics, and other related social science areas.

Finally, we would like to share that the results established in this paper do not only unify or improve the corresponding results in the existing literature for nonlinear analysis, but they can also be regarded as the continuation of (or) related work established recently by Yuan [144, 145].

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Author details
[1]Chongqing University of Technology, Chongqing, 400054, China. [2]Chengdu University, Chengdu 610601, China. [3]School of Mathematics, Sichuan University, Chengdu 610065, China. [4]Business School, East China University of Science and Technology, Shanghai 200237, China. [5]Business School, Sun Yat-sen University, Guangzhou 510275, China.

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References
1. Agarwal, R.P., Meehan, M., O'Regan, D.: Fixed Point Theory and Applications. Cambridge Tracts in Mathematics, vol. 141. Cambridge University Press, Cambridge (2001)
2. Agarwal, R.P., O'Regan, D.: Birkhoff-Kellogg theorems on invariant directions for multimaps. Abstr. Appl. Anal. **7**, 435–448 (2003)
3. Agarwal, R.P., O'Regan, D.: Essential U_c^k-type maps and Birkhoff-Kellogg theorems. J. Appl. Math. Stoch. Anal. **1**, 1–8 (2004)
4. Ageev, S.M., Repovš, D.: A selection theorem for strongly regular multivalued mappings. Set-Valued Anal. **6**(4), 345–362 (1998)
5. Alghamdi, M.A., O'Regan, D., Shahzad, N.: Krasnosel'skii type fixed point theorems for mappings on nonconvex sets. Abstr. Appl. Anal. **2020**, Article ID 267531 (2012)
6. Askoura, Y., Godet-Thobie, C.: Fixed points in contractible spaces and convex subsets of topological vector spaces. J. Convex Anal. **13**(2), 193–205 (2006)
7. Balachandran, V.K.: Topological Algebras, vol. 185. Elsevier, Amsterdam (2000)
8. Bayoumi, A.: Foundations of Complex Analysis in Nonlocally Convex Spaces. Function Theory Without Convexity Condition. North-Holland Mathematics Studies, vol. 193. Elsevier, Amsterdam (2003)

9. Bayoumi, A., Faried, N., Mostafa, R.: Regularity properties of p-distance transformations in image analysis. Int. J. Contemp. Math. Sci. **10**, 143–157 (2015)
10. Ben-El-Mechaiekh, H.: Approximation and selection methods for set-valued maps and fixed point theory. In: Fixed Point Theory, Variational Analysis, and Optimization, pp. 77–136. CRC Press, Boca Raton (2014)
11. Ben-El-Mechaiekh, H., Saidi, F.B.: On the continuous approximation of upper semicontinuous set-valued maps. Quest. Answ. Gen. Topol. **31**(2), 71–78 (2013)
12. Bernstein, S.: Sur les equations de calcul des variations. Ann. Sci. Éc. Norm. Supér. **29**, 431–485 (1912)
13. Bernuées, J., Pena, A.: On the shape of p-convex hulls $0 < p < 1$. Acta Math. Hung. **74**(4), 345–353 (1997)
14. Birkhoff, G.D., Kellogg, O.D.: Invariant points in function space. Trans. Am. Math. Soc. **23**(1), 96–115 (1922)
15. Browder, F.E.: Nonexpansive nonlinear operators in a Banach space. Proc. Natl. Acad. Sci. USA **54**, 1041–1044 (1965)
16. Browder, F.E.: Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces. Arch. Ration. Mech. Anal. **24**(1), 82–90 (1967)
17. Browder, F.E.: The fixed point theory of multi-valued mappings in topological vector spaces. Math. Ann. **177**, 283–301 (1968)
18. Browder, F.E.: Semicontractive and semiaccretive nonlinear mappings in Banach spaces. Bull. Am. Math. Soc. **74**, 660–665 (1968)
19. Browder, F.E.: Nonlinear Functional Analysis. Proc. Sympos. Pure Math., vol. 18. Am. Math. Soc., Providence (1976)
20. Browder, F.E.: Fixed point theory and nonlinear problems. Bull. Am. Math. Soc. (N.S.) **9**(1), 1–39 (1983)
21. Carbone, A., Conti, G.: Multivalued maps and existence of best approximations. J. Approx. Theory **64**, 203–208 (1991)
22. Cauty, R.: Rétractès absolus de voisinage algébriques. (French) [Algebraic absolute neighborhood retracts]. Serdica Math. J. **31**(4), 309–354 (2005)
23. Cauty, R.: Le théorème de Lefschetz–Hopf pour les applications compactes des espaces ULC. (French) [The Lefschetz–Hopf theorem for compact maps of uniformly locally contractible spaces]. J. Fixed Point Theory Appl. **1**(1), 123–134 (2007)
24. Cellina, A.: Approximation of set valued functions and fixed point theorems. Ann. Mat. Pura Appl. **82**(4), 17–24 (1969)
25. Chang, S.S.: Some problems and results in the study of nonlinear analysis. Nonlinear Anal. **30**(7), 4197–4208 (1997)
26. Chang, S.S., Cho, Y.J., Park, S., Yuan, G.Z.: Fixed point theorems for quasi upper semicontinuous set-valued mappings in *p*-vector spaces. In: Debnath, P., Torres, D.F.M., Cho, Y.J. (eds.) Advanced Mathematical Analysis and Applications. CRC Press, Boca Raton (2023)
27. Chang, S.S., Cho, Y.J., Zhang, Y.: The topological versions of KKM theorem and Fan's matching theorem with applications. Topol. Methods Nonlinear Anal. **1**(2), 231–245 (1993)
28. Chang, T.H., Huang, Y.Y., Jeng, J.C.: Fixed point theorems for multi-functions in S-KKM class. Nonlinear Anal. **44**, 1007–1017 (2001)
29. Chang, T.H., Huang, Y.Y., Jeng, J.C., Kuo, K.H.: On S-KKM property and related topics. J. Math. Anal. Appl. **229**, 212–227 (1999)
30. Chang, T.H., Yen, C.L.: KKM property and fixed point theorems. J. Math. Anal. Appl. **203**, 224–235 (1996)
31. Chen, Y.K., Singh, K.L.: Fixed points for nonexpansive multivalued mapping and the Opial's condition. Jñānābha **22**, 107–110 (1992)
32. Chen, Y.Q.: Fixed points for convex continuous mappings in topological vector spaces. Proc. Am. Math. Soc. **129**(7), 2157–2162 (2001)
33. Darbo, G.: Punti uniti in trasformazioni a condominio non compatto. Rend. Semin. Mat. Univ. Padova **24**, 84–92 (1955)
34. Ding, G.G.: New Theory in Functional Analysis. Academic Press, Beijing (2007)
35. Dobrowolski, T.: Revisiting Cauty's proof of the Schauder conjecture. Abstr. Appl. Anal. **7**, 407–433 (2003)
36. Dugundji, J.: Topology. Allyn and Bacon, Needham Heights (1978)
37. Ennassik, M., Maniar, L., Taoudi, M.A.: Fixed point theorems in r-normed and locally *r*-convex spaces and applications. Fixed Point Theory **22**(2), 625–644 (2021)
38. Ennassik, M., Taoudi, M.A.: On the conjecture of Schauder. J. Fixed Point Theory Appl. **23**(4), 52 (2021)
39. Ewert, J., Neubrunn, T.: On quasi-continuous multivalued maps. Demonstr. Math. **21**(3), 697–711 (1988)
40. Fan, K.: Fixed-point and minimax theorems in locally convex topological linear spaces. Proc. Natl. Acad. Sci. USA **38**, 121–126 (1952)
41. Fan, K.: A generalization of Tychonoff's fixed point theorem. Math. Ann. **142**, 305–310 (1960/61)
42. Fan, K.: Extensions of two fixed point theorems of F. E. Browder. Math. Z. **112**, 234–240 (1969)
43. Fan, K.: A minimax inequality and applications. In: Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969; dedicated to the Memory of Theodore S. Motzkin), pp. 103–113. Academic Press, New York (1972)
44. Furi, M., Pera, M.P.: A continuation method on locally convex spaces and applications to ordinary differential equations on noncompact intervals. Ann. Pol. Math. **47**(3), 331–346 (1987)
45. Gal, S.G., Goldstein, J.A.: Semigroups of linear operators on p-Fréchet spaces $0 < p < 1$. Acta Math. Hung. **114**(1–2), 13–36 (2007)
46. Gholizadeh, L., Karapinar, E., Roohi, M.: Some fixed point theorems in locally *p*-convex spaces. Fixed Point Theory Appl. **2013**, 312 (2013)
47. Goebel, K.: On a fixed point theorem for multivalued nonexpansive mappings. Ann. Univ. Mariae Curie-Skłodowska, Sect. A **29**, 69–72 (1975)
48. Goebel, K., Kirk, W.A.: Topics in Metric Fixed Point Theory. Cambridge Studies in Advanced Mathematics, vol. 28. Cambridge University Press, Cambridge (1990)
49. Goebel, K., Kirk, W.A.: Some problems in metric fixed point theory. J. Fixed Point Theory Appl. **4**(1), 13–25 (2008)
50. Göhde, D.: Zum Prinzip der kontraktiven Abbildung. Math. Nachr. **30**, 251–258 (1965)
51. Górniewicz, L.: Topological Fixed Point Theory of Multivalued Mappings. Mathematics and Its Applications, vol. 495. Kluwer Academic, Dordrecht (1999)
52. Górniewicz, L., Granas, A., Kryszewski, W.: On the homotopy method in the fixed point index theory of multivalued mappings of compact absolute neighborhood retracts. J. Math. Anal. Appl. **161**(2), 457–473 (1991)
53. Granas, A., Dugundji, J.: Fixed Point Theory. Springer Monographs in Mathematics. Springer, New York (2003)

54. Guo, T.X.: Survey of recent developments of random metric theory and its applications in China. I. Acta Anal. Funct. Appl. **3**(2), 129–158 (2001)
55. Guo, T.X., Zhang, R.X., Wang, Y.C., Guo, Z.C.: Two fixed point theorems in complete random normed modules and their applications to backward stochastic equations. J. Math. Anal. Appl. **483**(2), 123644 (2020)
56. Halpern, B.R., Bergman, G.H.: A fixed-point theorem for inward and outward maps. Trans. Am. Math. Soc. **130**, 353–358 (1965)
57. Huang, N.J., Lee, B.S., Kang, M.K.: Fixed point theorems for compatible mappings with applications to the solutions of functional equations arising in dynamic programmings. Int. J. Math. Math. Sci. **20**(4), 673–680 (1997)
58. Husain, T., Latif, A.: Fixed points of multivalued nonexpansive maps. Math. Jpn. **33**, 385–391 (1988)
59. Husain, T., Tarafdar, E.: Fixed point theorems for multivalued mappings of nonexpansive type. Yokohama Math. J. **28**(1–2), 1–6 (1980)
60. Isac, G.: Leray-Schauder Type Alternatives, Complementarity Problems and Variational Inequalities. Nonconvex Optimization and Its Applications, vol. 87. Springer, New York (2006)
61. Jarchow, H.: Locally Convex Spaces. Teubner, Stuttgart (1981)
62. Kalton, N.J.: Compact p-convex sets. Q. J. Math. Oxford Ser. (2) **28**(2), 301–308 (1977)
63. Kalton, N.J.: Universal spaces and universal bases in metric linear spaces. Stud. Math. **61**, 161–191 (1977)
64. Kalton, N.J., Peck, N.T., Roberts, J.W.: An F-Space Sampler. London Mathematical Society Lecture Note Series, vol. 89. Cambridge University Press, Cambridge (1984)
65. Kaniok, L.: On measures of noncompactness in general topological vector spaces. Comment. Math. Univ. Carol. **31**(3), 479–487 (1990)
66. Kelly, J.L.: General Topology. Van Nostrand, Princeton (1957)
67. Kim, I.S., Kim, K., Park, S.: Leray-Schauder alternatives for approximable maps in topological vector spaces. Math. Comput. Model. **35**, 385–391 (2002)
68. Kirk, W., Shahzad, N.: Fixed Point Theory in Distance Spaces. Springer, Cham (2014)
69. Klee, V.: Convexity of Chevyshev sets. Math. Ann. **142**, 292–304 (1960/61)
70. Knaster, H., Kuratowski, C., Mazurkiwiecz, S.: Ein beweis des fixpunktsatzes für n-dimensional simplexe. Fundam. Math. **63**, 132–137 (1929)
71. Ko, H.M., Tsai, Y.H.: Fixed point theorems for point to set mappings in locally convex spaces and a characterization of complete metric spaces. Bull. Acad. Sin. **7**(4), 461–470 (1979)
72. Kozlov, V., Thim, J., Turesson, B.: A fixed point theorem in locally convex spaces. Collect. Math. **61**(2), 223–239 (2010)
73. Kryszewsky, W.: Graph-approximation of set-valued maps on noncompact domains. Topol. Appl. **83**(1), 1–21 (1998)
74. Kuratowski, K.: Sur les espaces complets. Fundam. Math. **15**, 301–309 (1930)
75. Lami Dozo, E.: Multivalued nonexpansive mappings and Opial's condition. Proc. Am. Math. Soc. **38**, 286–292 (1973)
76. Leray, J., Schauder, J.: Topologie et equations fonctionnelles. Ann. Sci. Éc. Norm. Supér. **51**, 45–78 (1934)
77. Li, G.Z.: The fixed point index and the fixed point theorems of 1-set-contraction mappings. Proc. Am. Math. Soc. **104**, 1163–1170 (1988)
78. Li, G.Z., Xu, S.Y., Duan, H.G.: Fixed point theorems of 1-set-contractive operators in Banach spaces. Appl. Math. Lett. **19**(5), 403–412 (2006)
79. Li, J.L.: An extension of Tychonoff's fixed point theorem to pseudonorm adjoint topological vector spaces. Optimization **70**(5–6), 1217–1229 (2021)
80. Lim, T.C.: A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach space. Bull. Am. Math. Soc. **80**, 1123–1126 (1974)
81. Liu, L.S.: Approximation theorems and fixed point theorems for various classes of 1-set-contractive mappings in Banach spaces. Acta Math. Sin. Engl. Ser. **17**(1), 103–112 (2001)
82. Machrafi, N., Oubbi, L.: Real-valued noncompactness measures in topological vector spaces and applications. [Corrected title: real-valued noncompactness measures in topological vector spaces and applications]. Banach J. Math. Anal. **14**(4), 1305–1325 (2020)
83. Mańka, R.: The topological fixed point property–an elementary continuum-theoretic approach. Fixed point theory and its applications. Banach Cent. Publ. **77**, 183–200 (2007)
84. Mauldin, R.D.: The Scottish Book, Mathematics from the Scottish Café with Selected Problems from the New Scottish Book, 2nd edn. Birkhäuser, Basel (2015)
85. Muglia, L., Marino, G.: Some results on the approximation of solutions of variational inequalities for multivalued maps on Banach spaces. Mediterr. J. Math. **18**(4), 157 (2021)
86. Neubrunn, T.: Quasi-continuity. Real Anal. Exch. **14**(2), 259–306 (1988/89). https://doi.org/10.2307/44151947
87. Nhu, N.T.: The fixed point property for weakly admissible compact convex sets: searching for a solution to Schauder's conjecture. Topol. Appl. **68**(1), 1–12 (1996)
88. Nussbaum, R.D.: The fixed point index and asymptotic fixed point theorems for k-set-contractions. Bull. Am. Math. Soc. **75**, 490–495 (1969)
89. Okon, T.: The Kakutani fixed point theorem for Robert spaces. Topol. Appl. **123**(3), 461–470 (2002)
90. Opial, Z.: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. **73**, 595–597 (1967)
91. O'Regan, D.: Abstract Leray-Schauder type alternatives and extensions. An. Ştiinţ. Univ. 'Ovidius' Constanţa, Ser. Mat. **27**(1), 233–243 (2019)
92. O'Regan, D.: Continuation theorems for Monch countable compactness-type set-valued maps. Appl. Anal. **100**(7), 1432–1439 (2021)
93. O'Regan, D., Precup, R.: Theorems of Leray-Schauder Type and Applications. Gordon & Breach, New York (2001)
94. Oubbi, L.: Algebras of Gelfand-continuous functions into Arens-Michael algebras. Commun. Korean Math. Soc. **34**(2), 585–602 (2019)
95. Park, S.: Some coincidence theorems on acyclic multifunctions and applications to KKM theory. In: Fixed Point Theory and Applications, Halifax, NS, 1991, pp. 248–277. World Scientific, River Edge (1992)
96. Park, S.: Generalized Leray-Schauder principles for compact admissible multifunctions. Topol. Methods Nonlinear Anal. **5**(2), 271–277 (1995)

97. Park, S.: Acyclic maps, minimax inequalities and fixed points. Nonlinear Anal. **24**(11), 1549–1554 (1995)
98. Park, S.: Generalized Leray-Schauder principles for condensing admissible multifunctions. Ann. Mat. Pura Appl. **172**(4), 65–85 (1997)
99. Park, S.: The KKM principle in abstract convex spaces: equivalent formulations and applications. Nonlinear Anal. **73**(4), 1028–1042 (2010)
100. Park, S.: On the KKM theory of locally *p*-convex spaces (nonlinear analysis and convex analysis). In: Institute of Mathematical Research, Kyoto University, 2011, pp. 70–77. Kyoto University, Japan (2016). http://hdl.handle.net/2433/231597
101. Park, S.: One hundred years of the Brouwer fixed point theorem. J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **60**(1), 1–77 (2021)
102. Park, S.: Some new equivalents of the Brouwer fixed point theorem. Adv. Theory Nonlinear Anal. Appl. **6**(3), 300–309 (2022). https://doi.org/10.31197/atnaa.1086232
103. Petrusel, A., Rus, I.A., Serban, M.A.: Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem for a multivalued operator. J. Nonlinear Convex Anal. **15**(3), 493–513 (2014)
104. Petryshyn, W.V.: Construction of fixed points of demicompact mappings in Hilbert space. J. Math. Anal. Appl. **14**, 276–284 (1966)
105. Petryshyn, W.V.: Fixed point theorems for various classes of 1-set-contractive and 1-ball-contractive mappings in Banach spaces. Trans. Am. Math. Soc. **182**, 323–352 (1973)
106. Pietramala, P.: Convergence of approximating fixed points sets for multivalued nonexpansive mappings. Comment. Math. Univ. Carol. **32**(4), 697–701 (1991)
107. Poincare, H.: Sur un theoreme de geometric. Rend. Circ. Mat. Palermo **33**, 357–407 (1912)
108. Potter, A.J.B.: An elementary version of the Leray-Schauder theorem. J. Lond. Math. Soc. **5**(2), 414–416 (1972)
109. Qiu, J., Rolewicz, S.: Ekeland's variational principle in locally p-convex spaces and related results. Stud. Math. **186**(3), 219–235 (2008)
110. Reich, S.: Fixed points in locally convex spaces. Math. Z. **125**, 17–31 (1972)
111. Repovš, D., Semenov, P.V., Ščepin, E.V.: Approximation of upper semicontinuous maps on paracompact spaces. Rocky Mt. J. Math. **28**(3), 1089–1101 (1998)
112. Roberts, J.W.: A compact convex set with no extreme points. Stud. Math. **60**(3), 255–266 (1977)
113. Robertson, L.B.: Topological vector spaces. Publ. Inst. Math. **12**(26), 19–21 (1971)
114. Rolewicz, S.: Metric Linear Spaces. Polish Sci., Warsaw (1985)
115. Rothe, E.H.: Some homotopy theorems concerning Leray-Schauder maps. In: Dynamical Systems, II, Gainesville, Fla., 1981, pp. 327–348. Academic Press, New York (1982)
116. Rothe, E.H.: Introduction to Various Aspects of Degree Theory in Banach Spaces. Mathematical Surveys and Monographs, vol. 23. Am. Math. Soc., Providence (1986)
117. Sadovskii, B.N.: On a fixed point principle [in Russian]. Funkc. Anal. Prilozh. **1**(2), 74–76 (1967)
118. Schauder, J.: Der Fixpunktsatz in Funktionalraumen. Stud. Math. **2**, 171–180 (1930)
119. Sezer, S., Eken, Z., Tinaztepe, G., Adilov, G.: *p*-convex functions and some of their properties. Numer. Funct. Anal. Optim. **42**(4), 443–459 (2021)
120. Shahzad, N.: Fixed point and approximation results for multimaps in $S-KKM$ class. Nonlinear Anal. **56**(6), 905–918 (2004)
121. Shahzad, N.: Approximation and Leray-Schauder type results for $\mathfrak{U}_c^k$ maps. Topol. Methods Nonlinear Anal. **24**(2), 337–346 (2004)
122. Shahzad, N.: Approximation and Leray-Schauder type results for multimaps in the S-KKM class. Bull. Belg. Math. Soc. **13**(1), 113–121 (2006)
123. Silva, E.B., Fernandez, D.L., Nikolova, L.: Generalized quasi-Banach sequence spaces and measures of noncompactness. An. Acad. Bras. Ciênc. **85**(2), 443–456 (2013)
124. Simons, S.: Boundedness in linear topological spaces. Trans. Am. Math. Soc. **113**, 169–180 (1964)
125. Singh, S.P., Watson, B., Srivastava, F.: Fixed Point Theory and Best Approximation: The KKM-Map Principle. Mathematics and Its Applications, vol. 424. Kluwer Academic, Dordrecht (1997)
126. Smart, D.R.: Fixed Point Theorems. Cambridge University Press, Cambridge (1980)
127. Tabor, J.A., Tabor, J.O., Idak, M.: Stability of isometries in *p*-Banach spaces. Funct. Approx. **38**, 109–119 (2008)
128. Tan, D.N.: On extension of isometries on the unit spheres of L^p-spaces for $0 < p \leq 1$. Nonlinear Anal. **74**, 6981–6987 (2011)
129. Tan, K.K., Yuan, X.Z.: Random fixed-point theorems and approximation in cones. J. Math. Anal. Appl. **185**, 378–390 (1994)
130. Tychonoff, A.: Ein Fixpunktsatz. Math. Ann. **111**, 767–776 (1935)
131. Wang, J.Y.: An Introduction to Locally *p*-Convex Spaces pp. 26–64. Academic Press, Beijing (2013)
132. Weber, H.: Compact convex sets in non-locally convex linear spaces. Dedicated to the memory of Professor Gottfried Köthe. Note Mat. **12**, 271–289 (1992)
133. Weber, H.: Compact convex sets in non-locally convex linear spaces, Schauder-Tychonoff fixed point theorem. In: Topology, Measures, and Fractals, Warnemunde, 1991. Math. Res., vol. 66, pp. 37–40. Akademie Verlag, Berlin (1992)
134. Xiao, J.Z., Lu, Y.: Some fixed point theorems for *s*-convex subsets in *p*-normed spaces based on measures of noncompactness. J. Fixed Point Theory Appl. **20**(2), 83 (2018)
135. Xiao, J.Z., Zhu, X.H.: Some fixed point theorems for *s*-convex subsets in *p*-normed spaces. Nonlinear Anal. **74**(5), 1738–1748 (2011)
136. Xu, H.K.: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**(12), 1127–1138 (1991)
137. Xu, H.K.: Metric fixed point theory for multivalued mappings. Diss. Math. **389**, 39 (2000)
138. Xu, H.K., Muglia, L.: On solving variational inequalities defined on fixed point sets of multivalued mappings in Banach spaces. J. Fixed Point Theory Appl. **22**(4), 79 (2020)
139. Xu, S.Y.: New fixed point theorems for 1-set-contractive operators in Banach spaces. Nonlinear Anal. **67**(3), 938–944 (2007)
140. Xu, S.Y., Jia, B.G., Li, G.Z.: Fixed points for weakly inward mappings in Banach spaces. J. Math. Anal. Appl. **319**(2), 863–873 (2006)

141. Yanagi, K.: On some fixed point theorems for multivalued mappings. Pac. J. Math. **87**(1), 233–240 (1980)
142. Yuan, G.X.Z.: The study of minimax inequalities and applications to economies and variational inequalities. Mem. Am. Math. Soc. **132**, 625 (1998)
143. Yuan, G.X.Z.: KKM Theory and Applications in Nonlinear Analysis. Monographs and Textbooks in Pure and Applied Mathematics, vol. 218. Dekker, New York (1999)
144. Yuan, G.X.Z.: Nonlinear analysis by applying best approximation method in p-vector spaces. Fixed Point Theory Algorithms Sci. Eng. **2022**, 20 (2022). https://doi.org/10.1186/s13663-022-00730-x
145. Yuan, G.X.Z.: Nonlinear analysis in p-vector spaces for single-valued 1-set contractive mappings. Fixed Point Theory Algorithms Sci. Eng. **2022**, 26 (2022). https://doi.org/10.1186/s13663-022-00735-6
146. Zeidler, E.: Nonlinear Functional Analysis and Its Applications, Vol. I, Fixed-Point Theorems. Springer, New York (1986)

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