Fixed Point Theory and Algorithms for Sciences and Engineering

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Some applications of fixed point results for monotone multivalued and integral type contractive mappings

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Abstract

The motivation of the present paper is to introduce and establish some new fixed point results for monotone multivalued functions in partially ordered complete D^*-metric spaces, where the partial ordered set $(X, \preceq)$ is obtained via a pair of functions (Υ, Ω). Moreover, several existence and uniqueness coupled fixed point theorems of mappings satisfying contractive conditions have been investigated and verified in the setting of partially ordered complete D^*-metric spaces by using the concept of integral type contractions with respect to partially ordered D^*-metric space. Furthermore, we present appropriate examples as an application for our main results. Our results generalize the work of Ghasab, Majani, and Rad on the study of integral type contraction and coupled fixed point theorems in the ordered G-metric spaces.

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1 Introduction and preliminaries

The fixed point theorems in partially ordered metric spaces play a major role in verifying the existence and uniqueness of solutions for some differential and integral equations. The theory of multivalued mappings is a branch of mathematics that has received great attention in the last decades and has various applications in convex optimization, optimal control theory, and differential inclusions. Let (X, d) be a complete metric space. A mapping $T : X \to X$ is a contraction mapping if there exists a constant $q \in (0, 1)$ such that $d(T(x), T(y)) \le qd(x, y)$ for all $x, y \in X$. Then the Banach fixed point theorem states that T always has a unique fixed point in X. After witnessing the application of Banach fixed point theorem in giving the existence and uniqueness solutions for many integral and differential equations, various generalizations or extensions of Banach fixed point theorem were carried out. By considering subsequences in the sequence of iterates, Edelstein [1] weakened the condition in the Banach fixed point theorem, and later people knew this as Edelstein's fixed point theorem. Meanwhile, Boyd and Wong [2] introduced a continuous function $\Upsilon : [0,\infty) \to [0,\infty)$ to replace $qd(x,y)$ with $\psi(d(x,y))$ and hence generalized Banach fixed point theorem. In 1976, Caristi [3] proved some general fixed point theorems using the characterization of weakly inward mappings. An analogue version of Banach fixed point theorem in the setting of partially ordered sets was proved in [4] and is also known as the Ran–Reurings fixed point theorem.

However, there is also a failed attempt on generalizing Banach fixed point theorem by Dhage et al. [5] who introduced the D-metric space topology, see [6–8] for details. As a correction to the topology introduced by Dhage et al., in 2007, Sedghi et al. [9] established the meaning of D^*-metric spaces, which is a probable modification of D-metric spaces proved by the author in [5]. Afterwards, many authors [10, 11] proved several fixed point theorems in these spaces.

Some coupled fixed point results for a mixed monotone mapping in ordered metric spaces have been established (see [12, 13]); for more details on coupled fixed point and n tuples fixed point theorems, we refer the reader to paper [14] and the references therein. Fixed point problems have further considered the concept of partially ordered complete generalized D^*-metric spaces. Al. Jumaili in [15] used the meaning of D^*-metric spaces and presented some coincidence fixed point theorems for functions satisfying contractive conditions concerning nondecreasing φ-mappings in partially ordered complete generalized D^*-metric spaces.

Recently, Ghasab et al. [16] applied the idea of integral kind contractions and proved some coupled fixed point theorems for such contractions in ordered G-metric spaces. Our objective in this article is to introduce and investigate several new fixed point theorems for monotone multivalued functions in partially ordered complete D^*-metric spaces, wherever the partial ordered set $(X,\leq)$ is obtained via a pair of functions (Υ,Ω). Furthermore, we study and establish some existence and uniqueness coupled fixed point theorems for mappings satisfying contractive conditions in the setting of partially ordered complete D^*-metric spaces by using the concept of integral type contractions. In addition, we provide suitable examples as an application for our main results. In our research, in the beginning, we explain several definitions and fundamental conclusion under the concept of D^*-metric spaces, because we think these explanations give readers the opportunity to understand more easily in subsequent parts.

Definition 1.1 ([9, Definition 1.1]) Let X be a nonempty set. Let $D^* : X \times X \times X \to [0,\infty)$ be a function that satisfies the following conditions for all $x,y,z,b \in X$:

(i) $D^*(x,y,z) \geq 0$;
(ii) $D^*(x,y,z) = 0$ if and only if $x = y = z$;
(iii) $D^*(x,y,z) = D^*(P\{x,y,z\})$, where P is a permutation function;
(iv) $D^*(x,y,z) \leq D^*(x,y,b) + D^*(b,z,z)$.

Then the function D^* is called a D^*-metric on X, and the pair (X,D^*) is called a D^*-metric space.

It was remarked in [9, Remark 1.2] that the equality $D^*(x,x,y) = D^*(x,y,y)$ holds true for all $x,y \in X$. Also, some examples of D^* metric have been presented in the same reference. We need the following definition to study the D^*-metric spaces.

Definition 1.2 ([9, Definition 1.4]) Suppose that (X,D^*) is a D^*-metric space. Then we say that:

(i) A sequence $\{x_s\}$ in X converges to a point $x \in X$ if and only if $D^*(x_s, x_s, x) = D^*(x, x, x_s) \to 0$ as $s \to \infty$. That is, for each $\varepsilon > 0$, there exists a positive integer s_o such that for all $s \geq s_o \Rightarrow D^*(x, x, x_s) < \varepsilon$. This is equivalent to: for each $\varepsilon > 0$, there exists a positive integer s_o such that $D^*(x, x_s, x_r) < \varepsilon$ for all $s, r \geq s_o$.

(ii) A sequence $\{x_s\}$ in X is a Cauchy sequence if for given $\varepsilon > 0$ there exists a positive integer s_o such that, for each $s, r \geq s_o, D^*(x_s, x_s, x_r) < \varepsilon$.

(iii) A space (X, D^*) is a complete D^*-metric if every Cauchy sequence in (X, D^*) is convergent in (X, D^*).

Lemma 1.1 ([17, Lemma 1.9]) *Let (X, D^*) be a D^*-metric space. If a sequence $\{x_s\}$ is convergent to $x \in X$, then it is a Cauchy sequence.*

Lemma 1.2 ([9, Lemma 1.7]) *Let (X, D^*) be a D^*-metric space. Then D^* is a continuous function on X^3, that is,*

$$\lim_{s\to\infty} D^*(x_s, y_s, z_s) = D^*(x, y, z),$$

whenever a sequence $\{(x_s, y_s, z_s)\}$ in X^3 converges to a point $(x, y, z) \in X^3$.

Definition 1.3 ([3]) Let (X, d) be a metric space and $\Omega : X \to [0, \infty)$ be a functional. Define the relation $\preceq$ on X by $x \preceq y$ if and only if $d(x, y) \leq \Omega(x) - \Omega(y)$. Then $\preceq$ is a partial order relation on X induced by Ω and $(X, \preceq)$ is called an ordered metric space introduced by Ω.

Definition 1.4 ([3]) The following two classes of mappings Ψ and Φ are defined as $\Psi = (\Upsilon | \Upsilon : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing with $\Upsilon(\tau) = 0$ iff $\tau = 0\}$ and $\Phi = \{\Omega | \Omega : [0, \infty) \to [0, \infty)$ is lower semicontinuous, $\Omega(\tau) > 0$ for all $\tau > 0$ and $\Omega(0) = 0\}$.

Definition 1.5 ([18, Definition 1.2]) An element $(a, b) \in X^2$ is said to be a coupled fixed point of the mapping $F : X^2 \to X$ if $F(a, b) = a$ and $F(b, a) = b$.

Definition 1.6 ([18, Definition 1.1]) Suppose that $(X, \preceq)$ is an ordered partial metric space. If relation $\sqsubseteq$ is defined on X^2 by $(a, b) \sqsubseteq (u, v)$ iff $a \preceq u$ and $b \preceq y$, then $(X^2, \sqsubseteq)$ is an ordered partial metric space.

Definition 1.7 ([18, Definition 1.1]) Suppose that $(X, \preceq)$ is a partially ordered set. The mapping $F : X^2 \to X$ is said to have the mixed monotone property if F is monotone nondecreasing in its first argument and is monotone nonincreasing in its second argument; i.e., for all $x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$ for all $y \in x$, and for all $y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_2) \preceq F(x, y_1)$ for all $x \in X$.

Definition 1.8 ([19, definition 2.1]) A function $f : X \to \mathbb{R}$ is called lower semi-continuous if for any $\{x_n\} \subset X$ and $x \in X$

$$x_n \to x \quad \Longrightarrow \quad f(x) \leq \lim_{n\to\infty} f(x_n).$$

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2 Multivalued functions and D^*-metric spaces

This section is devoted to introducing and studying several new fixed point theorems for monotone multivalued functions in partially ordered complete D^*-metric spaces by using the setting of D^*-metric spaces. Throughout this article, we assume that the function $\Upsilon : [0,\infty) \longrightarrow [0,\infty)$ has the following properties:

(i) Υ is nondecreasing and continuous;
(ii) $\Upsilon^{-1}(\{0\}) = \{0\}$;
(iii) $\Upsilon(a+b) \leq \Upsilon(a) + \Upsilon(b)$ for all $a, b \in [0,+\infty)$.

Definition 2.1 Let $(X, D^*, \preceq)$ be a D^*-metric space, and let $\Upsilon : [0,\infty) \longrightarrow [0,\infty)$ be a real-valued mapping, and $\Omega : X \to [0,\infty)$ be a functional. We define a relation $\preceq$ as follows: $x \preceq y$ if and only if

$$\Upsilon\bigl(D^*(x,x,y)\bigr) \leq \Omega(x) - \Omega(y)$$

for all $x, y \in X$.

Proposition 2.1 *Let (X, D^*) be a D^*-metric space, then $\preceq$ is a partial order on X and $(X, \preceq)$ is a partially ordered set.*

Proof First, we prove that $\preceq$ is reflexive. Since $\Upsilon(D^*(x,x,x)) = \Omega(x) - \Omega(x)$ for all $x \in X$, it follows that $\preceq$ is reflexive. Secondly, we prove $\preceq$ is antisymmetric. If $x, y \in X$ such that $x \preceq y$ and $y \preceq x$, then $\Upsilon(D^*(x,x,y)) \leq \Omega(x) - \Omega(y)$ and $\Upsilon(D^*(y,y,x)) \leq \Omega(y) - \Omega(x)$. Hence $\Upsilon(D^*(x,x,y)) + \Upsilon(D^*(y,y,x)) = 0$. Thus, $\Upsilon(D^*(x,x,y)) = \Upsilon(D^*(y,y,x)) = 0$. So $\Upsilon(D^*(x,x,y)) = 0$, and so $x = y$, which shows that $\preceq$ is antisymmetric. In addition, $\preceq$ is transitive because if $x, y, z \in X$ such that $x \preceq y$ and $y \preceq z$, hence

$$\Upsilon\bigl(D^*(x,x,y)\bigr) \leq \Omega(x) - \Omega(y)$$

and

$$\Upsilon\bigl(D^*(y,y,z)\bigr) \leq \Omega(y) - \Omega(z).$$

Hence, $\Upsilon(D^*(x,x,y)) + \Upsilon(D^*(y,y,z)) \leq \Omega(x) - \Omega(z)$. Utilizing part (iii) of the definition of D^*-metric space and property (iii) of the function Υ, we obtain

$$\begin{aligned}\Upsilon\bigl(D^*(x,x,z)\bigr) &\leq \Upsilon\bigl\{D^*(x,x,y) + D^*(y,z,z)\bigr\} \\ &\leq \Upsilon\bigl(D^*(x,x,y)\bigr) + \Upsilon\bigl(D^*(y,y,z)\bigr)(symmetric) \leq \Omega(x) - \Omega(z).\end{aligned}$$

Thus, we get $x \preceq z$. □

Remark 2.1 From the proof of Proposition 2.1, we see that $(X, \Upsilon \circ D^*)$ is again a D^*-metric space.

In the next work we suppose that $(X, D^*, \preceq)$ is an ordered D^*-metric space induced via (Υ, Ω).

Definition 2.2 Suppose that $(X, D^*, \preceq)$ is an ordered D^*-metric space induced via (Υ, Ω), the ordered intervals in X are defined as follows:

(i) $[x, y] = \{z \in X : x \preceq z \preceq y\}$;

(ii) $[x, \infty) = \{z \in X : x \preceq z\}$;

(iii) $(-\infty, x] = \{z \in X : z \preceq x\}$.

Definition 2.3 ([19, Definition 2.3]) Suppose that $F : X \to 2^X$ is a multivalued mapping. We say that F is upper semicontinuous if whenever $\{x_s\} \in X$ and $\{y_s\} \in F(x_s)$ with $x_s \to p \in X$ and $y_s \to \omega \in X$, then $\omega \in F(p)$.

Definition 2.4 ([19, Definition 2.4]) An element $x \in X$ is said to be a fixed point of a multivalued mapping $F : X \to 2^X$ if $x \in F(x)$.

We illustrate our first result in the following theorem.

Theorem 2.1 *Assume that* $(X, D^*, \preceq)$ *is a partially ordered complete* D^**-metric space induced via* (Υ, Ω), *where* $\Omega : X \to [0, \infty)$ *is a mapping which is bounded below. Let* $F : X \to 2^X$ *be a multivalued mapping and* $M = \{x \in X : F(x) \cap [x, \infty) \neq \emptyset\}$. *Assume that*

(i) F *is upper semicontinuous*;

(ii) *If* $x \in M$, *then* $y \in M$ *for all* $y \in F(x) \cap [x, \infty)$;

(iii) $F(p) \cap [p, \infty) \neq \emptyset$ *for some* $p \in X$.

Then there exists a sequence $\{x_s\}$ *such that* $x_{s-1} \preceq x_s \in F(x_{s-1})$ *for all* $s \in \mathbb{N}$, *and* F *has a fixed point* x_o *such that* $\mathrm{x}_\mathrm{s} \to x_o$. *In addition, if* Ω *is lower semicontinuous, then* $x_s \preceq x_0$ *for all* s.

Proof By(iii), there exists $p \in X$ such that $p \in M$. Thus choose $q \in F(p) \cap [p, +\infty)$, and we have $p \preceq q$. By(ii), we have $q \in M$. Choose $\tau \in F(q) \cap [q, +\infty)$, and we have $q \preceq \tau$. Continuing in this manner, we obtain a sequence $\{x_s\}$ in X such that $x_{s-1} \preceq x_s \in F(x_{s-1})$ for all $s \in \mathbb{N}$.

Now, since $(X, D^*, \preceq)$ is a partially ordered D^*-metric space induced via (Υ, Ω), we obtain that

$$\Upsilon\big(D^*(x_{s-1}, x_{s-1}, x_s)\big) \le \Omega(x_{s-1}) - \Omega(x_s).$$

Since Υ is a nonnegative mapping, we have that $\Omega(x_{s-1}) - \Omega(x_s) \ge 0$ for all $s \in \mathbb{N}$. Therefore, $\Omega(x_{s-1}) \ge \Omega(x_s)$ for all $s \in \mathbb{N}$. Since Ω is a mapping that is bounded below, we get $\Omega(x_s)$ is a decreasing sequence that is bounded below. So, via the completeness property of R, we obtain $\lim_{s\to+\infty} \Omega(x_s) = \inf\{x_s : s \in \mathbb{N}\}$. Thus,

$$\lim_{s,r\to+\infty} \Upsilon\big(D^*(x_s, x_s, x_r)\big) \le \lim_{s\to+\infty} \Omega(x_s) - \lim_{r\to+\infty} \Omega(x_r).$$

Therefore, $\lim_{s,r\to+\infty} \Upsilon(D^*(x_s, x_s, x_r)) = 0$.

Now, utilizing the continuity of the mapping Υ and the property that $\Upsilon^{-1}(\{0\}) = \{0\}$, we obtain that $\lim_{s,r\to+\infty} D^*(x_s, x_s, x_r) = 0$. Therefore, $\{x_s\}$ is a Cauchy sequence in x. Since X is complete, there exists $x_0 \in X$ such that $\{x_s\}$ is D^*-convergent to x_o. Since $x_{s-1} \in X$, $x_s \in F(x_{s-1}), x_{s-1} \to x_o$, and $x_s \to x_o$, via the definition of upper semicontinuity of F, we have $\mathrm{x}_o \in F(\mathrm{x}_o)$.

Now, assume that Ω is lower semicontinuous, then for all $s \in \mathbb{N}$, we have

$$\begin{aligned}\Upsilon\big(D^*(x_s,x_s,x_o)\big) &= \lim_{r\to+\infty}\Upsilon\big(D^*(x_s,x_s,x_r)\big) \le \liminf_{r\to+\infty}\Omega(x_s)-\Omega(x_r)\\ &= \Omega(x_s)-\liminf_{r\to+\infty}\Omega(x_r) \le \Omega(x_s)-\Omega(x_o).\end{aligned}$$

Thus, $x_s \le x_o$ for all $s \in \mathbb{N}$. □

The next result shows that assumption (ii) can be replaced with a new assumption on F.

Corollary 2.1 *Suppose that $(X, D^*, \le)$ is a partially ordered complete D^*-metric space induced via (Υ, Ω), where $\Omega : X \to [0,\infty)$ is a mapping that is bounded below, and let $F : X \to 2^X$ be a multivalued mapping where*

(i) *F is upper semicontinuous;*
(ii) *F satisfies the condition of monotonic sequence: for all $x, y \in X$ with $x \le y$ and every $\alpha \in F(x)$, there exists $\beta \in F(y)$ such that $\alpha \le \beta$;*
(iii) *There exists $p \in X$ such that $F(p) \cap [p,+\infty) \ne \emptyset$.*

Then there exists a sequence $\{x_s\}$ in X with $x_{s-1} \le x_s \in F(x_{s-1})$ for all $s \in \mathbb{N}$, and F has a fixed point x_o such that $x_s \to x_0$. Furthermore, if Ω is lower semicontinuous, then $x_s \le x_0$ for all s.

Proof By(ii), we have $p \in M$. Let $y \in F(p) \cap [p,+\infty)$. Then the—condition of F implies that there exists $z \in F(y)$ such that $y \le z$. In other words, $z \in F(y) \cap [y,+\infty) \ne \emptyset$. Hence $y \in M$ and the proof is complete by applying Theorem 2.1. □

Corollary 2.2 *Suppose that $(X, D^*, \le)$ is a partially ordered complete D^*-metric space induced via (Υ, Ω) such that $\Omega : X \to [0,\infty)$ is a mapping which is bounded below, and let $f : X \to X$ be a mapping. Assume the following:*

(i) *f is continuous;*
(ii) *f satisfies the condition of monotonic increasing sequence: for any $\alpha \in f(x)$, there exists $\beta \in f(y)$ such that $\alpha \le \beta$;*
(iii) *There exists $p \in X$ such that $p \le f(p)$.*

Then there exists a sequence $\{x_s\}$ in X with $x_{s-1} \le x_s \in f(x_{s-1})$ for all $s \in \mathbb{N}$, and f has a fixed point x_o such that $x_s \to x_o$. As well, if Ω is lower semicontinuous, then $x_s \le x_o$ for all s.

Proof Define a multivalued mapping, $F : X \to 2^X$ via $F(x) = \{f(x)\}$ for all $x \in X$, then F and X satisfy all the assumption of Theorem 2.1. So, we obtain the result from Theorem 2.1. □

The following results are an analogous version of the previous results by replacing the condition of bounded below with the condition of bounded above. The proofs are similar and hence omitted.

Theorem 2.2 *Let $(X, D^*, \le)$ be a partially ordered complete D^*-metric space induced via (Υ, Ω), where $\Omega : X \to R$ is a mapping that is bounded above. Suppose that $F : X \to 2^X$ is a multivalued mapping and $M = \{x \in X : F(x) \cap (-\infty, x] \ne \emptyset\}$. Assume that*

(i) *F is upper semicontinuous;*
(ii) *For all $x \in M$, $F(x) \cap M \cap (-\infty, x] \ne \emptyset$.*

Then there exists a sequence $\{x_s\}$ *such that* $x_{s-1} \geq x_s \in F(x_{s-1})$ *for all* $s \in \mathbb{N}$, *and* F *has a fixed point* x_o *such that* $x_s \to x_o$. *Also, if* Ω *is lower semicontinuous, then* $x_s \geq x_o$ *for all* s.

Corollary 2.3 *Suppose that* $(X, D^*, \leq)$ *is a partially ordered complete* D^**-metric space induced via* (Υ, Ω), *where* $\Omega : X \to R$ *is a mapping that is bounded above, and let* $F : X \to 2^X$ *be a multivalued mapping, where*

(i) F *is upper semicontinuous*;
(ii) F *satisfies the condition of monotonic sequence*: *for all* $x, y \in X$ *with* $x \geq y$ *and every* $\alpha \in F(x)$, *there exists* $\beta \in F(y)$ *such that* $\alpha \geq \beta$;
(iii) *There exists* $p \in X$ *such that* $F(p) \cap (-\infty, p] \neq \emptyset$.

Then there exists a sequence $\{x_s\}$ *in* X *with* $x_{s-1} \geq x_s \in F(x_{s-1})$ *for all* $s \in \mathbb{N}$, *and* F *has a fixed point* x_o *such that* $x_s \to x_o$. *Furthermore, if* Ω *is lower semicontinuous, then* $x_s \geq x_o$ *for all* s.

Corollary 2.4 *Suppose that* $(X, D^*, \leq)$ *is a partially ordered complete* D^**-metric space induced via* (Υ, Ω) *such that* $\Omega : X \to [0, +\infty)$ *is a mapping that is bounded below, and let* f: $X \to X$ *be a mapping where*

(i) f *is continuous*;
(ii) f *satisfies the condition of monotonic increasing sequence*. $\alpha \in f(x)$, *there exists* $\beta \in f(y)$ *such that* $\alpha \leq \beta$;
(iii) *There exists* $p \in X$ *such that* $p \geq f(p)$.

Then there exists a sequence $\{x_s\}$ *in* X *with* $x_{s-1} \geq x_s \in f(x_{s-1})$ *for all* $s \in \mathbb{N}$, *and* f *has a fixed point* $x_\circ$ *such that* $x_s \to x_o$. *As well, if* Ω *is lower semicontinuous, then* $x_s \geq x_o$ *for all* s.

3 Coupled fixed point theorems in D^*-metric spaces

This section is devoted to introducing and studying several results of coupled fixed point satisfying contractive conditions in the setting of partially ordered complete D^*-metric spaces by using the idea of integral type contractions. For convenience, we explain the following properties of mappings $\Phi, \Psi : [0, \infty) \to [0, \infty)$:

(i) Ψ is nondecreasing on $[0, \infty)$;
(ii) $\Psi(t) \leq t$ for all $t > 0$;
(iii) Ψ is an additive mapping;
(iv) $\sum_{s=1}^{\infty} s\Psi^s(t) < \infty$ for all $t > 0$.

Also,

(i) Φ is nonincreasing on $[0, \infty)$;
(ii) Φ is Lebesgue integrable;
(iii) For every $\epsilon > 0$, $\int_0^{\epsilon} \Phi(t) \mathrm{d}t > 0$;
(iv) Φ is a continuous mapping.

Now, we introduce the main theorem in our manuscript.

Theorem 3.1 *Suppose that* $(X, D^*, \leq)$ *is a partially ordered complete* D^**-metric space, and let* $F : X^2 \to X$ *be a continuous mapping with the mixed monotone property on* X *such that*

$$\int_0^{D^*(F(x,y),F(p,q),F(d,z))} \Phi(t)\, dt \leq \Psi\left(\int_0^{D^*(x,p,d)+D^*(y,q,z)} \Phi(t)\, dt\right), \tag{3.1}$$

where $x, y, z, p, q, d \in X$ *and* $\Phi : [0, \infty) \to [0, \infty)$ *is a Lebesgue-integrable mapping that is summable* (*i.e., with finite integral*) *with* $d \leq p \leq x$ *and* $y \leq q \leq z$, *where either* $p \neq d$ *or*

$q \neq z$. If there exist $y_0, x_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then F has a coupled fixed point in X.

Proof By assumption, there exist $x_o, y_o \in X$ such that $x_o \leq F(x_o, y_o)$ and $F(x_o, y_o) \leq y_o$. Define $x_1, y_1 \in X$ as follows: $x_1 = x_\circ \leq F(x_o, y_o)$ and $y_1 = F(y_o, x_o) \leq y_o$. Suppose that $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$. Therefore we get

$$\begin{aligned} x_2 &= F(x_1, y_1) = F\big(F(x_o, y_o), F(y_o, x_o)\big) = F^2(x_o, y_o), \\ y_2 &= F(y_1, x_1) = F\big(F(y_o, x_o), F(x_o, y_o)\big) = F^2(y_o, x_o). \end{aligned}$$

By utilizing the property of mixed monotonicity for the mapping F, we obtain:

$$\begin{aligned} x_2 &= F^2(x_o, y_o) = F(x_1, y_1) \geq F(x_o, y_o) = x_1 \geq x_o, \\ y_2 &= F^2(y_o, x_o) = F(y_1, x_1) \leq F(y_o, x_o) = y_1 \leq y_o. \end{aligned}$$

If we continue the above proceedings for each $s \geq 0$, we obtain the following:

$$x_o \leq x_1 \leq x_2 \leq \cdots \leq x_{s+1} \leq \cdots, \qquad y_o \geq y_1 \geq y_2 \geq \cdots \geq y_{s+1} \geq \cdots$$

such that

$$\begin{aligned} x_{s+1} &= F^{s+1}(x_o, y_o) = F\big(F^s(x_o, y_o), F^s(y_o, x_o)\big), \\ y_{s+1} &= F^{s+1}(y_o, x_o) = F\big(F^s(y_o, x_o), F^s(x_o, y_o)\big). \end{aligned}$$

Notice that if $(x_{s+1}, y_{s+1}) = (x_s, y_s)$, then F has a coupled fixed point. Now, suppose that $(x_{s+1}, y_{s+1}) \neq (x_s, y_s)$ for all $s \geq 0$, that is, let either $(x_{s+1} = F(x_s, y_s) \neq x_s$ or $y_{s+1} = F(y_s, x_s) \neq y_s$. By (3.1), we obtain

$$\begin{aligned} \int_0^{D^*(x_s, x_s, x_{s+1})} \Phi(t)\,dt &= \int_0^{D^*(F(x_{s-1}, y_{s-1}), F(x_s, y_s), F(x_s, y_s))} \Phi(t)\,dt \\ &\leq \Psi\left(\int_0^{D^*(x_{s-1}, x_{s-1}, x_s) + D^*(y_{s-1}, y_{s-1}, y_s)} \Phi(t)\,dt\right). \end{aligned} \tag{3.2}$$

In the same way we get

$$\begin{aligned} \int_0^{D^*(y_s, y_s, y_{s+1})} \Phi(t)\,dt &= \int_0^{D^*(F(y_{s-1}, x_{s-1}), F(y_s, x_s), F(y_s, x_s))} \Phi(t)\,dt \\ &\leq \Psi\left(\int_0^{D^*(x_{s-1}, x_{s-1}, x_s) + D^*(y_{s-1}, y_{s-1}, y_s)} \Phi(t)\,dt\right). \end{aligned} \tag{3.3}$$

Since the mapping Φ is nonincreasing, for every $a, b \geq 0$, we get

$$\int_0^{a+b} \Phi(t)\,dt \leq \int_0^a \Phi(t)\,dt + \int_0^b \Phi(t)\,dt. \tag{3.4}$$

Also, since Ψ is linear and nondecreasing, it follows from (3.1), (3.2), and (3.4) that for all $s \geq 0$

$$\begin{aligned}
\int_0^{D^*(x_s,x_s,x_{s+1})} \Phi(t)\,dt &= \int_0^{D^*(F(x_{s-1},y_{s-1}),F(x_s,y_s),F(x_s,y_s))} \Phi(t)\,dt \\
&\leq \Psi\left(\int_0^{D^*(x_{s-1},x_{s-1},x_s)+D^*(y_{s-1},y_{s-1},y_s)} \Phi(t)\,dt\right) \\
&\leq \Psi\left(\int_0^{D^*(x_{s-1},x_{s-1},x_s)} \Phi(t)\,dt\right) + \Psi\left(\int_0^{D^*(y_{s-1},y_{s-1},y_s)} \Phi(t)\,dt\right) \\
&\leq \Psi^2\left(\int_0^{D^*(x_{s-2},x_{s-2},x_{s-1})} \Phi(t)\,dt\right) + \Psi^2\left(\int_0^{D^*(y_{s-2},y_{s-2},y_{s-1})} \Phi(t)\,dt\right) \\
&\leq 2\Psi^2\left(\int_0^{D^*(x_{s-2},x_{s-2},x_{s-1})+D^*(y_{s-2},y_{s-2},y_{s-1})} \Phi(t)\,dt\right) \\
&\vdots \\
&\leq s\Psi^s\left(\int_0^{D^*(x_o,x_o,x_1)+D^*(y_o,y_o,y_1)} \Phi(t)\,dt\right). \qquad (3.5)
\end{aligned}$$

In a similar manner, we get

$$\begin{aligned}
\int_0^{D^*(y_s,y_s,y_{s+1})} \Phi(t)\,dt &= \int_0^{D^*(F(y_{s-1},x_{s-1}),F(y_s,x_s),F(y_s,x_s))} \Phi(t)\,dt \\
&\leq \Psi\left(\int_0^{D^*(y_{s-1},y_{s-1},y_s)+D^*(x_{s-1},x_{s-1},x_s)} \Phi(t)\,dt\right) \\
&\leq \Psi\left(\int_0^{D^*(y_{s-1},y_{s-1},y_s)} \Phi(t)\,dt\right) + \psi\left(\int_0^{D^*(x_{s-1},x_{s-1},x_s)} \Phi(t)\,dt\right) \\
&\leq 2\Psi^2\left(\int_0^{D^*(y_{s-2},y_{s-2},y_{s-1})+D^*(x_{s_2},x_{s_2},x_{s-1})} \Phi(t)\,dt\right) \\
&\vdots \\
&\leq s\Psi^s\left(\int_0^{D^*(y_o,y_o,y_1)+D^*(x_o,x_o,x_1)} \Phi(t)\,dt\right). \qquad (3.6)
\end{aligned}$$

Assume that $r, s \in \mathbb{N}$ such that $r > s$. It then follows from Definition 1.1 that

$$\begin{aligned}
\int_0^{D^*(x_s,x_s,x_r)} \Phi(t)\,dt \leq{}& \int_0^{D^*(x_s,x_s,x_{s+1})} \Phi(t)\,dt \\
&+ \int_0^{D^*(x_{s+1},x_{s+1},x_{s+2})} \Phi(t)\,dt + \cdots + \int_0^{D^*(x_{r-1},x_{r-1},x_r)} \Phi(t)\,dt.
\end{aligned}$$

Consequently, it follows from (3.5) that

$$\int_0^{D^*(x_s,x_s,x_r)} \Phi(t)\,dt \leq \sum_{i=s}^{r-1} i\Psi^i\left(\int_0^{D^*(x_o,x_o,x_1)+D^*(y_o,y_o,y_1)} \Phi(t)\,dt\right)$$

$$\leq \sum_{i=s}^{\infty} i\Psi^i\left(\int_0^{D^*(x_o,x_o,x_1)+D^*(y_o,y_o,y_1)} \Phi(t)\,dt\right).$$

Since $\sum_{i=s}^{\infty} i\Psi^i(t) < \infty$ for all $t \in [0,+\infty)$, it follows that $\lim_{s,r\to\infty} D^*(x_s,x_s,x_r) = 0$ and the sequence $\{x_s\}$ is a Cauchy sequence in X. Similarly, we can deduce that $\{y_s\}$ is a Cauchy sequence in X. Since X is a complete D^*-metric space, there exist $x,y \in X$ such that $\lim_{s\to\infty} x_s = x$ and $\lim_{s\to\infty} y_s = y$. Since F is continuous, it follows that $F(x,y) = x$ and $F(y,x) = y$; that is, (x,y) is a coupled fixed point of the mapping F. □

Theorem 3.2 *Assume that $(X,D^*,\preceq)$ is a partially ordered complete D^*-metric space satisfying the following conditions*:

(i) *If $\{x_s\}$ is a nondecreasing convergent sequence to $x \in X$, then $x_s \preceq x$ for all s*;

(ii) *If $\{y_s\}$ is a nonincreasing convergent sequence to $y \in X$, then $y_s \succeq y$ for all s, and let $F : X^2 \to X$ be a mapping having the mixed monotone property on X such that*

$$\int_0^{D^*(F(x,y),F(p,q),F(d,z))} \Phi(t)\,dt \leq \Psi\left(\int_0^{D^*(x,p,d)+D^*(y,q,z)} \Phi(t)\,dt\right),$$

where $x,y,z,p,q,d \in X$ and $\Phi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping that is summable (*i.e., with finite integral*) *with $d \preceq p \preceq x$ and $y \preceq q \preceq z$, where either $p \neq d$ or $q \neq z$; if there exist $x_o, y_o \in X$ such that $x_o \preceq F(x_o,y_o)$ and $F(y_o,x_o) \preceq y_o$, then F has a coupled fixed point in X.*

Proof By using a similar approach as that in proving Theorem 3.1, we obtain two Cauchy sequences $\{x_s\}$ and $\{y_s\} \in X$. It then follows from conditions (i) and (ii) that there exist x and $y \in X$ such that $x_s \preceq x$ and $y_s \preceq y$ for all $s \geq 0$. If $x_s = x$ and $y_s = y$ for some s, then, as shown in the proof of Theorem 3.1, we obtain $x_{s+1} = x$ and $y_{s+1} = y$; that is, (x,y) is a coupled fixed point. So, without loss of generality, we assume that either $x_s \neq x$ or $y_s \neq y$. Therefore, by using 3.1, we obtain

$$\begin{aligned}
&\int_0^{D^*(F(x,y),F(x,y),x)} \Phi(t)\,dt \\
&\quad\leq \int_0^{D^*(F(x,y),F(x,y),F(x_s,y_s))+D^*(F(x_s,y_s),F(x_s,y_s),x)} \Phi(t)\,dt \\
&\quad\leq \int_0^{D^*(F(x,y),F(x,y),F(x_s,y_s))} \Phi(t)\,dt + \int_0^{D^*(F(x_s,y_s),F(x_s,y_s),x)} \Phi(t)\,dt \\
&\quad\leq \Psi\left(\int_0^{D^*(x,x,x_s)+D^*(y,y,y_s)} \Phi(t)\,dt\right) + \int_0^{D^*(x_{s+1},x_{s+1},x)} \Phi(t)\,dt.
\end{aligned} \tag{3.7}$$

Hence, via (3.7) with as $s \to \infty$, we get $D^*(F(x,y),F(x,y),x) = 0$, which gives that $F(x,y) = x$. In the same way, we can illustrate that $D^*(F(y,x),F(y,x),y) = 0$ and thus $F(y,x) = y$. Therefore, the proof of the theorem is completed. □

We illustrate in the following theorem that the coupled fixed point of F can be unique.

Theorem 3.3 *Let $(X,D^*,\preceq)$ be a partially ordered complete D^*-metric space satisfying the following conditions*:

(i) *If* $\{x_s\}$ *ia a nondecreasing convergent sequence to* $x \in X$, *then* $x_s \leq x$ *for all* s;
(ii) *If* $\{y_s\}$ *is a nonincreasing convergent sequence to* $y \in X$, *then* $y_s \geq y$ *for all* s;
(iii) *For all* $(x,y),(x_1,y_1) \in X^2$, *there exists* $(z_1,z_2) \in X^2$ *such that is comparable with* (x,y) *and* (x_1,y_1).

And let $F : X^2 \to X$ *be a continuous mapping having the mixed monotone property on* X *such that*

$$\int_0^{D^*(F(x,y),F(p,q),F(d,z))} \Phi(t)\,dt \leq \Psi\left(\int_0^{D^*(x,p,d)+D^*(y,q,z)} \Phi(t)\,dt\right),$$

where $x,y,z,p,q,d \in X$ *and* $\Phi : [0,\infty) \to [0,\infty)$ *is a Lebesgue-integrable mapping that is summable (i.e., with finite integral) with* $d \leq p \leq x$ *and* $y \leq q \leq z$, *where either* $p \neq d$ *or* $q \neq z$, *if there exist* $x_o, y_o \in X$ *such that* $x_\circ \leq F(x_o,y_o)$ *and* $F(y_o,x_o) \leq y_o$, *then* F *has a unique coupled fixed point in* (X,D^*).

Proof Suppose that (x_1,y_1) is another fixed point of F. Now we discuss the following cases. Case one. Assume that (x,y) and (x_1,y_1) are comparable with respect to the partial ordering $\sqsubseteq$ in X^2 as introduced in Definition 1.6. We assume that $x \leq x_1$ and $y \leq y_1$ without restriction of generality. Now, by using the conditions of Theorem 3.1, we obtain

$$\int_0^{D^*(F^s(x,y),F^s(x_1,y_1),F^s(x_1,y_1))} \Phi(t)\,dt \leq \sum_0^\infty s\Psi^s\left(\int_0^{D^*(x,x_1,x_1)+D^*(y,y_1,y_1)} \Phi(t)\,dt\right). \tag{3.8}$$

Suppose that $s \to \infty$, therefore via (3.8) we get $x = x_1$. By using same method, we can verify that $y = y_1$. Case two. Let (x,y) be not comparable with (x_1,y_1). So, by condition (iii) there exists $(z_1,z_2) \in X^2$, which is comparable to (x,y) and (x_1,y_1). We can assume that $z_1 \leq x, z_2 \leq y, z_1 \leq x_1$, and $z_2 \leq y_1$ without restriction of generality. By using the conditions of Theorem 3.1, we get

$$\int_0^{D^*(F^s(x,y),F^s(z_1,z_2),F^s(z_1,z_2))} \Phi(t)\,dt \leq \sum_0^\infty s\Psi^s\left(\int_0^{D^*(x,z_1,z_1)+D^*(y,z_2,z_2)} \Phi(t)\,dt\right). \tag{3.9}$$

Assume that $s \to \infty$, then by (3.9) we obtain $D^*(F^s(x,y),F^s(z_1,z_2),F^s(z_1,z_2)) = 0$. Thus, $\lim_{s\to\infty} F^s(x,y) = \lim_{s\to\infty} F^s(z_1,z_2) = x$. By means of above, we get $\lim_{s\to\infty} F^s(x_1,y_1) = \lim_{s\to\infty} F^s(z_1,z_2) = x_1$. Thus, $x = x_1$. In the same way, we obtain $y = y_1$. Then, in all cases, we have $(x,y) = (x_1,y_1)$, that is, a mapping F has a unique coupled fixed point. □

Theorem 3.4 *Let* $(X,D^*,\leq)$ *be a partially ordered complete* D^**-metric space satisfying the following conditions*:
(i) *If* $\{x_s\}$ *is a nondecreasing convergent sequence to* $x \in X$, *then* $x_s \leq x$ *for all* s;
(ii) *If* $\{y_s\}$ *is a nonincreasing convergent sequence to* $y \in X$, *then* $y_s \geq y$ *for all* s;
(iii) *Every pair of element of* X *has an upper and a lower bound in* X.

Also, let $F : X^2 \to X$ *be a continuous mapping having the mixed monotone property on* X *such that*:

$$\int_0^{D^*(F(x,y),F(p,q),F(d,z))} \Phi(t)\,dt \leq \Psi\left(\int_0^{D^*(x,p,d)+D^*(y,q,z)} \Phi(t)\,dt\right),$$

where $x, y, z, p, q, d \in X$ *and* $\Phi : [0, \infty) \to [0, \infty)$ *is a Lebesgue-integrable mapping that is summable (i.e., with finite integral) with* $d \leq p \leq x$ *and* $y \leq q \leq z$*, where either* $p \neq d$ *or* $q \neq z$*. If there exist* $x_o, y_o \in X$ *such that* $x_\circ \leq F(x_o, y_o)$ *and* $F(y_o, x_o) \leq y_o$*, then* $x = y$*.*

Proof We first assume that x and y are comparable with respect to the partial ordering $\sqsubset$ in X. We can assume that $x \leq y$ and $y \leq y$ without restriction of generality. So, by the same method of Theorem 3.3, we obtain $x = y$. Now, suppose that x and y are not comparable. Then there exists an upper bound or a lower bound of x and y; that is, there exists $z \in X$ comparable with x and y. This means that we can assume that $x \leq z$ and $y \leq z$. By applying Theorem 3.3, we obtain $(x, y) = (z, z)$. Thus, we have $x = y$. □

Example 3.1 Let $X = [0, 1]$ and $D^* : X \times X \times X \to \mathbb{R}^+$ be a mapping defined by $D^*(a, b, c) = |a - b| + |a - c| + |b - c|$ for all $a, b, c \in X$. Therefore (X, D^*) is a complete D^*-metric space. Now, assume that $\Psi(t) = \frac{1}{2}t$ for all $t \in [0, \infty)$, and let $F : X \times X \to X$ be a mapping defined by $F(a, b) = \frac{a+b}{16}$. Since $|a + b - (m + n)| \leq |a - m| + |b - n|$ holds for all $a, b, m, n \in X$, it follows that the conditions of Theorem 3.1 hold. In fact, we have

$$\begin{aligned}
&\int_0^{D^*(F(a,b),F(m,n),F(c,\ell))} \Phi(t)\,dt \\
&\quad = \int_0^{|F(a,b)-F(m,n)|+|F(a,b)-F(c,\ell)|+|F(m,n)-F(c,\ell)|} \Phi(t)\,dt \\
&\quad = \int_0^{|\frac{a+b}{16}-\frac{m+n}{16}|+|\frac{a+b}{16}-\frac{c+\ell}{16}|+|\frac{m+n}{16}-\frac{c+\ell}{16}|} \Phi(t)\,dt \\
&\quad = \int_0^{\frac{1}{16}(|a-m|+|b-n|+|a-c|+|b-\ell|+|m-c|+|n-\ell|)} \Phi(t)\,dt \\
&\quad \leq \frac{1}{16}\int_0^{|a-m|+|b-n|+|a-c|+|b-\ell|+|m-c|+|n-\ell|} \Phi(t)\,dt \leq \Psi\left(\int_0^{D^*(a,m,c)+D^*(b,n,\ell)} \Phi(t)\,dt\right),
\end{aligned}$$

where $a, b, c, m, n, \ell \in X$. It is obvious that the mapping F satisfies all the conditions of Theorem 3.1. Hence, F has a coupled fixed point.

4 Conclusion

The fixed point results in partially ordered metric spaces play an essential role in constructing methods in mathematics to solve several problems in the pure and applied mathematical sciences. On the other hand, the study of partially ordered metric spaces plays the most important role in many fields both in pure and applied science such as biology, medicine, physics, and computer science (see [20, 21]). Therefore, we introduced and investigated some of new fixed point theorems for monotone multivalued functions in partially ordered complete generalized D^*-metric spaces. In addition, we presented and verified some of coupled fixed point theorems for mappings satisfying integral type conditions in partially ordered D^*-metric spaces. We hope that our results will be useful for the future studies on generalized metric spaces to carry out general framework for the practical applications and to solve the complicated problems containing uncertainties in environment, medical, engineering, economics, and in dynamical systems of various types.

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