Shukla *Fixed Point Theory Algorithms Sci Eng* (2023) 2023:12
https://doi.org/10.1186/s13663-023-00749-8

Fixed Point Theory and Algorithms
for Sciences and Engineering

RESEARCH Open Access

Some fixed-point theorems of convex orbital (α,β)-contraction mappings in geodesic spaces

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Abstract

The aim of this paper is to broaden the applicability of convex orbital (α,β)-contraction mappings to geodesic spaces. This class of mappings is a natural extension of iterated contraction mappings. The paper derives fixed-point theorems both with and without assuming continuity. Furthermore, the paper investigates monotone convex orbital (α,β)-contraction mappings and establishes a fixed-point theorem for this class of mappings.

Mathematics Subject Classification: Primary 47H10; secondary 54H25; 47H09

Keywords: Iterated contraction; Geodesic space; Partial order

1 Introduction

Picard operators that are most commonly utilized and play a crucial role in nonlinear analysis are a specific type of mapping referred to as Picard–Banach contractions. This class of mappings was first introduced by Banach in [2] and has since been widely recognized in the literature as a valuable tool in the study of nonlinear problems. The fundamental idea behind the Banach contraction principle (BCP) is that, within a complete metric space (X,Ω), any mapping $\Phi : X \to X$ that satisfies the condition of being a contraction, that is, there exists $\beta \in [0,1)$ such that $\Omega(\Phi(\xi),\Phi(\varrho)) \leq \beta\Omega(\xi,\varrho)$ $\forall\, \xi,\varrho \in X$, will have a unique fixed point.

Over the past 100 years, an extensive body of literature has emerged following the introduction of the Picard–Banach fixed-point theorem. This includes several monographs, as well as numerous references, such as [6, 7, 20]. The theorem, along with its various extensions, has proven to be a valuable and adaptable tool for solving a variety of nonlinear problems, including differential equations, integral equations, integrodifferential equations, optimization problems, and variational inequalities. This is evidenced by the vast amount of literature cited in [4, 9, 18, 19].

In [11, pp. 400] the following definition was considered.

Definition 1.1 The mapping $\Phi : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be an iterated contraction on the set $D_0 \subset D$ if there exists a $\beta < 1$ such that

$$\|\Phi(\Phi(\xi)) - \Phi(\xi)\| \le \beta \|\Phi(\xi) - \xi\|,$$

whenever ξ and $\Phi(\xi)$ are in D_0.

Although iterated contractions do not necessarily possess the properties of continuity or unique fixed points, they prove to be highly valuable in analyzing specific iterative procedures. Nonetheless, if an iterated contraction is continuous, the conventional proof of Banach's theorem remains applicable, enabling us to establish the presence of a fixed point.

Theorem 1.2 ([16] [11, Chap. 12]) *Let X be a complete metric space, $\Phi : X \to X$ be a continuous mapping. Suppose there exists $\beta \in [0,1)$ such that*

$$\Omega(\Phi^2(\xi), \Phi(\xi)) \le \beta \Omega(\xi, \Phi(\xi)) \quad \text{for all } \xi \in X.$$

Then, for each $\xi \in X$, the sequence $\{\Phi^n(\xi)\}$ converges to a fixed point of Φ.

Recently, Petrușel and Petrușel [13] considered the class of convex orbital β-Lipschitz mappings (see Definition 3.1) in the setting of Hilbert spaces. They showed that many important contraction mappings were properly contained in this class (see Remark 3.2). They obtained fixed-point results that are closely related to the admissible perturbations approach. Popescu [14] generalized the convex orbital β-Lipschitz mapping and considered the class of convex orbital (α,β)-Lipschitz mappings. He generalized and complemented the results in [13] for convex orbital (α,β)-Lipschitz mappings in Hilbert spaces.

In this paper, we extend the class of convex orbital (α,β)-contraction mappings in the setting of geodesic spaces. We present an example to illustrate that this class of mappings is a natural extension of the class of iterated contraction mappings. We obtain some fixed-point theorems with and without continuous assumptions. Further, we consider the class of monotone convex orbital (α,β)-contraction mappings and obtain a fixed-point theorem.

2 Preliminaries

Let ξ and ϱ be a pair of points in metric space (X,Ω). A path $\zeta : [0,1] \to X$ joins ξ and ϱ if

$$\zeta(0) = \xi \quad \text{and} \quad \zeta(1) = \varrho.$$

A path ζ is considered to be a geodesic if the following holds for all $s,t \in [0,1]$

$$\Omega(\zeta(s), \zeta(t)) = \Omega(\zeta(0), \zeta(1))|s-t|.$$

If every two points $\xi,\varrho \in X$ are connected by a geodesic, then the metric space (X,Ω) is called a geodesic space. If the geodesics in a geodesic space are unique, then the space is classified as a Busemann space, as per [3]. Some well-known spaces, such as all normed spaces, the CAT(0)-spaces, Hadamard manifolds, and the Hilbert open unit ball equipped with the hyperbolic metric, are special cases of these spaces (cf. [1, 8]). Kohlenbach [8] introduced a precise formulation of hyperbolic spaces, which is presented below.

Definition 2.1 If a function $W : X \times X \times [0,1] \to X$ exists such that (X,Ω) is a metric space and (X,Ω,W) satisfies the following conditions, then it is referred to as a hyperbolic metric space:

(i) $\Omega(z, W(\xi,\varrho,\theta)) \leq (1-\theta)\Omega(z,\xi) + \theta\Omega(z,\varrho)$;
(ii) $\Omega(W(\xi,\varrho,\theta), W(\xi,\varrho,\overline{\theta})) = |\theta - \overline{\theta}|\Omega(\xi,\varrho)$;
(iii) $W(\xi,\varrho,\theta) = W(\varrho,\xi,1-\theta)$;
(iv) $\Omega(W(\xi,z,\theta), W(\varrho,\zeta,\theta)) \leq (1-\theta)\Omega(\xi,\varrho) + \theta\Omega(z,\zeta)$,

for all $\xi,\varrho,z,\zeta \in X$ and $\theta,\overline{\theta} \in [0,1]$.

Remark 2.2 If $W(\xi,\varrho,\theta) = (1-\theta)\xi + \theta\varrho$ for all $\xi,\varrho \in X, \theta \in [0,1]$, then these spaces include all normed linear spaces.

For $\xi,\varrho \in X$,

$$[\xi,\varrho] = \{(1-\theta)\xi \oplus \theta\varrho : \theta \in [0,1]\}$$

denotes geodesic segments.

A map $x : [a,b] \to X$ is an affinely reparametrized geodesic if there exist an interval $[c,d]$ and a geodesic $x' : [c,d] \to X$ such that $x = x' o \psi$, where $\psi : [a,b] \to [c,d]$ is the unique affine homeomorphism between the intervals $[a,b]$ and $[c,d]$ or x is a constant path. A geodesic space (X,Ω) is a Busemann space if for any two affinely reparametrized geodesics $x : [a,b] \to X$ and $x' : [c,d] \to X$, the map $D_{x,x'} : [a,b] \times [c,d] \to \mathbb{R}$ is defined as

$$D_{x,x'}(s,t) = \Omega(x(s), x'(t))$$

and is convex, see [3].

There exists a unique convexity mapping W such that (X,Ω,W) is a uniquely geodesic W-hyperbolic space if (X,Ω) is a Busemann space. This means that for any $\xi \neq \varrho \in X$ and any $\theta \in [0,1]$, there is a unique element $\zeta \in X$ (which is $\zeta = W(\xi,\varrho,\theta)$) such that

$$\Omega(\xi,\zeta) = \theta\Omega(\xi,\varrho) \quad \text{and} \quad \Omega(\varrho,\zeta) = (1-\theta)\Omega(\xi,\varrho).$$

3 Convex orbital (α,β)-contraction mapping

Petrușel and Petrușel [13] considered the following class of mappings.

Definition 3.1 Let X be a normed space and Y a convex subset of X such that $Y \neq \emptyset$. Let $\Phi : Y \to Y$ be a mapping and $\alpha \in (0,1]$. The mapping Φ is a convex orbital β-Lipschitz mapping if $\beta > 0$ and

$$\|\Phi((1-\alpha)\xi + \alpha\Phi(\xi)) - \Phi(\xi)\| \leq \alpha\beta\|\xi - \Phi(\xi)\|$$

for all $\xi \in Y$.

Remark 3.2 It is shown in [13] that this class of mappings includes the following class of mappings:

(i) Banach contraction mappings;
(ii) Kannan contraction mappings;

(iii) Ćirić–Reich–Rus contraction mappings;
(iv) Berinde contraction mappings;
(v) nonexpansive mappings;
(vi) enriched (β,θ)-contraction mappings;
(vii) Lipschitz mappings.

Popescu [14] generalized the convex orbital β-Lipschitz mapping and considered the following class of mappings:

Definition 3.3 Let X and Y be the same as in Definition 3.1. Let $\Phi : Y \to Y$ be a mapping. The mapping Φ is said to be a convex orbital (α,β)-Lipschitz mapping if there exist $\alpha \in (0,1]$ and $\beta > 0$ such that

$$\left\|\Phi\big((1-\alpha)\xi + \alpha\Phi(\xi)\big) - \Phi(\xi)\right\| \le \alpha\beta\left\|\xi - \Phi(\xi)\right\|$$

for all $\xi \in Y$.

Thus, if $T : Y \to Y$ is a convex orbital β-Lipschitz mapping then T is a convex orbital (α,β)-Lipschitz mapping. Now, we extend Definition 3.3 in the setting of geodesic spaces as follows:

Definition 3.4 Let (X,Ω,W) be a W-hyperbolic space, and $\Phi : X \to X$ be a mapping. The mapping Φ is called a convex orbital (α,β)-contraction if there exist $\alpha,\beta \in (0,1)$ such that

$$\Omega\big(\Phi(\xi), \Phi\big(W\big(\xi,\Phi(\xi),\alpha\big)\big)\big) \le \alpha\beta\Omega\big(\xi,\Phi(\xi)\big)$$

for all $\xi \in X$.

If we consider $\alpha = 1$, then convex orbital (α,β)-contraction is an iterated contraction mapping. Thus, we take $\alpha \in (0,1)$. In the following example we show that a convex orbital (α,β)-contraction is a natural extension of an iterated contraction mapping.

Example 3.5 Let $\Upsilon = [0,3] \subset \mathbb{R}$ with the usual metric. Define $\Psi : \Upsilon \to \Upsilon$ by

$$\Psi(\xi) = \begin{cases} 0 & \text{if } \xi \neq 3, \xi \neq 2 \text{ and } \xi \neq \frac{12}{11}, \\ \frac{1}{10} & \text{if } \xi = \frac{12}{11}, \\ 1 & \text{if } \xi = 2, \\ \frac{19}{10} & \text{if } \xi = 3. \end{cases}$$

First, we show that Ψ is a convex orbital (α,β)-contraction mapping for $\alpha = \frac{10}{11}$ and $\beta = \frac{99}{100}$. We consider the following cases:

(1) If $\xi \neq \frac{12}{11}$, $\xi \neq 2$ and $\xi \neq 3$, then the condition is trivially satisfied.
(2) If $\xi = \frac{12}{11}$, then

$$\left|\Phi\big((1-\alpha)\xi + \alpha\Phi(\xi)\big) - \Phi(\xi)\right| = \left|\Phi\left(\left(1-\frac{10}{11}\right)\times\frac{12}{11} + \frac{10}{11}\Phi\left(\frac{12}{11}\right)\right) - \Phi\left(\frac{12}{11}\right)\right|$$

$$\begin{aligned}&= \left|\Phi\left(\frac{23}{121}\right) - \Phi\left(\frac{12}{11}\right)\right| \\ &= \frac{1}{10} < \frac{10791}{12100} = \frac{10}{11} \times \frac{99}{100}\left|\frac{12}{11} - \frac{1}{10}\right| \\ &= \alpha\beta\left|\xi - \Phi(\xi)\right|.\end{aligned}$$

(3) If $\xi = 2$, then

$$\begin{aligned}\left|\Psi\big((1-\alpha)\xi + \alpha\Psi(\xi)\big) - \Psi(\xi)\right| &= \left|\Psi\left(\left(1 - \frac{10}{11}\right) \times 2 + \frac{10}{11}\Psi(2)\right) - \Psi(2)\right| \\ &= \left|\Psi\left(\frac{12}{11}\right) - 1\right| = \frac{9}{10} = \frac{10}{11} \times \frac{99}{100}\left|2 - \Phi(2)\right| \\ &= \alpha\beta\left|\xi - \Psi(\xi)\right|.\end{aligned}$$

(3) If $\xi = 3$, then

$$\begin{aligned}\left|\Psi\big((1-\alpha)\xi + \alpha\Psi(\xi)\big) - \Psi(\xi)\right| &= \left|\Psi\left(\left(1 - \frac{10}{11}\right) \times 3 + \left(1 - \frac{10}{11}\right)\Psi(3)\right) - \Psi(3)\right| \\ &= \left|\Psi\left(\frac{22}{11}\right) - \frac{19}{10}\right| = \left|\Psi(2) - \frac{19}{10}\right| \\ &= \frac{9}{10} < \frac{99}{100} = \frac{10}{11} \times \frac{99}{100}\left|3 - \Phi(3)\right| \\ &= \alpha\beta\left|\xi - \Psi(\xi)\right|.\end{aligned}$$

On the other hand, Ψ is not an iterated contraction mapping. Indeed, at $\xi = 3$

$$\begin{aligned}\left|\Psi^2(\xi) - \Psi(\xi)\right| &= \left|\Psi^2(3) - \Psi(3)\right| \\ &= \left|0 - \frac{19}{10}\right| > \beta\frac{11}{10} = \beta\left|3 - \frac{19}{10}\right| \\ &= \beta\left|\xi - \Psi(\xi)\right|\end{aligned}$$

for any $\beta \in (0,1)$.

Motivated by the condition (E) (see [5]) and condition considered in [12], the following definition can be considered:

Definition 3.6 Let X be a metric space. A mapping $\Phi : X \to X$ is said to be a $(E\text{-}\mu, s)$-contraction mapping on X if there exist $\mu \geq 1$ and $s \in (0,1)$ such that for all $\xi, \varrho \in X$,

$$\Omega\big(\xi, \Phi(\varrho)\big) \leq \mu\Omega\big(\xi, \Phi(\xi)\big) + s\Omega(\xi, \varrho).$$

Theorem 3.7 *Let X be a complete Busemann space, and $\Phi : X \to X$ be a convex orbital (α, β)-contraction and a $(E\text{-}\mu, s)$-contraction. Then, Φ has a unique fixed point in X.*

Proof Let $\xi_0 \in X$ and define the following sequence

$$\xi_{n+1} = W\big(\xi_n, \Phi(\xi_n), \alpha\big) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.1}$$

From the condition on space, we have

$$\Omega(\xi_n,\xi_{n+1}) = \alpha\Omega\bigl(\xi_n,\Phi(\xi_n)\bigr). \tag{3.2}$$

From the definition of mapping Φ

$$\begin{aligned}\Omega\bigl(\Phi(\xi_n),\Phi(\xi_{n+1})\bigr) &= \Omega(\Phi(\xi_n),\Phi\bigl(W\bigl(\xi_n,\Phi(\xi_n),\alpha\bigr)\bigr) \le \alpha\beta\Omega\bigl(\xi_n,\Phi(\xi_n)\bigr)\\ &= \beta\Omega(\xi_n,\xi_{n+1}).\end{aligned} \tag{3.3}$$

Again, from Definition 2.1,

$$\begin{aligned}\Omega(\xi_{n+2},\xi_{n+1}) &= \Omega\bigl(W\bigl(\xi_{n+1},\Phi(\xi_{n+1}),\alpha\bigr),W\bigl(\xi_n,\Phi(\xi_n),\alpha\bigr)\bigr)\\ &\le (1-\alpha)\Omega(\xi_n,\xi_{n+1}) + \alpha\Omega\bigl(\Phi(\xi_{n+1}),\Phi(\xi_n)\bigr).\end{aligned}$$

From (3.3), we obtain

$$\begin{aligned}\Omega(\xi_{n+2},\xi_{n+1}) &\le (1-\alpha)\Omega(\xi_n,\xi_{n+1}) + \alpha\beta\Omega(\xi_n,\xi_{n+1})\\ &\le (1-\alpha+\alpha\beta)\Omega(\xi_n,\xi_{n+1}).\end{aligned}$$

Using the successive approximation method,

$$\Omega(\xi_{n+1},\xi_n) \le (1-\alpha+\alpha\beta)^n\Omega(\xi_1,\xi_0). \tag{3.4}$$

Take $c = (1-\alpha+\alpha\beta) < 1$. Let $m,n \in \mathbb{N}$ with $n < m$. From (3.4) and by the triangle inequality,

$$\begin{aligned}\Omega(\xi_m,\xi_n) &\le \Omega(\xi_m,\xi_{m-1}) + \Omega(\xi_{m-1},\xi_{m-2}) + \cdots + \Omega(\xi_{n+1},\xi_n)\\ &\le \bigl(c^{m-1} + c^{m-2} + \cdots + c^n\bigr)\Omega(\xi_1,\xi_0)\\ &\le c^n\bigl(c^{m-n-1} + c^{m-n-2} + \cdots + 1\bigr)\Omega(\xi_1,\xi_0)\\ &\le \frac{c^n}{1-c}\Omega(\xi_1,\xi_0).\end{aligned}$$

Since $\lim_{n\to\infty} c^n = 0$ and $\Omega(\xi_1,\xi_0)$ is fixed. It follows that $\{\xi_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $\xi^\dagger \in X$ such that $\xi_n \to \xi^\dagger$ as $n \to \infty$. We show that this limit point $\xi^\dagger$ is a fixed point of Φ. Since $\xi_n \to \xi^\dagger$, $\lim_{n\to\infty}\Omega(\xi_{n+1},\xi_n) = 0$. From (3.2), we have

$$\lim_{n\to\infty}\Omega\bigl(\xi_n,\Phi(\xi_n)\bigr) = 0. \tag{3.5}$$

From the condition on mapping Φ,

$$\Omega\bigl(\xi_n,\Phi\bigl(\xi^\dagger\bigr)\bigr) \le \mu\Omega\bigl(\xi_n,\Phi(\xi_n)\bigr) + s\Omega\bigl(\xi_n,\xi^\dagger\bigr)$$

and from (3.5) $\lim_{n\to\infty}\Omega(\xi_n,\Phi(\xi^\dagger)) = 0$. Therefore, $\xi^\dagger$ is a fixed point of Φ. To prove the uniqueness, let q be the other fixed point of Φ. Then,

$$0 < \Omega\bigl(\xi^\dagger,q\bigr) = \Omega\bigl(\xi^\dagger,\Phi(q)\bigr) \le \mu\Omega\bigl(\xi^\dagger,\Phi\bigl(\xi^\dagger\bigr)\bigr) + s\Omega\bigl(\xi^\dagger,q\bigr)$$

$$= s\Omega(\xi^\dagger, q) < \Omega(\xi^\dagger, q),$$

which is a contradiction unless $\xi^\dagger = q$. Hence, Φ admits a unique fixed point. □

Theorem 3.8 *Let X be a complete Busemann space, and $\Phi : X \to X$ be a convex orbital (α, β)-contraction mapping. If Φ is continuous, then Φ has a fixed point in X.*

Proof Following the same proof techniques as in Theorem 3.7, one can show that $\{\xi_n\}$ is a Cauchy sequence in X, there exists $\xi^\dagger \in X$ such that $\xi_n \to \xi^\dagger$ and

$$\lim_{n\to\infty} \Omega(\xi_n, \Phi(\xi_n)) = 0. \tag{3.6}$$

Since $\xi_n \to \xi^\dagger$ as $n \to \infty$, the continuity of Φ yields

$$\lim_{n\to\infty} \Omega(\Phi(\xi_n), \Phi(\xi^\dagger)) = 0. \tag{3.7}$$

Now,

$$\Omega(\xi_n, \Phi(\xi^\dagger)) \le \Omega(\xi_n, \Phi(\xi_n)) + \Omega(\Phi(\xi_n), \Phi(\xi^\dagger)).$$

From (3.6) and (3.7) $\xi_n \to \Phi(\xi^\dagger)$ as $n \to \infty$ and $\xi^\dagger$ is a fixed point of Φ. □

As demonstrated by the following example, the absence of continuity can result in a deficiency of fixed points, regardless of the domain's compactness. In fact, we drop an additional condition on mapping, that is, a $(E\text{-}\mu, s)$-contraction condition, then a convex orbital (α, β)-contraction mapping yields a lack of a fixed point.

Example 3.9 Let $\Upsilon = [0, 3] \subset \mathbb{R}$ with the usual metric. Define $\Psi : \Upsilon \to \Upsilon$ by

$$\Psi(\xi) = \begin{cases} \frac{3}{2} & \text{if } \xi = 0, \\ \frac{\xi}{2} & \text{if } \xi \in (0, 3]. \end{cases}$$

We shall show that Φ is a convex orbital $(\frac{3}{4}, \frac{5}{6})$-contraction mapping.

Case 1. If $\xi = 0$, then

$$\begin{aligned} \left|\Phi((1-\alpha)\xi + \alpha\Phi(\xi)) - \Phi(\xi)\right| &= \left|\Phi\left(\left(1 - \frac{3}{4}\right) \times 0 + \frac{3}{4}\Phi(0)\right) - \Phi(0)\right| \\ &= \left|\Phi\left(\frac{9}{8}\right) - \frac{3}{2}\right| = \left|\frac{9}{16} - \frac{3}{2}\right| = \frac{15}{16} \\ &= \frac{3}{4} \times \frac{5}{6}\left|0 - \Phi(0)\right| = \alpha\beta\left|\xi - \Phi(\xi)\right|. \end{aligned}$$

Case 2. If $\xi \in (0, 3]$ then

$$\begin{aligned} \left|\Phi((1-\alpha)\xi + \alpha\Phi(\xi)) - \Phi(\xi)\right| &= \left|\Phi\left(\frac{1}{4}\xi + \frac{3}{4}\Phi(\xi)\right) - \Phi(\xi)\right| \\ &= \left|\Phi\left(\frac{1}{4}\xi + \frac{3}{8}\xi\right) - \frac{\xi}{2}\right| \end{aligned}$$

$$= \left|\frac{5}{16}\xi - \frac{\xi}{2}\right| = \frac{3}{16}\xi \leq \frac{5}{16}\xi = \frac{3}{4} \times \frac{5}{6}\left|\xi - \frac{\xi}{2}\right|$$
$$= \alpha\beta\left|\xi - \Phi(\xi)\right|.$$

Now, we show that Φ is not a $(E\text{-}\mu, s)$-contraction mapping. Indeed, let $\xi_n = \frac{1}{n}$ and $\varrho_n = 0$ for all $n \in \mathbb{N}\backslash\{1\}$. Then,

$$\begin{aligned}\frac{|\xi_n - \Phi(\varrho_n)| - s|\xi_n - \varrho_n|}{|\xi_n - \Phi(\xi_n)|} &= \frac{|\frac{1}{n} - \frac{3}{2}| - s|\frac{1}{n} - 0|}{|\frac{1}{n} - \frac{1}{2n}|} \\ &= \frac{\frac{3}{2} - \frac{(1+s)}{n}}{\frac{1}{2n}} \\ &= 3n - 2(1+s) \to +\infty.\end{aligned}$$

Hence, Φ is not a $(E\text{-}\mu, s)$-contraction mapping. The mapping is not continuous and is fixed point free.

This example illustrates that there exists a mapping Ψ that does not satisfy the contraction condition, yet a specific iterate of the same mapping can satisfy the contraction condition.

Example 3.10 ([17, Example 1.3.1]) Suppose $X = \mathbb{R}$ and $\Phi : X \to X$ is defined by

$$\Psi(\xi) = \begin{cases} 0 & \text{if } \xi \in (-\infty, 2], \\ -\frac{1}{3} & \text{if } \xi \in (2, +\infty). \end{cases}$$

Although Φ is discontinuous and therefore not a contraction, Φ^2 can be considered a contraction.

If we encounter a situation where the classical Picard–Banach contraction mapping principle cannot be applied, we may find the following fixed-point theorem to be a useful alternative. This theorem is discussed in various sources.

Theorem 3.11 ([17, Theorem 1.3.2]) *Let X be a complete metric space and $\Phi : X \to X$ a mapping. If there exists a $N \in \mathbb{N}$ such that Φ^N is a contraction, then $F(\Phi) = \{\xi^*\}$, where $F(\Phi)$ is the fixed-point set of Φ.*

Theorem 3.12 *Let X be a complete Busemann space, $G : X \to X$ be a mapping and there exists a $N \in \mathbb{N}$ such that G^N is a convex orbital (α, β)-contraction and G^N is a $(E\text{-}\mu, s)$-contraction mapping. For given $\xi_0 \in X$, define a sequence*

$$\xi_{n+1} = W(\xi_n, G^N(\xi_n), \alpha), \quad n \geq 0. \tag{3.8}$$

Then, the sequence $\{\xi_n\}$ converges to a unique fixed point of G.

Proof Suppose that $\Phi = G^N$ then the sequence (3.8) becomes

$$\xi_{n+1} = W(\xi_n, \Phi(\xi_n), \alpha).$$

We use Theorem 3.7, then $\{\xi_n\}$ converges to a unique fixed point of G^N, say p, and $F(\Phi) = F(G^N) = \{p\}$. We have

$$G^N\bigl(G(p)\bigr) = G^{N+1}(p) = G\bigl(G^N(p)\bigr) = G(p),$$

hence, it follows that $G(p)$ is a fixed point of G^N. However, $F(G^N) = \{p\}$; thus, $G(p) = p$ and hence $p \in F(G)$. □

4 Monotone convex orbital (α, β)-contraction mapping

Ran and Reurings [15] extended BCP in partially ordered metric spaces and employed BCP to obtain a positive solution of matrix equations. The main fixed-point theorem of [15] was expanded by Nieto and Rodríguez-López [10], who utilized it to discover solutions for a selection of differential equations. In this section, we extend the convex orbital (α, β)-contraction in partially ordered Busemann spaces.

Let X be a partially ordered Busemann space. A subset K of X is said to be convex if $[\xi, \varrho] \subset K$ whenever $\xi, \varrho \in K$. In this section, we denote the order intervals in X by

$$[\xi, \rightarrow) := \{z \in X : \xi \preceq z\} \quad \text{and} \quad (\rightarrow, \varrho] := \{z \in X : z \preceq \varrho\},$$

we also assume the following hypothesis in the framework of partially ordered Busemann spaces:

$$(H): \quad \text{For any } \xi \in X, \text{ the order interval } [\xi, \rightarrow) \text{ is a closed and convex subset of } X.$$

Definition 4.1 Let X be the same as above and K a convex subset of X. A mapping $\Phi : K \to K$ is said to be monotone if

$$\xi \preceq \varrho \quad \text{implies } \Phi(\xi) \preceq \Phi(\varrho), \text{ for all } \xi, \varrho \in K.$$

Definition 4.2 Let X and K be the same as in Definition 4.1. The mapping $\Phi : K \to K$ is called a monotone convex orbital (α, β)-contraction if Φ is monotone and there exist $\alpha, \beta \in (0, 1)$ such that

$$\Omega\bigl(\Phi(\xi), \Phi\bigl(W\bigl(\xi, \Phi(\xi)\alpha\bigr)\bigr)\bigr) \le \alpha\beta\Omega\bigl(\xi, \Phi(\xi)\bigr)$$

for all $\xi \in K$ with $\xi \preceq \Phi(\xi)$.

Definition 4.3 Let X and K be the same as in Definition 4.1. A mapping $\Phi : K \to K$ is said to be a monotone $(E\text{-}\mu, s)$-contraction on K if Φ is monotone and there exist $\mu \ge 1$ and $s \in (0, 1)$ such that

$$\Omega\bigl(\xi, \Phi(\varrho)\bigr) \le \mu\Omega\bigl(\xi, \Phi(\xi)\bigr) + s\Omega(\xi, \varrho)$$

for all $\xi, \varrho \in K$ with $\xi \preceq \varrho$.

Theorem 4.4 *Let X be a complete Busemann space, K a convex closed subset of X, and $\Phi : K \to K$ be a monotone convex orbital (α, β)-contraction and a monotone $(E\text{-}\mu, s)$-contraction mapping. Suppose that $\xi_0 \in K$ such that $\xi_0 \preceq \Phi(\xi_0)$. Then, Φ has a fixed point in K.*

Proof Let $\xi_0 \in X$ and define the following sequence

$$\xi_{n+1} = W\bigl(\xi_n, \Phi(\xi_n), \alpha\bigr) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{4.1}$$

From the condition on space, we have

$$\Omega(\xi_n, \xi_{n+1}) = \alpha \Omega\bigl(\xi_n, \Phi(\xi_n)\bigr). \tag{4.2}$$

Now, we show that

$$\xi_n \preceq \xi_{n+1} \preceq \Phi(\xi_n) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{4.3}$$

We shall use induction to prove the above claim. Since $\xi_0 \preceq \Phi(\xi_0)$, in view of the convexity of order interval, we obtain

$$\xi_0 \preceq \xi_1 \preceq \Phi(\xi_0).$$

Since Φ is monotone, $\Phi(\xi_0) \preceq \Phi(\xi_1)$ and

$$\xi_1 \preceq \Phi(\xi_0) \preceq \Phi(\xi_1).$$

By the convexity of order interval

$$\xi_1 \preceq \xi_2 \preceq \Phi(\xi_1).$$

Thus, (4.3) is true for $n = 1$. Suppose it is true for a fixed $k \in \mathbb{N}$, that is, $\xi_k \preceq \xi_{k+1} \preceq \Phi(\xi_k)$. Again Φ is monotone, $\Phi(\xi_k) \preceq \Phi(\xi_{k+1})$ and by convexity

$$\xi_{k+1} \preceq \xi_{k+2} \preceq \Phi(\xi_{k+1}).$$

This proves the claim. From the definition of mapping Φ,

$$\begin{aligned} \Omega\bigl(\Phi(\xi_n), \Phi(\xi_{n+1})\bigr) = \Omega\bigl(\Phi(\xi_n), \Phi\bigl(W\bigl(\xi_n, \Phi(\xi_n), \alpha\bigr)\bigr)\bigr) &\leq \alpha\beta\Omega\bigl(\xi_n, \Phi(\xi_n)\bigr) \\ &= \beta\Omega(\xi_n, \xi_{n+1}). \end{aligned} \tag{4.4}$$

Again, by Definition 2.1,

$$\begin{aligned} \Omega(\xi_{n+2}, \xi_{n+1}) &= \Omega\bigl(W\bigl(\xi_{n+1}, \Phi(\xi_{n+1}), \alpha\bigr), W\bigl(\xi_n, \Phi(\xi_n), \alpha\bigr)\bigr) \\ &\leq (1-\alpha)\Omega(\xi_n, \xi_{n+1}) + \alpha\Omega\bigl(\Phi(\xi_{n+1}), \Phi(\xi_n)\bigr). \end{aligned}$$

From (4.4), we obtain

$$\begin{aligned}\Omega(\xi_{n+2},\xi_{n+1}) &\leq (1-\alpha)\Omega(\xi_n,\xi_{n+1}) + \alpha\beta\Omega(\xi_n,\xi_{n+1}) \\ &\leq (1-\alpha+\alpha\beta)\Omega(\xi_n,\xi_{n+1}).\end{aligned}$$

Using the successive approximation method,

$$\Omega(\xi_{n+1},\xi_n) \leq (1-\alpha+\alpha\beta)^n\Omega(\xi_1,\xi_0). \tag{4.5}$$

Take $c = (1-\alpha+\alpha\beta) < 1$. Following largely from the proof of Theorem 3.7, one can show that $\{\xi_n\}$ is a Cauchy sequence, since K is closed, the sequence $\xi_n \to \xi^\dagger \in K$ as $n \to \infty$ with $\xi_n \preceq \xi^\dagger$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \Omega(\xi_{n+1},\xi_n) = 0$. From (4.2), we have

$$\lim_{n\to\infty} \Omega\big(\xi_n,\Phi(\xi_n)\big) = 0. \tag{4.6}$$

From the condition on mapping Φ,

$$\Omega\big(\xi_n,\Phi\big(\xi^\dagger\big)\big) \leq \mu\Omega\big(\xi_n,\Phi(\xi_n)\big) + k\Omega\big(\xi_n,\xi^\dagger\big)$$

and from (4.6) $\lim_{n\to\infty} \Omega(\xi_n,\Phi(\xi^\dagger)) = 0$. Therefore, $\xi^\dagger$ is a fixed point of Φ. □

Theorem 4.5 *Let X, K, and Φ be the same as in Theorem 4.4. Then, the fixed point of Φ is unique, if*

$$\textit{for all } \xi,\varrho \in K,\ \textit{there exists } w \in K \textit{ such that } w \preceq \Phi(w), \xi \preceq w \textit{ and } \varrho \preceq w.$$

Proof Let u and v be two fixed points of Φ such that $u \neq v$. In view of the assumption, there exists $w \in K$ such that $u \preceq w$ and $v \preceq w$. Let $w_0 = w \in K$ and define the following sequence

$$w_{n+1} = W\big(w_n,\Phi(w_n),\alpha\big) \quad \text{for all } n \in \mathbb{N}\cup\{0\} \tag{4.7}$$

and

$$\Omega(w_n,w_{n+1}) = \alpha\Omega\big(w_n,\Phi(w_n)\big). \tag{4.8}$$

Following largely the proof of Theorem 4.4 one can show that

$$\lim_{n\to\infty} \Omega\big(w_n,\Phi(w_n)\big) = 0. \tag{4.9}$$

Since $v \preceq w = w_0$ and $w_0 \preceq \Phi(w_0)$, one can see that $v \preceq w_n$ and $u \preceq w_n$ for all $n \in \mathbb{N}$.

Case 1. If $v = w_{n_0}$ for some $n_0 \geq 0$, then $v = \Phi(v) = \Phi(w_{n_0})$ and

$$\begin{aligned}w_{n_0+1} &= (1-\alpha)w_{n_0} + \alpha\Phi(w_{n_0}) \\ &= (1-\alpha)v + \alpha v = v.\end{aligned}$$

Thus, $w_n = v$ for all $n \geq n_0$.

Case 2. If $v \preceq w_n$ and $v \neq w_n$ for all $n \geq 0$, then

$$\begin{aligned}\Omega\bigl(v,\Phi(w_n)\bigr) &\leq \mu\Omega\bigl(v,\Phi(v)\bigr) + s\Omega(v,w_n)\\ &= s\Omega\bigl(v,\Phi(w_n)\bigr) + s\Omega(w_n,\Phi(w_n)).\end{aligned}$$

Thus, from (4.9)

$$(1-s)\Omega\bigl(v,\Phi(w_n)\bigr) \leq s\Omega\bigl(w_n,\Phi(w_n)\bigr)$$

and $\Omega(v,\Phi(w_n)) \to 0$ as $n \to \infty$. In view of (4.9), $w_n \to v$. Similarly $w_n \to u$, by the uniqueness of the limit, it follows that $u = v$. Hence, Φ has a unique fixed point. □

Acknowledgements
The author is very thankful to the reviewers for their constructive comments and suggestions that have been useful for the improvement of this paper.

Funding
No funding was received.

Availability of data and materials
All data generated or analyzed during this study are included in this manuscript.

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Competing interests
The author declares no competing interests.

Author contributions
The authors contributed equally to this work.

Received: 23 March 2023 Accepted: 24 August 2023 Published online: 12 September 2023

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