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Fixed point theorems for enriched Kannan mappings in CAT(0) spaces

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Abstract

We present enriched Kannan and enriched Bianchini mappings in the framework of unique geodesic spaces. For such mappings, we establish the existence and uniqueness of a fixed point in the setting of CAT(0) spaces and show that an appropriate Krasnoselskij scheme converges with certain rate to the fixed point. We proved some inclusion relations between enriched Kannan mapping and some applicable mappings such as strongly demicontractive mapping. Finally, we give an example in a nonlinear CAT(0) space and perform numerical experiments to support the theoretical results.

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1 Introduction

Let $(\mathcal{H}, d)$ be a metric space, and let D be a nonempty subset of $\mathcal{H}$. Given a mapping $T : D \to \mathcal{H}$, a point $p \in D$ is called a *fixed point* of T if $Tp = p$. The set of all fixed points of T is denoted by $Fix(T)$, that is,

$$Fix(T) = \{u \in D : Tu = u\}.$$

If T sends D to itself, that is, $T : D \to D$, then T is called a *self-mapping*. For several years, real-world phenomena are transformed into problems that require finding fixed point(s) of certain mapping(s). These problems have been extensively studied (see, for example, the monographs [6, 15, 28, 38] and references therein). Recall that the mapping, T is called a *contraction* if there exists $k \in [0, 1)$ such that

$$d(Tu, Tw) \le kd(u, w) \quad \text{for all } u, w \in D. \tag{1.1}$$

An important fact is that any contraction self-mapping on a complete metric space $(\mathcal{H}, d)$ has a unique fixed point and the sequence $\{u_n\}$ defined by $u_{n+1} = Tu_n (u_1 \in \mathcal{H})$ converges to the fixed point. This substantial result is known as the *Banach contraction mapping* *principle*, which can be traced to [4], and the sequence $\{u_n\}$ is known as a *Picard iteration*. Following the Banach contraction mapping principle, scholars introduced various contraction-type mappings with certain fixed point properties among which a Kannan mapping is introduced in [24, 25]. A mapping $T : D \to \mathcal{H}$ is called a *Kannan mapping* if there exists $h \in [0, 1/2)$ such that

$$d(Tu, Tv) \le h\big(d(u, Tu) + d(v, Tv)\big) \quad \text{for all } u, v \in D. \tag{1.2}$$

The Kannan mapping in (1.2) and the contraction mapping in (1.1) may have a similar structure but are independent in the sense that there are contraction mappings (resp., Kannan mappings) that are not Kannan mappings (resp., contraction mappings). Moreover, it has been observed that the class of contraction mappings on a given space does not characterize the completeness of the space. In fact, an example of a noncomplete metric space in which every contraction self-mapping has a fixed point is given in [16]. However, it has been established in [36] that a metric space $(\mathcal{H}, d)$ is complete if and only if every Kannan mapping on $\mathcal{H}$ has a fixed point. Further details about contraction and Kannan mappings can be found in [1, 2, 26, 35, 37].

Berinde [7] introduced a superclass of contraction mappings in a normed linear space $(H, \|\cdot\|)$, which is referred to as the class of enriched contraction mappings. A mapping $T : H \to H$ is said to be an *enriched contraction* (or (α, β)-enriched contraction) if there exist $\alpha \in [0, +\infty)$ and $\beta \in [0, \alpha + 1)$ such that

$$\big\|\alpha(u - w) + Tu - Tw\big\| \le \beta\|u - w\| \quad \forall u, w \in H. \tag{1.3}$$

Thereafter, the same author, together with an associate, introduced in [8] the enriched Kannan mappings as a generalization of the Kannan mappings in linear spaces.

Definition 1.1 Let $(H, \|\cdot\|)$ be a normed linear space, and let D be a nonempty subset of H. A mapping $T : D \to H$ is said to be an *enriched Kannan mapping* if there exist $\eta \in [0, 1/2)$ and $\gamma \in [0, +\infty)$ such that

$$\big\|\gamma(u - v) + Tu - Tv\big\| \le \eta\big(\|u - Tu\| + \|v - Tv\|\big) \quad \forall u, v \in D. \tag{1.4}$$

To specify the nonnegative numbers involved, T is called a (γ, η)-enriched Kannan mapping. We now state the main result for enriched Kannan mapping in [8].

Theorem 1.2 *Let $(H, \|\cdot\|)$ be a complete normed linear space, and let $T : H \to H$ be a (γ, η)-enriched Kannan mapping. Then*

(i) $Fix(T) = \{p\}$;

(ii) *there exists $\delta \in (0, 1]$ such that the sequence $\{u_n\}$ defined iteratively by*

$$\begin{cases} u_1 \in H, \\ u_{n+1} = (1 - \delta)u_n + \delta Tu_n, \quad n \ge 0, \end{cases}$$

converges to p;

(iii) *for* $q \in [0,1)$, *we have the following estimate*:

$$\|u_{n+j-1} - p\| \leq \frac{q^j}{1-q}\|u_n - u_{n-1}\|, \quad n \geq 0, j \geq 1, \tag{1.5}$$

where $q = \frac{\gamma}{1-\gamma}$.

Theorem 1.2 is applicable in solving split feasibility problems, variational inequality problems, and many other problems (see [8] and the references therein). In the same paper the authors introduced and analyzed the class of enriched Bianchini mappings as a superclass of enriched Kannan mappings. It is important to note that the class of enriched Bianchini mappings is a superclass of Bianchini mappings introduced in [9] and further studied in [10, 22, 23].

Definition 1.3 Let $(H, \|\cdot\|)$ be a normed linear space, and let D be a nonempty subset of H. Then a mapping $T : D \to H$ is said to be an *enriched Bianchini mapping* if there exist $\alpha^* \in [0,1)$ and $\gamma^* \in [0,+\infty)$ such that

$$\left\|\gamma^*(u-v) + Tu - Tv\right\| \leq \alpha^* \max\left\{\|u - Tu\|, \|v - Tv\|\right\} \quad \forall u, v \in D. \tag{1.6}$$

The authors obtained similar results as in the Theorem 1.2 for the enriched Bianchini mappings.

Although the Banach contraction mapping principle and the Kannan and Bianchini mapping theorems are established for complete metric spaces, which need not have a linear structure, the enriched classes of Kannan and Bianchini mappings have been introduced in the setting of linear spaces. All these classes of mappings are immensely substantial in nonlinear convex analysis and optimizations as many optimization problems can be reduced to finding fixed point(s) of such mappings. Moreover, most of the real-world problems, which usually resulted in optimization problems, have some nonlinear structure with constraints that are not necessarily convex or smooth. Recently, many scholars argued that the setting of CAT(0) spaces allows the transformation from nonsmooth nonconvex constrained optimization problems into smooth and convex unconstrained problems. These, among other reasons, resulted in recent significant extensions of fixed point theory from linear spaces to CAT(0) spaces (see, for example, [3, 11, 12, 17, 19, 28, 30] and the references therein). To our knowledge, Kirk is one of the first scholars to study fixed point theory in the setting of CAT(0) spaces (see [27, 29]). The author analyzed the class of nonexpansive mappings. Thereafter, fixed points of enriched contraction and enriched nonexpansive mappings are analysed in CAT(0) spaces [34].

Having the results in [8] and considering the notion of enriched contraction mappings in [34], the purpose of this work is twofold. The first is to introduce and analyze two classes of mappings in the setting of unique geodesic spaces, namely, the classes of enriched Kannan mappings and Bianchini mappings. The second is to establish an inclusion relation between enriched Kannan mappings with strongly demicontractive mappings and quasi-expansive mappings. The results complement the results in [8, 24, 31] and extend several further results in the literature. This contributes to a unified understanding of contraction mappings and their generalizations, especially Kannan-type and Bianchini-type mappings, in metric spaces. The results are applicable to all $\mathbb{R}$-trees, Hadamard manifolds, and all CAT(κ) spaces with $\kappa \leq 0$. Moreover, this opens new avenues for applying metric convexity to expand several classes of contraction mappings, particularly when the metric is endowed with certain convex structures. Furthermore, this technique can be effectively applied to address other nonlinear problems, such as equilibrium problems, split problems, split variational inequality problems, and common null point problems.

To this end, in this paper, we discuss some basic concepts in CAT(0) spaces attributed to [11] and state some known results that will be used to analyze our results.

2 CAT(0) spaces

Let $(\mathcal{H}, d)$ be a metric space. A *geodesic path* from u to v is a map $\tau : [0,1] \to \mathcal{H}$ such that

$$\tau(0) = u, \qquad \tau(1) = v, \quad \text{and} \quad d\big(\tau(s), \tau(t)\big) = |s-t| d(u,v) \quad \forall s,t \in [0,1].$$

In the literature, the image of τ is often called a *geodesic segment* connecting u and v. If such a segment is unique, then we write $[u,v]$ to mean $\tau([0,1])$. The space in which every two points are connected by a geodesic segment (resp., unique geodesic segment) is called a *geodesic space* (resp., *unique geodesic space*). For $u, v \in \mathcal{H}$ having unique geodesic segment and for any $\delta \in [0,1]$, there exists a unique point $w \in [u,v]$ such that

$$d(u,w) = \delta d(u,v) \quad \text{and} \quad d(w,v) = (1-\delta) d(u,v). \tag{2.1}$$

We will henceforth denote such a point w by $(1-\delta)u \oplus \delta v$. Also, for $x, y \in \mathcal{H}$, $d^2(x,y)$ means $[d(x,y)]^2$.

A geodesic space is called a *CAT(0) space* if the CN-inequality of Bruhat and Tits [13] holds. This inequality states that for $u, v \in \mathcal{H}$,

$$d^2\left(\frac{1}{2}u \oplus \frac{1}{2}v, y\right) \leq \frac{1}{2}d^2(v,y) + \frac{1}{2}d^2(v,y) - \frac{1}{4}d^2(u,v) \tag{2.2}$$

for every $y \in \mathcal{H}$. A complete CAT(0) space is called a *Hadamard space*. For a precise definition and detailed discussion on CAT(0) spaces, see, for example, [11, 14]. It is known that Hadamard manifolds, Hilbert spaces, classical hyperbolic spaces, $\mathbb{R}$-trees, complex Hilbert balls, and Euclidean buildings are all examples of CAT(0) spaces [12, 20, 28, 33]. It is worth mentioning that CAT(0) spaces are unique geodesic spaces (see, for example, [11]).

Let u, v be in a CAT(0) space $(\mathcal{H}, d)$. For every $w \in \mathcal{H}$, we have the following important facts (see [18]):

$$d\big((1-t)u \oplus tv, w\big) \leq (1-t)d(u,w) + td(v,w) \tag{2.3}$$

and

$$d^2\big((1-t)u \oplus tv, w\big) \leq (1-t)d^2(u,w) + td^2(v,w) - t(1-t)d^2(u,v) \tag{2.4}$$

for all $t \in [0,1]$. Berg and Nikolaev [5] denoted $(u,w) \in \mathcal{H} \times \mathcal{H}$ by $\overrightarrow{uw}$ and defined a quasi-linearization map $\langle \cdot,\cdot \rangle : (\mathcal{H} \times \mathcal{H}) \times (\mathcal{H} \times \mathcal{H}) \to \mathbb{R}$ by

$$\langle \overrightarrow{uw}, \overrightarrow{vz} \rangle = \frac{1}{2}\bigl(d^2(u,z) + d^2(w,v) - d^2(u,v) - d^2(w,z)\bigr) \tag{2.5}$$

for all points $u,v,w,z \in \mathcal{H}$. Thereafter, this concept of quasi-linearization is used in [34] to analyze enriched contraction mappings.

Definition 2.1 For a nonempty subset D of a CAT(0) space $(\mathcal{H},d)$, a mapping $T : D \to \mathcal{H}$ is called an enriched contraction if there exist $\alpha \in [0,+\infty)$ and $\beta \in [0,\alpha+1)$ such that

$$d^2(Tu,Tw) + \alpha^2 d^2(u,w) + 2\alpha \langle \overrightarrow{uw}, \overrightarrow{TuTw} \rangle \leq \beta^2 d^2(u,w) \quad \forall u,w \in \mathcal{H}. \tag{2.6}$$

The following lemma is substantial in obtaining the main results of [34].

Lemma 2.2 ([34, Lemma 3.5]) *Let $(\mathcal{H},d)$ be a Hadamard space, and let $T : \mathcal{H} \to \mathcal{H}$ be a mapping. Then for any $\sigma \in (0,1]$,*

(i) $u = Tu \iff u = (1-\sigma)u \oplus \sigma Tu$;

(ii) $$d^2\bigl((1-\sigma)u \oplus \sigma Tu, (1-\sigma)w \oplus \sigma Tw\bigr) \leq \sigma^2 d^2(Tu,Tw)^2 + (1-\sigma)d^2(u,w) + 2\sigma(1-\sigma)\langle \overrightarrow{uw}, \overrightarrow{TuTw} \rangle.$$

3 Enriched Kannan mappings

Following the definition of an enriched Kannan mapping, we observe that (1.4) can be rewritten as follows:

$$\left\| \frac{\gamma}{\gamma+1}(u-v) + \frac{1}{\gamma+1}(Tu-Tv) \right\| \leq \frac{\eta}{\gamma+1}\bigl(\|u-Tu\| + \|v-Tv\|\bigr) \quad \forall u,v \in D,$$

which is equivalent to

$$\|T_\gamma u - T_\gamma v\| \leq \eta\bigl(\|u-T_\gamma u\| + \|v-T_\gamma v\|\bigr) \quad \forall u,v \in D,$$

where $T_\gamma x := \frac{\gamma}{\gamma+1}x + \frac{1}{\gamma+1}Tx$. Thus we can have the following definition in the setting of a unique geodesic space.

Definition 3.1 Let $(\mathcal{H},d)$ be a unique geodesic space, and let D be a nonempty set of $\mathcal{H}$. A mapping $T : D \to \mathcal{H}$ is said to be an *enriched Kannan mapping* if there exist $\eta \in [0,1/2)$ and $\gamma \in [0,+\infty)$ such that for all $u,v \in D$, we have

$$d(T_\gamma u, T_\gamma v) \leq \eta\bigl(d(u,T_\gamma u) + d(v,T_\gamma v)\bigr), \tag{3.1}$$

where $T_\gamma x := \frac{\gamma}{\gamma+1}x \oplus \frac{1}{\gamma+1}Tx$ for $x \in D$.

We further call T a (γ,η)-enriched Kannan mapping to indicate the nonnegative numbers involved.

Example 1 Every Kannan mapping $T: D \to \mathcal{H}$ is (γ,η)-enriched Kannan with $\gamma = 0$. Indeed, for $\gamma = 0$, $T_\gamma \equiv T$, and, consequently, (3.1) reduces to (1.4).

Example 2 Every (α,β)-enriched contraction mapping $T: D \to \mathcal{H}$ with $\beta < \frac{\alpha+1}{3}$ is (γ,η)-enriched Kannan with $\gamma = \alpha$ and $\eta = \frac{\beta}{\alpha+1-\beta}$. Indeed, it follows from Lemma 2.2(ii) that

$$\begin{aligned}
&\left(1 - \frac{\beta}{\alpha+1}\right) d(T_\alpha u, T_\alpha v) \\
&= \left[d^2(T_\alpha u, T_\alpha v)\right]^{1/2} - \left(\frac{\beta}{\alpha+1}\right) d(T_\alpha u, T_\alpha v) \\
&\leq \left[\frac{1}{(\alpha+1)^2} d(Tu, Tv)^2 + \frac{\alpha^2}{(\alpha+1)^2} d(u,v)^2 \right. \\
&\quad \left. + \frac{2\alpha}{(\alpha+1)^2} \langle \overrightarrow{uv}, \overrightarrow{TuTv} \rangle \right]^{1/2} - \left(\frac{\beta}{\alpha+1}\right) d(T_\alpha u, T_\alpha v) \\
&\leq \left[\frac{\beta^2}{(\alpha+1)^2} d(u,v)^2\right]^{1/2} - \left(\frac{\beta}{\alpha+1}\right) d(T_\alpha u, T_\alpha v) \\
&= \frac{\beta}{\alpha+1} d(u,v) - \left(\frac{\beta}{\alpha+1}\right) d(T_\alpha u, T_\alpha v) \\
&\leq \frac{\beta}{\alpha+1} \left[d(u, T_\alpha u) + d(v, T_\alpha v)\right].
\end{aligned}$$

Consequently, we have

$$d(T_\alpha u, T_\alpha v) \leq \frac{\beta}{\alpha+1-\beta} \left[d(u, T_\alpha u) + d(v, T_\alpha v)\right].$$

Since every contraction mapping with constant κ is a $(0,\kappa)$-enriched contraction, an immediate example follows.

Example 3 Every contraction mapping $T: D \to \mathcal{H}$ with constant $k < \frac{1}{3}$ is (γ,η)-enriched Kannan with $\gamma = 0$ and $\eta = \frac{k}{1-k}$.

We now state the main theorem of this section.

Theorem 3.2 *Let $(\mathcal{H}, d)$ be a Hadamard space, and let $T: \mathcal{H} \to \mathcal{H}$ be a (γ,η)-enriched Kannan mapping. Then there exists $p \in \mathcal{H}$ such that*

(i) *$Fix(T) = \{p\}$;*

(ii) *there exists $\delta \in (0,1]$ such that the sequence $\{u_n\}$ defined iteratively by*

$$\begin{cases} u_1 \in \mathcal{H}, \\ u_{n+1} = (1-\delta)u_n \oplus \delta T u_n, \quad n \geq 1, \end{cases} \tag{3.2}$$

converges to p;

(iii) *for* $q \in [0,1)$, *we have the following estimate*:

$$d(u_{n+j-1}, p) \leq \frac{q^j}{1-q} d(u_n, u_{n-1}), \quad n > 1, j \geq 1. \tag{3.3}$$

Proof Let $T_\gamma x := \frac{\gamma}{\gamma+1} x \oplus \frac{1}{\gamma+1} Tx$ and take $\delta = \frac{1}{\gamma+1} \in (0,1]$. Then $\{u_n\}$ defined in (3.2) corresponds to the Picard iteration of the mapping T_γ, that is,

$$u_1 \in \mathcal{H}, \qquad u_{n+1} = T_\gamma u_n, \quad n \geq 1.$$

Let $n > 1$, $u = u_n$, and $v = u_{n-1}$. Then (3.1) implies that

$$d(u_{n+1}, u_n) \leq \eta \bigl(d(u_n, u_{n+1}) + d(u_n, u_{n-1}) \bigr).$$

Consequently, we have

$$d(u_{n+1}, u_n) \leq \frac{\eta}{1-\eta} d(u_n, u_{n-1}).$$

Let $q = \frac{\eta}{1-\eta}$. Then $q < 1$, and the sequence $\{u_n\}$ satisfies

$$d(u_{n+1}, u_n) \leq q d(u_n, u_{n-1}) \quad \text{for all } n \geq 2. \tag{3.4}$$

Inductively, we obtain

$$d(u_{n+1}, u_n) \leq q^{n-1} d(u_2, u_1), \quad n \geq 1. \tag{3.5}$$

Thus, for all $m, n \geq 1$, we have

$$\begin{aligned} d(u_{n+m}, u_n) &\leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + \cdots + d(u_{n+m-1}, u_{n+m}) \\ &\leq q^{n-1} d(u_2, u_1) + q^n d(u_2, u_1) + \cdots + q^{n+m-2} d(u_2, u_1) \\ &= q^{n-1} \bigl[1 + q + \cdots + q^{m-1} \bigr] d(u_2, u_1) \\ &\leq q^{n-1} d(u_2, u_1) \sum_{j=1}^{+\infty} q^j \\ &\leq \frac{q^{n-1}}{1-q} d(u_2, u_1). \end{aligned}$$

Therefore $\{u_n\}$ is a Cauchy sequence. Since $\mathcal{H}$ is a complete CAT(0) space, there exists $p \in \mathcal{H}$ such that $\{u_n\}$ converges to p, that is,

$$\lim_{n \to \infty} u_n = p. \tag{3.6}$$

By Definition 3.1 we get

$$d(p, T_\gamma p) \leq d(p, u_{n+1}) + d(u_{n+1}, T_\gamma p)$$

$$= d(u_{n+1}, p) + d(T_\gamma u_n, T_\gamma p)$$
$$\leq d(u_{n+1}, p) + \eta\big(d(u_n, T_\gamma u_n) + d(p, T_\gamma p)\big).$$

Thus

$$d(p, T_\gamma p) \leq \frac{1}{1-\eta} d(u_{n+1}, p) + q d(u_{n+1}, u_n). \tag{3.7}$$

Consequently, letting $n \to \infty$, we get $p = T_\gamma p$.

Suppose that T_γ has a fixed point u' different from p. Then it follows from (3.1) that

$$0 < d(u', p) \leq \eta(0) = 0,$$

which is a contradiction. Hence p is the unique fixed point of T_γ. Consequently, Lemma 2.2(i) implies that

$$Fix(T) = Fix(T_\gamma) = \{p\}.$$

Therefore we have (i), and (ii) follows from (3.6).

To prove (iii), we estimate using (3.4) as follows:

$$\begin{aligned} d(u_{n+m}, u_n) &\leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + \cdots + d(u_{n+m-1}, u_{n+m}) \\ &\leq q d(u_n, u_{n-1}) + q^2 d(u_n, u_{n-1}) + \cdots + q^m d(u_n, u_{n-1}) \\ &= \left(\sum_{j=1}^{m} q^j\right) d(u_n, u_{n-1}) \\ &\leq q \frac{1-q^m}{1-q} d(u_n, u_{n-1}) \\ &\leq \frac{q}{1-q} d(u_n, u_{n-1}), \quad n > 1, m \geq 1. \end{aligned}$$

Hence we have

$$d(u_{n+m}, u_n) \leq \frac{q}{1-q} d(u_n, u_{n-1}).$$

Moreover, letting $m \to \infty$, we get

$$d(p, u_n) \leq \frac{q}{1-q} d(u_n, u_{n-1}). \tag{3.8}$$

This and (3.4) imply

$$\begin{aligned} d(u_{n+j-1}, p) &\leq \frac{q}{1-q} d(u_{n+j-1}, u_{n+j-2}) \\ &\leq \frac{q^2}{1-k} d(u_{n+j-2}, u_{n+j-3}) \\ &\vdots \end{aligned}$$

$$\leq \frac{q^j}{1-q} d(u_n, u_{n-1}), \quad n > 1, j \geq 1,$$

as desired. □

4 Enriched Bianchini mappings

Following the definition of enriched Bianchini mappings, we can rewrite (1.6) as follows:

$$\left\| \frac{\ell}{\ell+1}(u-v) + \frac{1}{\ell+1}(Tu - Tv) \right\| \leq \frac{h}{\ell+1} \max\bigl\{\|u - Tu\|, \|v - Tv\|\bigr\} \quad \forall u, v \in D,$$

which is equivalent to

$$\|T_\ell u - T_\ell v\| \leq h \max\bigl\{\|u - T_\ell u\|, \|v - T_\ell v\|\bigr\} \quad \forall u, v \in D,$$

where $T_\ell x := \frac{\ell}{\ell+1}x + \frac{1}{\ell+1}Tx$ for $x \in D$. This leads to the following definition in the setting of a unique geodesic space.

Definition 4.1 Let $(\mathcal{H}, d)$ be a unique geodesic space, and let D be a nonempty subset of $\mathcal{H}$. A mapping $T : D \to \mathcal{H}$ is said to be an *enriched Bianchini mapping* if there exist $h \in [0, 1)$ and $\ell \in [0, +\infty)$ such that for all $u, v \in D$,

$$d(T_\ell u, T_\ell v) \leq h \max\bigl\{d(u, T_\ell u), d(v, T_\ell v)\bigr\}, \tag{4.1}$$

where $T_\ell x := \frac{\ell}{\ell+1}x \oplus \frac{1}{\ell+1}Tx$ for $x \in \mathcal{H}$.

We henceforth refer T, using the numbers involved, as (ℓ, h)-enriched Bianchini mapping.

Example 4 Every Bianchini mapping $T : D \to \mathcal{H}$ is (ℓ, h)-enriched Bianchini with $\ell = 0$. Indeed, for $\ell = 0$, $T_\ell \equiv T$, and, consequently, (4.1) reduces to

$$d(Tu, Tv) \leq h \max\bigl\{d(u, Tu), d(v, Tv)\bigr\} \quad \text{for all } u, v \in D.$$

Example 5 Every (γ, η)-enriched Kannan mapping $T : D \to \mathcal{H}$ is an (ℓ, h)-enriched Bianchini mapping with $\ell = \gamma$ and $h = 2\eta$. Indeed, for all $u, v \in D$,

$$\begin{aligned} d(T_\gamma u, T_\gamma v) &\leq \eta\bigl(d(u, T_\gamma u) + d(v, T_\gamma v)\bigr) \\ &\leq 2\eta \max\bigl\{d(u, T_\gamma u), d(v, T_\gamma v)\bigr\}. \end{aligned}$$

We now state the main theorem of this section.

Theorem 4.2 *Let $(\mathcal{H}, d)$ be a Hadamard space, and let $T : \mathcal{H} \to \mathcal{H}$ be an (ℓ, h)-enriched Bianchini mapping. Then*

(i) *T has a unique fixed point;*

(ii) *there exists* $\delta \in (0,1]$ *such that the sequence* $\{u_n\}$ *defined iteratively by*

$$\begin{cases} u_1 \in \mathcal{H}, \\ u_{n+1} = (1-\delta)u_n \oplus \delta T u_n, \quad n \geq 1, \end{cases} \tag{4.2}$$

converges strongly to the fixed point of T;

(iii) *there exists* $k \in [0,1)$ *such that for a fixed point* p,

$$d(u_{n+j-1}, p) \leq \frac{k^j}{1-k} d(u_n, u_{n-1}), \quad n > 1, j \geq 1. \tag{4.3}$$

Proof Let $T_\ell x := \frac{\ell}{\ell+1} x \oplus \frac{1}{\ell+1} Tx$ and take $\delta = \frac{1}{\ell+1} \in (0,1]$. Then $\{u_n\}$ defined in (4.2) corresponds to

$$u_1 \in \mathcal{H}, \qquad u_{n+1} = T_\ell u_n, \quad n \geq 1.$$

Now let $n \geq 2$, $u = u_n$, and $v = u_{n-1}$. Then (4.1) implies that

$$d(u_{n+1}, u_n) \leq h \max\bigl\{d(u_n, u_{n+1}), d(u_n, u_{n-1})\bigr\}, \tag{4.4}$$

Suppose that $\max\{d(u_n, u_{n+1}), d(u_n, u_{n-1})\} = d(u_n, u_{n+1})$. Then (4.4) implies that $d(u_n, u_{n+1}) < d(u_n, u_{n+1})$, which is a contradiction. Therefore

$$d(u_{n+1}, u_n) \leq h \max\bigl\{d(u_n, u_{n+1}), d(u_n, u_{n-1})\bigr\} = h d(u_n, u_{n-1}) \quad \text{for all } n > 1. \tag{4.5}$$

Inductively, we obtain

$$d(u_{n+1}, u_n) \leq h^{n-1} d(u_2, u_1), \quad n \geq 1. \tag{4.6}$$

Thus for all $m, n \geq 1$, we have

$$\begin{aligned} d(u_{n+m}, u_n) &\leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + \cdots + d(u_{n+m-1}, u_{n+m}) \\ &\leq h^{n-1} d(u_2, u_1) + h^n d(u_2, u_1) + \cdots + h^{n+m-2} d(u_2, u_1) \\ &= h^{n-1}\bigl[1 + q + \cdots + h^{m-1}\bigr] d(u_2, u_1) \\ &\leq h^{n-1} d(u_2, u_1) \sum_{j=1}^{+\infty} h^j \\ &\leq \frac{h^{n-1}}{1-h} d(u_2, u_1). \end{aligned}$$

Therefore $\{u_n\}$ is a Cauchy sequence. Since $\mathcal{H}$ is a complete CAT(0) space, there exists $p \in \mathcal{H}$ such that $\{u_n\}$ converges to p, that is,

$$\lim_{n\to\infty} u_n = p. \tag{4.7}$$

Let $n \geq 1$. If $\max\{d(u_n,u_{n+1}),d(p,T_\ell p)\} = d(p,T_\ell p)$, then

$$\begin{aligned} d(p,T_\ell p) &\leq d(p,u_{n+1}) + d(u_{n+1},T_\ell p) \\ &= d(u_{n+1},p) + d(T_\ell u_n,T_\ell p) \\ &\leq d(u_{n+1},p) + h\max\bigl\{d(u_n,u_{n+1}),d(p,T_\ell p)\bigr\} \\ &\leq d(u_{n+1},p) + hd(p,T_\ell p), \end{aligned}$$

which implies

$$d(p,T_\ell p) \leq \frac{1}{1-h} d(u_{n+1},p). \tag{4.8}$$

Also, if $\max\{d(u_n,u_{n+1}),d(p,T_\ell p)\} = d(u_n,u_{n+1})$, then we obtain from (4.6) that

$$\begin{aligned} d(p,T_\ell p) &\leq d(p,u_{n+1}) + d(u_{n+1},T_\ell p) \\ &= d(u_{n+1},p) + d(T_\ell u_n,T_\ell p) \\ &\leq d(u_{n+1},p) + h\max\bigl\{d(u_n,u_{n+1}),d(p,T_\ell p)\bigr\} \\ &\leq d(u_{n+1},p) + hd(u_n,u_{n+1}) \\ &\leq d(u_{n+1},p) + h^{n-1}d(u_2,u_1). \end{aligned}$$

This and (4.8) imply that, in any case, $p \in Fix(T_\ell)$. Suppose that T_ℓ has a fixed point u' that is different from p. Then it follows from (4.1) that

$$0 < d\bigl(u',p\bigr) \leq h \cdot 0,$$

which is a contradiction. Hence p is the unique fixed point of T_ℓ. Finally, Lemma 2.2(i) implies that

$$Fix(T) = Fix(T_\ell) = \{p\}.$$

Therefore we have (i), and (ii) follows from (4.7).

Part (iii) follows similarly to the proof of Theorem 3.2. □

5 Stongly demicontractive mapping as enriched Kannan

Hicks and Kubicek [21] introduced demicontractive mappings in the setting of a linear space $(H,\|\cdot\|)$ as a mapping $T : H \to H$ with $Fix(T) \neq \emptyset$ and

$$\|Tu-p\|^2 \leq \|u-p\|^2 + k\|u-Tu\|^2 \quad \forall u \in H, p \in Fix(T), \tag{5.1}$$

where $k \in [0,1)$. In [31] a strongly demicontractive mapping is introduced by strengthening inequality (5.1) in the following sense:

$$\|Tu-p\|^2 \leq \alpha\|u-p\|^2 + k\|u-Tu\|^2 \quad \forall u \in H, p \in Fix(T), \tag{5.2}$$

where $\alpha \in (0,1)$ and $k \geq 0$. It is worth noting that if T is strongly demicontractive, then T has a unique fixed point. We now show some relationship between strongly demicontractive mappings and enriched Kannan mappings.

Given a metric space $(\mathcal{H}, d)$ with a nonempty set D, a mapping $T : D \to \mathcal{H}$ is called *strongly demicontractive* if for $p \in Fix(T)$,

$$d^2(Tu,p) \leq \alpha d^2(u,p) + kd^2(u,Tu) \quad \forall u \in D, \tag{5.3}$$

where $\alpha \in (0,1)$ and $k \geq 0$. In the sequel, we consider $(\mathcal{H}, d)$ as a CAT(0) space.

It is worth mentioning that the fixed point of a strongly demicontractive mapping T is unique. Indeed, if there exist two distinct fixed points of T, say p and q, then (5.3) yields

$$d^2(q,p) \leq \alpha d^2(q,p), \tag{5.4}$$

which is a contradiction. Demicontractive mappings can also be defined using quasi-linearization as in the following proposition.

Proposition 5.1 *For any $\alpha, k \in \mathbb{R}$, $u \in D$, $p \in \mathcal{H}$, and $T : D \to \mathcal{H}$, the following inequalities are equivalent*:

(i) $d^2(Tu,p) \leq \alpha d^2(u,p) + kd^2(u,Tu)$;
(ii) $(1-\alpha)d^2(u,p) + (1-k)d^2(u,Tu) \leq 2\langle \overrightarrow{uTu}, \overrightarrow{up} \rangle$.

Proof The proof follows from the definition of quasi-linearization in (2.5). □

It is worth noting from Proposition 5.1 that $T : D \to \mathcal{H}$ is demicontractive if and only if the following inequality holds for $p \in Fix(T)$:

$$\frac{(1-k)}{2} d^2(u,Tu) \leq \langle \overrightarrow{uTu}, \overrightarrow{up} \rangle \quad \forall u \in D.$$

Theorem 5.2 *For $\lambda \in (\frac{8}{9}, 1]$, let $T : D \to \mathcal{H}$ be a strongly demicontractive mapping with constants α and κ. If*

$$\alpha < \frac{1}{9\lambda} \quad \textit{and} \quad \kappa < \frac{9}{4}(1-\alpha)\lambda^2 - 2\lambda,$$

then T is (γ, η)-enriched Kannan mapping with $\gamma = \frac{1}{\lambda} - 1$ and

$$\eta = \frac{1 + \alpha\lambda - \lambda + \sqrt{1 + (1-\alpha)(k-\lambda)}}{\lambda(1-\alpha)}.$$

Proof Let $u \in D$ and $p \in Fix(T)$. From (2.4), (2.1), and the assumption that T is strongly demicontractive mapping, we have

$$\begin{aligned} d^2(T_\gamma u, p) &\leq \frac{\gamma}{\gamma+1} d^2(u,p) + \frac{1}{\gamma+1} d^2(Tu,p) \\ &\leq \frac{\gamma}{\gamma+1} d^2(u,p) + \frac{\alpha}{\gamma+1} d^2(u,p) + \frac{\kappa}{\gamma+1} d^2(Tu,u) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\gamma+\alpha}{\gamma+1} d^2(u,p) + \kappa(\gamma+1) d^2(T_\gamma u, u) \\
&\leq \frac{\gamma+\alpha}{\gamma+1} \bigl[d(u,T_\gamma u) + d(T_\gamma u, p)\bigr]^2 + \kappa(\alpha+1) d^2(T_\gamma u, u) \\
&= \left(\frac{\gamma+\alpha}{\gamma+1} + \kappa(\gamma+1)\right) d^2(u, T_\gamma u) + \frac{\gamma+\alpha}{\gamma+1} d^2(T_\gamma u, p) \\
&\quad + 2\frac{\gamma+\alpha}{\gamma+1} d(T_\gamma u, u) d(T_\gamma u, p).
\end{aligned}
$$

This means

$$
\frac{1-\alpha}{\gamma+1} d^2(T_\gamma u, p) \leq \left(\frac{\gamma+\alpha}{\gamma+1} + \kappa(\gamma+1)\right) d^2(u, T_\gamma u) + 2\frac{\gamma+\alpha}{\gamma+1} d(T_\gamma u, u) d(T_\gamma u, p).
$$

Thus by simple calculations we get

$$
\begin{aligned}
&\left(\frac{1-\alpha}{\gamma+1}\right)^2 \left[d(T_\gamma u, p) - \left(\frac{\gamma+1}{1-\alpha} - 1\right) d(u, T_\gamma u)\right]^2 \\
&\quad \leq \left[1 + (1-\alpha)\left(k - \frac{1}{\gamma+1}\right)\right] d^2(u, T_\gamma u).
\end{aligned}
$$

So,

$$
\frac{1-\alpha}{\gamma+1} \left| d(T_\gamma u, p) - \left(\frac{\gamma+1}{1-\alpha} - 1\right) d(u, T_\gamma u)\right| \leq \sqrt{1 + (1-\alpha)\left(k - \frac{1}{\gamma+1}\right)}\, d(u, T_\gamma u).
$$

Consequently, we have

$$
d(T_\gamma u, p) \leq \frac{\gamma + \alpha + (\gamma+1)\sqrt{1 + (1-\alpha)(k - \frac{1}{\gamma+1})}}{1-\alpha} d(u, T_\gamma u). \tag{5.5}
$$

For $\lambda \in (\frac{8}{9}, 1]$, observe that if $\gamma = \frac{1}{\lambda} - 1$, then (5.5) reduces to

$$
d(T_\gamma u, p) \leq \frac{1 + \alpha\lambda - \lambda + \sqrt{1 + (1-\alpha)(k-\lambda)}}{\lambda(1-\alpha)} d(u, T_\gamma u). \tag{5.6}
$$

Now for this λ, let

$$
\eta := \frac{1 + \alpha\lambda - \lambda + \sqrt{1 + (1-\alpha)(k-\lambda)}}{\lambda(1-\alpha)}.
$$

Then

$$
\begin{aligned}
d(T_\gamma u, T_\gamma w) &\leq d(T_\gamma u, p) + d(p, T_\gamma w) \\
&\leq \eta\bigl[d(u, T_\gamma u) + d(w, T_\gamma w)\bigr].
\end{aligned}
$$

We now show that $\eta \in (0, 1/2)$. Note that for any $\alpha \in (0,1)$,

$$
\frac{1 + \alpha\lambda - \lambda + \sqrt{1 + (1-\alpha)(k-\lambda)}}{\lambda(1-\alpha)} < \frac{1}{2}
$$

if and only if

$$\sqrt{1+(1-\alpha)(k-\lambda)} < \frac{3}{2}\lambda(1-\alpha) - 1. \tag{5.7}$$

This is equivalent to

$$k < \frac{9}{4}(1-\alpha)\lambda^2 - 2\lambda. \tag{5.8}$$

The class of strongly demicontractive mappings contains several other important classes of mappings. For instance, every contraction mapping with constant c is a strongly demicontractive mapping with constants $\alpha = c$ and $k = 0$. We now recall an important class of mappings and discuss its inclusion with regards to enriched Kannan mapping.

Following [31, 32], a mapping $T : \mathcal{H} \to \mathcal{H}$ is called a *quasi-(L,m)-contraction* if there exist $L \geq 0$ and $m \in [0,1)$ such that $Fix(T) \neq \emptyset$ and

$$d(Tu,Tw) \leq Ld(u,Tu) + md(u,w) \quad \forall u,w \in H. \tag{5.9}$$

Theorem 5.3 *The class of quasi-(L,m)-contractions coincides with the class of strongly demicontractive mappings.*

Proof Let T be a quasi-(L,m)-contraction. It follows from (5.9) that for any $\alpha \in (m^2,1)$,

$$\begin{aligned}
d^2(u,p) &\leq \big(md(u,p) + Ld(u,Tu)\big)^2 \\
&= m^2d^2(u,p) + L^2d^2(u,Tu) + 2mLd(u,p)d(u,Tu) \\
&= \alpha d^2(u,p) + \left(\frac{m^2L^2}{\alpha - m^2} + L^2\right)d^2(u,Tu) \\
&\quad - \left(\sqrt{\alpha - m^2}d(u,p) - \frac{mL}{\sqrt{\alpha - m^2}}d^2(u,Tu)\right)^2 \\
&\leq \alpha d^2(u,p) + \left(\frac{m^2L^2}{\alpha - m^2} + L^2\right)d^2(u,Tu).
\end{aligned}$$

Now let T be a strongly demicontractive mapping. It follows from (5.2) that

$$\begin{aligned}
d^2(Tu,p) &\leq \alpha d^2(u,p) + \kappa d^2(u,Tu) \\
&= \big(\sqrt{\alpha}d(u,p) + \sqrt{\kappa}d(u,Tu)\big)^2 - 2\sqrt{\alpha\kappa}d(u,p)d(u,Tu) \\
&\leq \big(\sqrt{\alpha}d(u,p) + \sqrt{\kappa}d(u,Tu)\big)^2.
\end{aligned}$$

□

According to [31], a mapping $T : \mathcal{H} \to \mathcal{H}$ having a unique fixed point $p \in \mathcal{H}$ is called *quasi-expansive* if there exists a closed convex subset $D \subset \mathcal{H}$ containing p such that

$$d(u,p) \leq \beta d(u,Tu), \quad u \in D,$$

where $\beta > 0$. The next theorem connects strongly demicontractive and quasi-expansive mappings with enriched Kannan mapping. The proof of the theorem is based on the fact (see [31, p. 179]) that for $\alpha, \kappa \in [\frac{1}{10}, 1)$ and $\beta = \frac{5(1-\kappa)}{9}$, there exists $\lambda \in (0,1)$ such that

$$\frac{(1-\lambda+\alpha\lambda)\beta^2+\lambda^2-\lambda+\lambda\kappa}{\lambda^2} < \frac{1}{4}. \tag{5.10}$$

Theorem 5.4 *Suppose that $T : \mathcal{H} \to \mathcal{H}$ is strongly demicontractive with $\alpha, \kappa \in [\frac{1}{10}, 1)$ and quasi-expansive with $\beta = \frac{5(1-\kappa)}{9}$ on some closed convex subset $D \subset \mathcal{H}$. Then T restricted to D is an enriched Kannan mapping.*

Proof Let $u \in D$ and $p \in Fix(T)$. It follows from (2.4) that

$$\begin{aligned} d^2(T_\gamma u, p) &\le \frac{\gamma}{\gamma+1} d^2(u,p) + \frac{1}{\gamma+1} d^2(Tu,p) - \frac{\gamma}{(\gamma+1)^2} d^2(u,Tu) \\ &\le \frac{\gamma}{\gamma+1} d^2(u,p) + \frac{\alpha}{\gamma+1} d^2(u,p) + \frac{\kappa}{\gamma+1} d^2(Tu,u) - \frac{\gamma}{(\gamma+1)^2} d^2(u,Tu) \\ &= \frac{\gamma+\alpha}{\gamma+1} d^2(u,p) + \frac{\kappa(\gamma+1)-\gamma}{(\gamma+1)^2} d^2(Tu,u). \end{aligned}$$

Taking into account that

$$d(u,p) \le \beta d(u,Tu) \quad \text{and} \quad d(u,Tu) = (1+\gamma) d(u, T_\gamma u),$$

we obtain

$$\begin{aligned} d^2(T_\gamma u, p) &\le \left(\frac{\gamma+\alpha}{\gamma+1}\beta^2 + \frac{\kappa(\gamma+1)-\gamma}{(\gamma+1)^2} \right) d^2(Tu,u) \\ &= \big((\gamma+\alpha)(\gamma+1)\beta^2 + \kappa(\gamma+1) - \gamma \big) d^2(T_\gamma u, u). \end{aligned} \tag{5.11}$$

If $\gamma = \frac{1}{\lambda} - 1$, then (5.11) is equivalent to

$$d^2(T_\gamma u, p) \le \frac{(1-\lambda+\alpha\lambda)\beta^2+\lambda^2-\lambda+\lambda\kappa}{\lambda^2} d^2(T_\gamma u, u).$$

This implies that

$$d(T_\gamma u, p) \le \frac{\sqrt{(1-\lambda+\alpha\lambda)\beta^2+\lambda^2-\lambda+\lambda\kappa}}{\lambda} d(T_\gamma u, u).$$

Consequently, we get

$$\begin{aligned} d(T_\gamma u, T_\gamma w) &\le d(T_\gamma u, p) + d(T_\gamma w, p) \\ &\le \frac{\sqrt{(1-\lambda+\alpha\lambda)\beta^2+\lambda^2-\lambda+\lambda\kappa}}{\lambda} \big(d(T_\gamma u, u) + d(T_\gamma w, w) \big). \end{aligned}$$

This and (5.10) complete the proof. □

Remark 1 Note that a quasi-expansive mapping does not contradict a strongly demicontractive mapping. For examples of mappings that are both quasi-expansive and strongly demicontractive, see [31].

6 Numerical experiment

Example 6 Let $\mathcal{H} = \mathbb{R}^2$ be endowed with the following distance metric:

$$d(u,v) := \sqrt{(u_1 - v_1)^2 + \big(u_1^2 - u_2 - v_1^2 + v_2\big)^2} \quad \forall u = (u_1,u_2), v = (v_1,v_2) \in \mathcal{H}.$$

For $w = (w_1,w_2)$ and $z = (z_1,z_2)$, the map $\tau : t \mapsto (1-t)w \oplus tz$ of the explicit form

$$\big((1-t)w_1 + tz_1, \big((1-t)w_1 + tz_1\big)^2 - (1-t)\big(w_1^2 - w_2\big) - t\big(z_1^2 - z_2\big)\big)$$

defines a geodesic path in $(\mathcal{H}, d)$. Moreover, it is easy to see that with this path, the CN-inequlity (2.2) is satisfied. Indeed, for any $u = (u_1,u_2), v = (v_1,v_2), w = (w_1,w_2) \in \mathcal{H}$, we have

$$\frac{1}{2}u \oplus \frac{1}{2}v = \left(\frac{u_1+v_1}{2}, \left(\frac{u_1+v_1}{2}\right)^2 - \frac{u_1^2 - u_2}{2} - \frac{v_1^2 - v_2}{2}\right).$$

Thus

$$\begin{aligned}
&d^2\left(\frac{1}{2}u \oplus \frac{1}{2}v, w\right) \\
&\quad = \left[\frac{u_1 - v_1}{2} - w_1\right]^2 + \left[\left(\frac{u_1+v_1}{2}\right)^2 - \left(\left(\frac{u_1+v_1}{2}\right)^2 - \frac{u_1^2 - u_2}{2}\right.\right. \\
&\qquad \left.\left. - \frac{v_1^2 - v_2}{2}\right) - w_1^2 + w_2\right]^2 \\
&\quad = \left[\frac{u_1 - w_1}{2}\right]^2 + \left[\frac{w_1 - v_1}{2}\right]^2 + \left[\frac{u_1^2 - u_2 - w_1^2 + w_2}{2} - \frac{w_1^2 - w_2 - v_1^2 + v_2}{2}\right]^2 \\
&\quad = \frac{(u_1 - w_1)^2}{2} - \frac{(w_1 - v_1)^2}{2} - \frac{(u_1 - v_1)^2}{4} + \frac{(u_1^2 - u_2 - w_1^2 + w_2)^2}{2} \\
&\qquad + \frac{(w_1^2 - w_2 - v_1^2 + v_2)^2}{2} - \frac{(u_1^2 - u_2 - v_1^2 + v_2)^2}{4} \\
&\quad = \frac{1}{2}\big[(u_1 - w_1)^2 + \big(u_1^2 - u_2 - w_1^2 + w_2\big)^2\big] + \frac{1}{2}\big[(w_1 - v_1)^2 \\
&\qquad + \big(w_1^2 - w_2 - v_1^2 + v_2\big)^2\big] - \frac{1}{4}\big[(u_1 - v_1)^2 + \big(u_1^2 - u_2 - v_1^2 + v_2\big)^2\big] \\
&\quad \le \frac{1}{2}d^2(u,w) + \frac{1}{2}d^2(v,w) - \frac{1}{4}d^2(u,v).
\end{aligned}$$

This metric allows the transformations from nonconvex functions (in the usual distance) to convex functions, (see for example, [19, Example 5.2]).

Consider $T : \mathcal{H} \to \mathcal{H}$ defined by

$$T(x_1,x_2) = \big(1 - x_1, x_1^2 - 2x_1 + 1\big).$$

Then T is not a Kannan mapping, for if it were, then there would exist $h \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) = h\big(d(x,Tx) + d(y,Ty)\big)$$

for all $x, y \in \mathcal{H}$, would imply that

$$|x_1 - y_1| \le h\Big[\sqrt{|2x_1 - 1|^2 + \big|x_1^2 - x_2\big|^2} + \sqrt{|2y_1 - 1|^2 + \big|y_1^2 - y_2\big|^2}\Big]. \tag{6.1}$$

This is a contradiction, since for $x = (\frac{1}{2}, \frac{1}{4})$ and $y = (1, 1)$, (6.1) gives $\frac{1}{2} \le h$.

However, T is an enriched Kannan mapping. To see this, we take $\lambda = \frac{1}{\gamma+1}$ and $x = (x_1, x_2) \in \mathcal{H}$. Then we have

$$T_\gamma x = (1-\lambda)x \oplus \lambda Tx = \big((1-2\lambda)x_1 + \lambda, \big((1-2\lambda)x_1 + \lambda\big)^2 - (1-\lambda)\big(x_1^2 - x_2\big)\big),$$

and for all $y = (y_1, y_2) \in \mathcal{H}$, we get

$$\begin{aligned}
&d(T_\gamma x, T_\gamma y)\\
&\quad= \Bigg((1-2\lambda)^2\bigg[\frac{1}{2}(2x_1 - 1) + \frac{1}{2}(1 - 2y_1)\bigg]^2\\
&\qquad+ (1-\lambda)^2\bigg[\frac{1}{2}\big(2\big(x_1^2 - x_2\big)\big) + \frac{1}{2}\big(2\big(y_2 - y_1^2\big)\big)\bigg]^2\Bigg)^{1/2}\\
&\quad\le \Bigg((1-2\lambda)^2\bigg[\frac{1}{2}(2x_1 - 1)^2 + \frac{1}{2}(1 - 2y_1)^2\bigg]\\
&\qquad+ (1-\lambda)^2\bigg[\frac{1}{2}\big(2\big(x_1^2 - x_2\big)\big)^2 + \frac{1}{2}\big(2\big(y_2 - y_1^2\big)\big)^2\bigg]\Bigg)^{1/2}\\
&\quad\le \Bigg(\frac{(1-2\lambda)^2}{2}|2x_1 - 1|^2 + 2(1-\lambda)^2\big|x_1^2 - x_2\big|^2\\
&\qquad+ \frac{(1-2\lambda)^2}{2}|2y_1 - 1|^2 + 2(1-\lambda)^2\big|y_1^2 - y_2\big|^2\Bigg)^{1/2}\\
&\quad\le \sqrt{\frac{(1-2\lambda)^2}{2}|2x_1 - 1|^2 + 2(1-\lambda)^2\big|x_1^2 - x_2\big|^2}\\
&\qquad+ \sqrt{\frac{(1-2\lambda)^2}{2}|2y_1 - 1|^2 + 2(1-\lambda)^2\big|y_1^2 - y_2\big|^2}\\
&\quad\le \max\bigg\{\frac{|1-2\lambda|}{\sqrt{2}}, \sqrt{2}|1-\lambda|\bigg\}\Big[\sqrt{|2x_1 - 1|^2 + \big|x_1^2 - x_2\big|^2}\\
&\qquad+ \sqrt{|2y_1 - 1|^2 + \big|y_1^2 - y_2\big|^2}\Big]\\
&\quad\le \max\bigg\{\frac{|1-2\lambda|}{\sqrt{2}}, \sqrt{2}|1-\lambda|\bigg\}\big[d(x, Tx) + d(y, Ty)\big]\\
&\quad\le \max\bigg\{\frac{|1-2\lambda|}{\sqrt{2}}, \sqrt{2}|1-\lambda|\bigg\}\big[d(x, T_\gamma x) + d(y, T_\gamma y)\big].
\end{aligned}$$

This implies that T is an (γ, η)-enriched Kannan mapping with $\gamma = \frac{1}{\lambda} - 1 = \frac{1}{4}$ and $\eta = \max\{\frac{|1-2\lambda|}{\sqrt{2}}, \sqrt{2}|1-\lambda|\} = \frac{3}{5\sqrt{2}} \in [0, \frac{1}{2})$.

Table 1 Few values of the sequence $\{d(u_n,p)\}$ for Example 6

n	$d(u_n,p)$					
	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
1	7527.497011	114.6832595	4506.49068	2440.522536	250,500.498	5814.503268
2	1506.294384	24.00187493	902.0828399	488.9397611	50,100.8964	1163.705568
3	302.6860849	6.406527921	181.8229391	99.27876107	10,021.6134	234.1849807
4	63.04806509	2.849867365	38.80441727	22.36101214	2006.902261	49.35999109
5	16.45331986	1.630236106	11.23633173	7.620736384	405.9941947	13.59521876
6	7.144497966	0.972684322	5.368313326	4.003752637	89.07445314	6.232792607
7	4.064393128	0.583245636	3.115997596	2.361297334	28.28661806	3.588527534
8	2.423362363	0.349923042	1.86246768	1.414021758	14.34572362	2.142803073
9	1.452995616	0.209952203	1.117004205	0.84822908	8.414154905	1.284992441
10	0.871729223	0.125971214	0.670170755	0.508925181	5.035442799	0.770949491
⋮	⋮	⋮	⋮	⋮	⋮	⋮
40	1.93E-07	2.78E-08	1.48E-07	1.13E-07	1.11E-06	1.70E-07
41	1.16E-07	1.67E-08	8.89E-08	6.75E-08	6.68E-07	1.02E-07
42	6.94E-08	1.00E-08	5.33E-08	4.05E-08	4.01E-07	6.14E-08
43	4.16E-08	6.02E-09	3.20E-08	2.43E-08	2.40E-07	3.68E-08
44	2.50E-08	3.61E-09	1.92E-08	1.46E-08	1.44E-07	2.21E-08
45	1.50E-08	2.17E-09	1.15E-08	8.75E-09	8.65E-08	1.33E-08
46	8.99E-09	1.30E-09	6.91E-09	5.25E-09	5.19E-08	7.95E-09
47	5.39E-09	7.80E-10	4.15E-09	3.15E-09	3.12E-08	4.77E-09
48	3.24E-09	4.68E-10	2.49E-09	1.89E-09	1.87E-08	2.86E-09
49	1.94E-09	2.81E-10	1.49E-09	1.13E-09	1.12E-08	1.72E-09
50	1.17E-09	1.68E-10	8.96E-10	6.80E-10	6.73E-09	1.03E-09

For $\delta = \frac{4}{5}$, (3.2) reduces to

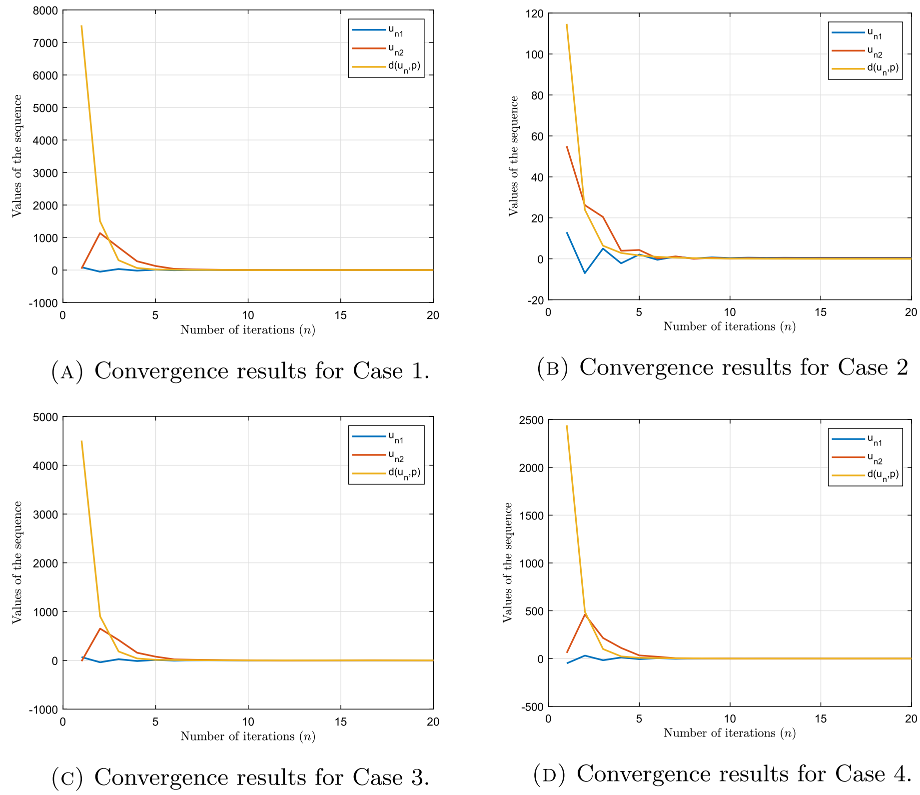
$$\begin{cases} u_1 \in \mathcal{H}, \\ u_{n+1} = \frac{1}{5}u_n \oplus \frac{4}{5}Tu_n, \quad n \geq 1, \end{cases} \tag{6.2}$$

which yields the results in Table 1 and Fig. 1. The results are obtained for six different starting points of $\{u_n\}$: Case 1 for $u_1 = (87,42)$, Case 2 for $u_1 = (13,55)$, Case 3 for $u_1 = (67,-17)$, Case 4 for $u_1 = (-50,60)$, Case 5 for $u_1 = (500,-500)$, and Case 6 for $u_1 = (-76,-38)$.

7 Conclusions

In a unique geodesic space, we introduced the classes of enriched Kannan and Bianchini mappings. The existence and uniqueness of the fixed point of such a mapping were established in the setting of a complete CAT(0) space. Also, a Krasnoselskij scheme is shown to converge with a certain rate to the fixed point. Furthermore, conditions that guarantee the inclusions of the class of strongly demicontractive mappings and the class of quasi-expansive mappings to the class of enriched Kannan mappings were derived. Finally, an example of na enriched Kannan mapping that is not Kannan was given in a nonlinear CAT(0) space. The result herein extended several results in the literature. In particular, we can observe the following:

1. Theorem 3.2 complements the results of [8, Theorem 2.1] from linear setting to CAT(0) spaces and the results in [24, 25] from Kannan mappings to enriched Kannan mappings.
2. Theorem 4.2 extends Theorem 3.1 of [8] from linear spaces to CAT(0) spaces and the results in [9] to the class of enriched Kannan mapping.



(A) Convergence results for Case 1.
(B) Convergence results for Case 2
(C) Convergence results for Case 3.
(D) Convergence results for Case 4.

Figure 1 Report of numerical experiment for Example 6

3. Theorem 5.2 generalizes [31, Theorem 2.2] from linear setting to CAT(0) setting and from Kannan mapping to enriched Kannan mapping.
4. Theorem 5.3 complements Theorem 2.1 of [31] to CAT(0) spaces.
5. Theorem 5.4 generalizes Theorem 3.4 of [31] to CAT(0) spaces.

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Declarations

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The authors declare no competing interests.

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