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Convergence results on the general inertial Mann–Halpern and general inertial Mann algorithms

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Abstract

In this paper, we prove strong convergence theorem of the general inertial Mann–Halpern algorithm for nonexpansive mappings in the setting of Hilbert spaces. We also prove weak convergence theorem of the general inertial Mann algorithm for k-strict pseudo-contractive mappings in the setting of Hilbert spaces. These convergence results extend and generalize some existing results in the literature. Finally, we provide examples to verify our main results.

Keywords: Mann algorithm; Inertia; Halpern algorithm; Nonexpansive mapping; k-strict pseudo-contractive mapping

1 Introduction

Let D be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. A self-mapping S on D is said to be a k-strict pseudo-contractive mapping if there exists $k \in [0,1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\big\|(I - S)x - (I - S)y\big\|^2$$

for all $x, y \in D$. The set of fixed points of the mapping $S : D \to D$ is defined by $\operatorname{Fix}(S) = \{y \in D : Sy = y\}$. S is nonexpansive if and only if S is a 0-strict pseudo-contractive mapping.

The development of various iterative methods for finding the approximate solution of nonlinear equations has become an active area of research in many scientific fields, and as a result various iteration methods for fixed point problems have been developed (see [3–6]). One of the most popular methods is the Mann algorithm [16], which is described as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \tag{1}$$

where $\{\alpha_n\} \subset [0,1)$ satisfying the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$; (b) $\sum_{n=1}^{\infty} \alpha_n = \infty$, where T is a nonexpansive mapping. But the convergence rate of the Mann algorithm is slow in general. Due to the fact that fast convergence is required in many practical applications (see [9, 12, 13, 17]), many researchers constructed fast iterative algorithms by using inertial extrapolation methods (see [2, 4, 7, 8, 11, 14, 15, 18–20]). Specifically, Mainge [15] developed the following algorithm by employing the Mann algorithm together with inertial extrapolation method:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = w_n + \lambda_n[Tw_n - w_n], \end{cases} \tag{2}$$

for each $n \geq 1$. He showed weak convergence of the iterative sequence $\{x_n\}$ to a fixed point of a nonexpansive mapping T under the conditions listed below:

(A1) $\alpha_n \in [0,\alpha]$ for any $\alpha \in [0,1)$; (A2) $\sum_{n=1}^{\infty} \alpha_n\|x_n - x_{n-1}\|^2 < \infty$; (A3) $0 < \inf_{n\geq 1} \lambda_n \leq \sup_{n\geq 1} \lambda_n < 1$. In 2018, Dong et al. [11] introduced the general inertial Mann algorithm for a nonexpansive mapping T, which is shown below:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = x_n + \beta_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \lambda_n)y_n + \lambda_n Tz_n, \end{cases} \tag{3}$$

for each $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, and λ_n satisfy:

(D1) $\{\alpha_n\} \subset [0,\alpha]$ and $\{\beta_n\} \subset [0,\beta]$ are nondecreasing with $\alpha_1 = \beta_1 = 0$ and $\alpha, \beta \in [0,1)$; (D2) For any $\lambda, \sigma, \delta > 0$, $\delta > \frac{\alpha\xi(1+\xi)+\alpha\sigma}{1-\alpha^2}$, $0 < \lambda \leq \lambda_n \leq \frac{\delta-\alpha[\xi(1+\xi)+\alpha\delta+\sigma]}{\delta[1+\xi(1+\xi)+\alpha\delta+\sigma]}$, where $\xi = \max\{\alpha, \beta\}$.

Inspired by the above work, in this paper, we extend the works of Dong et al. [11] for k-strict pseudo-contractive mappings. Moreover, we combine their algorithm with the Halpern algorithm to obtain strong convergence result for nonexpansive mappings.

The structure of this paper is as follows: In Sect. 2, we present some notations and lemmas that will be used in the main results. In Sect. 3, we prove strong convergence result by combining the general inertial Mann algorithm with the Halpern algorithm for nonexpansive mappings. In Sect. 4, we prove the weak convergence of the general inertial Mann algorithm for k-strict pseudo-contractive mappings. In the final section, conclusions are provided.

2 Preliminaries

In this section, we provide some useful notations and lemmas that will be used in the sequel.

We use the notation:

1. "$\rightharpoonup$" for weak convergence and
2. "$\to$" for strong convergence.

Lemma 1 [1] *Let* $\{\psi_n\}$, $\{\delta_n\}$, *and* $\{\alpha_n\}$ *be sequences in* $[0,\infty)$ *satisfying* $\psi_{n+1} \leq \psi_n + \alpha_n(\psi_n - \psi_{n-1}) + \delta_n$ *for each* $n \geq 1$, *where* $\sum_{n=1}^{\infty} \delta_n < \infty$. *Moreover, suppose there exists a real number* α *with* $0 \leq \alpha_n \leq \alpha < 1$ *for all* $n \in \mathbb{N}$. *Then the following hold*:

1. $\sum_{n\geq 1}[\psi_n - \psi_{n-1}]_+ < \infty$, *where* $[t]_+ = \max\{t, 0\}$;
2. *There exists* $\psi^* \in [0,\infty)$ *such that* $\lim_{n\to\infty} \psi_n = \psi^*$.

Lemma 2 [3] *Let* D *be a nonempty closed convex subset of* $\mathcal{H}$ *and* $S : D \to \mathcal{H}$ *be a nonexpansive mapping. Let* $\{x_n\}$ *be a sequence in* D *such that* $x_n \rightharpoonup x \in \mathcal{H}$ *and* $Sx_n - x_n \to 0$ *as* $n \to \infty$. *Then* $x \in \mathrm{Fix}(S)$.

Lemma 3 [3] *Let D be a nonempty subset of $\mathcal{H}$ and $\{x_n\}$ be a sequence in $\mathcal{H}$, then the sequence $\{x_n\}$ converges weakly to a point in D if for all $x \in D$, $\lim_{n\to\infty} \|x_n - x\|$ exists and every sequential weak cluster point of $\{x_n\}$ is in D.*

Lemma 4 [10] *Suppose that $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \le (1-\gamma_n)s_n + \gamma_n \varrho_n \quad \text{and} \quad s_{n+1} \le s_n - \mu_n + \varphi_n$$

for all $n \ge 0$, where $\{\gamma_n\}$ is a sequence in $(0,1)$, $\{\mu_n\}$ is a sequence of nonnegative real numbers, $\{\varrho_n\}$ and $\{\varphi_n\}$ are real sequences such that (i) $\sum_{n=0}^{\infty} \gamma_n = \infty$; (ii) $\lim_{n\to\infty} \varphi_n = 0$; (iii) $\lim_{k\to\infty} \mu_{n_k} = 0$ implies $\limsup_{k\to\infty} \varrho_{n_k} \le 0$ for any subsequence $\{n_k\}$ of $\{n\}$. Then $\lim_{n\to\infty} s_n = 0$.

3 General inertial Mann–Halpern algorithm for nonexpansive mappings

In this section, we introduce a general inertial Mann–Halpern algorithm and prove its strong convergence under some assumptions.

Theorem 1 *Assume that D is a nonempty closed and convex subset of a Hilbert space $\mathcal{H}$ and $S : D \to \mathcal{H}$ is a nonexpansive mapping with at least one fixed point. Given a fixed element v in D and sequences $\{\theta_n\}$, $\{\phi_n\}$ in $[0,1)$ and $\{\psi_n\}$, $\{\gamma_n\}$ in $(0,1)$. In addition, suppose the following conditions hold*:

(H1) $\sum_{n=0}^{\infty} \gamma_n = \infty$ *and* $\lim_{n\to\infty} \gamma_n = 0$;

(H2) $\lim_{n\to\infty} \frac{\theta_n}{\gamma_n} \|x_n - x_{n-1}\| = \lim_{n\to\infty} \frac{\phi_n}{\gamma_n} \|x_n - x_{n-1}\| = 0$;

(H3) $\inf_n^{\psi_n} > 0$, $\sup_n^{\psi_n} < 1$.

Let $x_{-1}, x_0 \in C$ be arbitrary. Define a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = x_n + \phi_n(x_n - x_{n-1}), \\ w_n = (1-\psi_n)y_n + \psi_n S z_n, \\ x_{n+1} = \gamma_n v + (1-\gamma_n)w_n. \end{cases} \tag{4}$$

Then the iterative sequence $\{x_n\}$ defined by (4) converges strongly to $q = P_{\mathrm{Fix}(S)} v$.

Proof Take arbitrary $q \in \mathrm{Fix}(S)$. Using (4), we have

$$\begin{aligned} \|x_{n+1} - q\| &\le \gamma_n \|v - q\| + (1-\gamma_n)\|w_n - q\| \\ &\le \gamma_n \|v - q\| + (1-\gamma_n)\big[(1-\psi_n)\|y_n - q\| + \psi_n \|S z_n - q\|\big] \\ &\le \gamma_n \|v - q\| + (1-\gamma_n)\big[(1-\psi_n)\|y_n - q\| + \psi_n \|z_n - q\|\big]. \end{aligned} \tag{5}$$

Again from (4), we get

$$\|y_n - q\| \le \|x_n - q\| + \theta_n \|x_n - x_{n-1}\|. \tag{6}$$

Similarly, we get

$$\|z_n - q\| \le \|x_n - q\| + \phi_n \|x_n - x_{n-1}\|. \tag{7}$$

Substituting (6) and (7) into (5), we get

$$\begin{aligned} \|x_{n+1}-q\| &\leq \gamma_n\|v-q\| + (1-\gamma_n)\|x_n-q\| + \theta_n\|x_n-x_{n-1}\| \\ &\quad + \phi_n\|x_n-x_{n-1}\|. \end{aligned} \tag{8}$$

Let $M = 3\max\{\|v-q\|, \sup_{n\geq 1}\frac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|, \sup_{n\geq 1}\frac{\phi_n}{\gamma_n}\|x_n-x_{n-1}\|\}$. Then (8) reduces to

$$\begin{aligned} \|x_{n+1}-q\| &\leq (1-\gamma_n)\|x_n-q\| + \gamma_n M \\ &\leq \max\{\|x_n-q\|, M\} \\ &\quad\vdots \\ &\leq \max\{\|x_0-q\|, M\}. \end{aligned} \tag{9}$$

Hence $\{x_n\}$ is bounded, and consequently $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are bounded.

From (4), we get

$$\begin{aligned} \|x_{n+1}-q\|^2 &= \left\|(1-\gamma_n)(w_n-q) + \gamma_n(v-q)\right\|^2 \\ &\leq (1-\gamma_n)^2\|w_n-q\|^2 + 2\gamma_n\langle v-q, x_{n+1}-q\rangle \\ &\leq (1-\gamma_n)\|w_n-q\|^2 + 2\gamma_n\langle v-q, x_{n+1}-q\rangle. \end{aligned} \tag{10}$$

Again from (4), we get

$$\begin{aligned} \|w_n-q\|^2 &= \left\|(1-\psi_n)(y_n-q) + \psi_n(Sz_n-q)\right\|^2 \\ &\leq (1-\psi_n)\|y_n-q\|^2 + \psi_n\|Sz_n-q\|^2 \\ &\quad - \psi_n(1-\psi_n)\|Sz_n-y_n\|^2 \\ &\leq (1-\psi_n)\|y_n-q\|^2 + \psi_n\|z_n-q\|^2 \\ &\quad - \psi_n(1-\psi_n)\|Sz_n-y_n\|^2. \end{aligned} \tag{11}$$

Substituting (11) into (10), we get

$$\begin{aligned} \|x_{n+1}-q\|^2 \leq& (1-\gamma_n)(1-\psi_n)\|y_n-q\|^2 + (1-\gamma_n)\psi_n\|z_n-q\|^2 \\ &- (1-\gamma_n)\psi_n(1-\psi_n)\|Sz_n-y_n\|^2 \\ &+ 2\gamma_n\langle v-q, x_{n+1}-q\rangle. \end{aligned} \tag{12}$$

Again from (4), we obtain

$$\begin{aligned} \|y_n-q\|^2 &= \left\|x_n-q+\theta_n(x_n-x_{n-1})\right\|^2 \\ &\leq \left\|x_n-q\right\|^2 + 2\theta_n\langle x_n-x_{n-1}, y_n-q\rangle \\ &\leq \left\|x_n-q\right\|^2 + 2\theta_n\|x_n-x_{n-1}\|\|y_n-q\|. \end{aligned} \tag{13}$$

Similarly, we get

$$\begin{aligned} \|z_n - q\|^2 &\leq \|x_n - q\|^2 + 2\phi_n \langle x_n - x_{n-1}, z_n - q\rangle \\ &\leq \|x_n - q\|^2 + 2\phi_n \|x_n - x_{n-1}\| \|z_n - q\|. \end{aligned} \tag{14}$$

Substituting (13) and (14) into (12), we get

$$\begin{aligned} \|x_{n+1} - q\|^2 \leq{} & (1-\gamma_n)\|x_n - q\|^2 + 2\theta_n \|x_n - x_{n-1}\| \|y_n - q\| \\ & + 2\phi_n \|x_n - x_{n-1}\| \|z_n - q\| - \psi_n(1-\psi_n)(1-\gamma_n)\|Sz_n - y_n\|^2 \\ & + 2\gamma_n \langle \nu - q, x_{n+1} - q\rangle. \end{aligned} \tag{15}$$

Now, letting

$$\begin{aligned} & s_n = \|x_n - q\|^2, \\ & \varrho_n = 2\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\| \|y_n - q\| + 2\frac{\phi_n}{\gamma_n}\|x_n - x_{n-1}\| \|z_n - q\| + 2\langle \nu - q, x_{n+1} - q\rangle, \\ & \varphi_n = \gamma_n \varrho_n, \quad \text{and} \\ & \mu_n = \psi_n(1-\psi_n)(1-\gamma_n)\|Sz_n - y_n\|^2, \end{aligned}$$

(15) reduces to

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n \varrho_n \quad \text{and} \quad s_{n+1} \leq s_n - \mu_n + \varphi_n.$$

We know that $\{\gamma_n\} \subset (0,1)$, and from conditions (H1) and (H2), we see that $\sum_{n=0}^{\infty} \gamma_n = \infty$ and $\lim_{n\to\infty} \varphi_n = 0$.

Now, if we first assume that $\lim_{k\to\infty} \mu_{n_k} = 0$ and then show that $\limsup_{k\to\infty} \varrho_{n_k} \leq 0$ for any subsequence $\{n_k\}$ of $\{n\}$, then by Lemma 4 we can conclude that $\lim_{n\to\infty} s_n = 0$. For this reason, assume $\lim_{k\to\infty} \mu_{n_k} = 0$. Using this assumption together with condition (H3), we obtain

$$\lim_{k\to\infty} \|Sz_{n_k} - y_{n_k}\| = 0. \tag{16}$$

It follows that

$$\begin{aligned} \|Sz_{n_k} - z_{n_k}\| &\leq \|Sz_{n_k} - y_{n_k}\| + \|y_{n_k} - z_{n_k}\| \\ &= \|Sz_{n_k} - y_{n_k}\| + |\theta_n - \phi_n| \|x_{n_k} - x_{n_{k-1}}\|. \end{aligned}$$

Now, applying (16) and (H2), we obtain

$$\lim_{k\to\infty} \|Sz_{n_k} - z_{n_k}\| = 0.$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup \tilde{x}$ as $j \to \infty$ and

$$\limsup_{k\to\infty} \langle \nu - q, x_{n_k} - q\rangle = \lim_{j\to\infty} \langle \nu - q, x_{n_{k_j}} - q\rangle.$$

Since $\|z_{n_k} - x_{n_k}\| = \phi_n \|x_{n_k} - x_{n_{k-1}}\|$, we can see that

$$\lim_{k\to\infty} \|z_{n_k} - x_{n_k}\| = 0.$$

Hence, we have $z_{n_k} \rightharpoonup \tilde{x}$ as $j \to \infty$. So, applying Lemma 2, we get $\tilde{x} \in \mathrm{Fix}(S)$.

Combining the projection property and $q = p_{\mathrm{Fix}(S)} v$, it follows that

$$\limsup_{k\to\infty} \langle v - q, x_{n_k} - q\rangle = \lim_{j\to\infty} \langle v - q, x_{n_{k_j}} - q\rangle = \langle v - q, \tilde{x} - q\rangle \leq 0. \tag{17}$$

From (4), we see that

$$\begin{aligned}
\|w_{n_k} - x_{n_k}\| &\leq (1 - \psi_{n_k})\|y_{n_k} - x_{n_k}\| + \psi_{n_k}\|Sz_{n_k} - x_{n_k}\| \\
&\leq (1 - \psi_{n_k})\|y_{n_k} - x_{n_k}\| + \psi_{n_k}\big[\|Sz_{n_k} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\|\big] \\
&= \|y_{n_k} - x_{n_k}\| + \psi_{n_k}\|Sz_{n_k} - y_{n_k}\|,
\end{aligned}$$

which implies that

$$\lim_{k\to\infty} \|w_{n_k} - x_{n_k}\| = 0.$$

Again from (4), we get

$$\|x_{n_{k+1}} - x_{n_k}\| \leq \gamma_n \|v - x_{n_k}\| + (1 - \gamma_n)\|w_{n_k} - x_{n_k}\|,$$

which implies that

$$\lim_{k\to\infty} \|x_{n_{k+1}} - x_{n_k}\| = 0. \tag{18}$$

Combining (17) and (18), we conclude that

$$\limsup_{k\to\infty} \langle v - q, x_{n_{k+1}} - q\rangle \leq 0,$$

and taking condition (H2) into account, we conclude that $\limsup_{k\to\infty} \varrho_{n_k} \leq 0$ for any subsequence $\{n_k\}$ of $\{n\}$. As a result, we have $\lim_{n\to\infty} s_n = 0$ by means of Lemma 4, and hence x_n converges strongly to q as $n \to \infty$. □

Next, we derive the following corollary from Theorem 1 by putting $\theta_n = \phi_n$ in (4).

Corollary *Assume that D is a nonempty closed and convex subset of a Hilbert space $\mathcal{H}$ and $S : D \to \mathcal{H}$ is a nonexpansive mapping with at least one fixed point. Given a fixed element v in D and sequences $\{\theta_n\}$ in $[0,1)$ and $\{\psi_n\}$, $\{\gamma_n\}$ in $(0,1)$. In addition, suppose the following conditions hold*:

(H1) $\sum_{n=0}^{\infty} \gamma_n = \infty$ *and* $\lim_{n\to\infty} \gamma_n = 0$;
(H2) $\lim_{n\to\infty} \frac{\theta_n}{\gamma_n} \|x_n - x_{n-1}\| = 0$;
(H3) $\inf_n^{\psi_n} > 0$, $\sup_n^{\psi_n} < 1$.

Let $x_{-1}, x_0 \in C$ be arbitrary. Define a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ w_n = (1-\psi_n)y_n + \psi_n S y_n, \\ x_{n+1} = \gamma_n v + (1-\gamma_n)w_n. \end{cases} \tag{19}$$

Then the iterative sequence $\{x_n\}$ defined by (19) converges strongly to $q = P_{\mathrm{Fix}(S)}v$.

Remark We can also derive other three corollaries by by considering three cases (that is, Case 1: when $\theta_n = 0$ and $\phi_n \neq 0$, Case 2: when $\phi_n = 0$ and $\theta_n \neq 0$, and Case 3: when $\theta_n = \phi_n = 0$).

Now, we illustrate Theorem 1 using the following examples.

Let the projection of v onto $\mathrm{Fix}(S)$, that is, $P_{\mathrm{Fix}(S)}v$ be the Euclidean projection.

Example 1 Let $S : \mathbb{R} \to \mathbb{R}$ be defined as $Sx = -\frac{1}{2}x$, which is a nonexpansive mapping. Take $v = 0$, $\psi_n = \frac{1}{2}$, $\theta_n = \frac{1}{5}$, $\phi_n = \frac{2}{5}$, and $\gamma_n = \frac{3}{4}$, algorithm (4) becomes

$$x_{n+1} = \left(\frac{1}{4}\right)^{2n-2} x_1,$$

which goes to $0 = P_{\mathrm{Fix}(S)}v$.

Example 2 Let $S : \mathbb{R} \to \mathbb{R}$ be given by $Sx = -\frac{1}{2}x + 1$, which is a nonexpansive mapping. Take $v = \frac{4}{3}$, $\psi_n = \frac{2}{3}$, $\theta_n = \phi_n = \frac{4}{5}$, and $\gamma_n = \frac{1}{n+1}$, algorithm (4) becomes

$$x_{n+1} = \frac{1}{n+2} + \frac{2}{3},$$

which goes to $\frac{2}{3} = P_{\mathrm{Fix}(S)}v$.

4 General inertial Mann algorithm for *k*-strict pseudo-contractive mappings

In this section, we study the weak convergence of the general inertial Mann algorithm for k-strict pseudo-contractive mappings under the conditions (E1)–(E5).

Theorem 2 *Suppose that $S : \mathcal{H} \to \mathcal{H}$ is a k-strict pseudo-contractive mapping with at least one fixed point. Suppose that the following conditions hold:*

(E1) $\{\theta_n\} \subset [0, \theta]$ *and* $\{\phi_n\} \subset [0, \phi]$ *are nondecreasing with* $\theta_1 = \phi_1 = 0$ *and* $\theta, \phi \in [0, 1)$;

(E2) *For any* $\lambda, \xi, \psi > 0$,

$$\lambda > \frac{\theta[\eta(1+\eta) + \theta\xi]}{(1-k) - \theta^2}, \quad 1 - k \neq \theta^2,$$

$$0 < \psi \leq \psi_n \leq \frac{\lambda(1-k) - \theta[\eta(1+\eta) + \theta\lambda + \xi]}{\lambda[1 + \eta(1+\eta) + \theta\lambda + \xi]},$$

where $\eta = \max\{\theta, \phi\}$,

(E3) $k \leq 1 - \psi_n$,

(E4) $\{Sz_n - z_n\}$ *is bounded*,

(E5) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$.

Let $x_{-1}, x_0 \in C$ be arbitrary. Define a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = x_n + \phi_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \psi_n)y_n + \psi_n S z_n. \end{cases} \tag{20}$$

Then the sequence $\{x_n\}$ generated by the general Mann algorithm (20) converges weakly to a point of $\mathrm{Fix}(S)$.

Proof Take arbitrary $q \in \mathrm{Fix}(S)$. From (20), it follows that

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \big\|(1 - \psi_n)(y_n - q) + \psi_n(Sz_n - q)\big\| \\ &= (1 - \psi_n)\|y_n - q\|^2 + \psi_n\|Sz_n - q\|^2 \\ &\quad - \psi_n(1 - \psi_n)\|Sz_n - y_n\|^2 \\ &= (1 - \psi_n)\|y_n - q\|^2 + \psi_n\big[\|z_n - q\|^2 + k\|z_n - Sz_n\|^2\big] \\ &\quad - \psi_n(1 - \psi_n)\|Sz_n - y_n\|^2. \end{aligned} \tag{21}$$

Again using (20), we get

$$\begin{aligned} \|y_n - q\|^2 &= \big\|(1 + \theta_n)x_n - \theta_n(x_{n-1} - q)\big\|^2 \\ &= (1 + \theta_n)\|x_n - q\|^2 - \theta_n\|x_{n-1} - q\|^2 \\ &\quad + \theta_n(1 + \theta_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{22}$$

Similarly, we have

$$\begin{aligned} \|z_n - q\|^2 &= (1 + \phi_n)\|x_n - q\|^2 - \phi_n\|x_{n-1} - q\|^2 \\ &\quad + \phi_n(1 + \phi_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{23}$$

Substituting (22) and (23) into (21), we get

$$\begin{aligned} &\|x_{n+1} - q\|^2 - (1 + \Omega_n)\|x_n - q\|^2 + \Omega_n\|x_{n-1} - q\|^2 \\ &\quad \le k\psi_n\|Sz_n - z_n\|^2 - \psi_n(1 - \psi_n)\|Sz_n - y_n\|^2 \\ &\qquad + \big[(1 - \psi_n)\theta_n(1 + \theta_n) + \psi_n\phi_n(1 + \phi_n)\big]\|x_n - x_{n-1}\|^2, \end{aligned} \tag{24}$$

where $\Omega_n = \theta_n(1 - \psi_n) + \phi_n\psi_n$.

Observe that

$$\begin{aligned} \|Sz_n - z_n\|^2 &= \|Sz_n - y_n + y_n - z_n\|^2 \\ &\le \|Sz_n - y_n\|^2 + 2\langle Sz_n - z_n, y_n - z_n\rangle \\ &\le \|Sz_n - y_n\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|Sz_n - z_n\|. \end{aligned} \tag{25}$$

Substituting (25) into (24) and rearranging, we get

$$\begin{aligned} &\|x_{n+1}-q\|^2-(1+\Omega_n)\|x_n-q\|^2+\Omega_n\|x_{n-1}-q\|^2 \\ &\quad\le \big[(1-\psi_n)\theta_n(1+\theta_n)+\psi_n\phi_n(1+\phi_n)\big]\|x_n-x_{n-1}\|^2 \\ &\qquad+\psi_n\big[k-(1-\psi_n)\big]\|Sz_n-y_n\|^2+2k\psi_n\theta_n\|x_n-x_{n-1}\|\|Sz_n-z_n\|. \end{aligned} \tag{26}$$

Since $\psi_n \in (0,1)$ and using (E1) and (E2), we see that $\Omega_n \subset [0,\eta)$ is nondecreasing with $\Omega_1 = 0$, where $\eta = \max\{\theta,\phi\}$.

Again from (20), we get

$$\begin{aligned} \|Sz_n-y_n\|^2 &= \left\|\frac{1}{\psi_n}(x_{n+1}-x_n)+\frac{\theta_n}{\psi_n}(x_{n-1}-x_n)\right\|^2 \\ &= \frac{1}{\psi_n^2}\|x_{n+1}-x_n\|^2+\frac{\theta_n^2}{\psi_n^2}\|x_{n-1}-x_n\|^2 \\ &\quad+2\frac{\theta_n}{\psi_n^2}\langle x_{n+1}-x_n,x_{n-1}-x_n\rangle \\ &\ge \frac{1}{\psi_n^2}\|x_{n+1}-x_n\|^2+\frac{\theta_n^2}{\psi_n^2}\|x_{n-1}-x_n\|^2 \\ &\quad+\frac{\theta_n}{\psi_n^2}\left(-\nu_n\|x_{n+1}-x_n\|^2-\frac{1}{\nu_n}\|x_{n-1}-x_n\|^2\right), \end{aligned} \tag{27}$$

where $\nu_n = \frac{1}{\theta_n+\lambda\psi_n}$.

Now, substituting (27) into (26), we get

$$\begin{aligned} &\|x_{n+1}-q\|^2-(1+\Omega_n)\|x_n-q\|^2+\Omega_n\|x_{n-1}-q\|^2 \\ &\quad\le \frac{[k-(1-\psi_n)](1-\nu_n\theta_n)}{\psi_n}\|x_{n+1}-x_n\|^2+\zeta_n\|x_n-x_{n-1}\|^2 \\ &\qquad+\theta_n\|x_n-x_{n-1}\|\pi_n, \end{aligned} \tag{28}$$

where $\pi_n = 2k\psi_n\|Sz_n-z_n\|^2$ and

$$\zeta_n = (1-\psi_n)\theta_n(1+\theta_n)+\psi_n\phi_n(1+\phi_n)+\theta_n\big[k-(1-\psi_n)\big]\frac{\nu_n\theta_n-1}{\nu_n\psi_n} \ge 0, \tag{29}$$

applying condition (E3) and the fact that $\nu_n\theta_n < 1$.

We can also observe that π_n is bounded taking into account condition (E4).

Again, taking into account the choice of ν_n, we have

$$\lambda = \frac{1-\nu_n\theta_n}{\nu_n\psi_n},$$

and from (29), we have

$$\zeta_n = (1-\psi_n)\theta_n(1+\theta_n)+\psi_n\phi_n(1+\phi_n)-\theta_n\big[k-(1-\psi_n)\big]\lambda \le \eta(1+\eta)+\theta\lambda. \tag{30}$$

Next, we adapted some techniques from [2, 5] to show

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < \infty.$$

So, first, we let

$$\sigma_n = \|x_n - q\|^2$$

for all $n \geq 1$ and

$$\tau_n = \sigma_n - \Omega_n \sigma_{n-1} + \zeta_n \|x_n - x_{n-1}\|^2 + \theta_n \|x_n - x_{n-1}\| \pi_n.$$

Using the monotonicity of $\{\Omega_n\}$ and the fact that $\sigma_n \geq 0$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \tau_{n+1} - \tau_n = {} & \sigma_{n+1} - (1 + \Omega_n)\sigma_n + \Omega_n \sigma_{n-1} + \zeta_{n+1} \|x_{n+1} - x_n\|^2 - \zeta_n \|x_n - x_{n-1}\|^2 \\ & + \theta_{n+1} \|x_{n+1} - x_n\| \pi_{n+1} - \theta_n \|x_n - x_{n-1}\| \pi_n. \end{aligned} \tag{31}$$

Rearranging (28), we have

$$\begin{aligned} & \sigma_{n+1} - (1 + \Omega_n)\sigma_n + \Omega_n \sigma_{n-1} - \zeta_n \|x_n - x_{n-1}\|^2 - \theta_n \|x_n - x_{n-1}\| \pi_n \\ & \quad \leq \frac{[k - (1 - \psi_n)](1 - \nu_n \theta_n)}{\psi_n} \|x_{n+1} - x_n\|^2. \end{aligned} \tag{32}$$

Combining (31) and (32), we get

$$\begin{aligned} \tau_{n+1} - \tau_n \leq {} & \left(\frac{[k - (1 - \psi_n)](1 - \nu_n \theta_n)}{\psi_n} + \zeta_{n+1} \right) \|x_{n+1} - x_n\|^2 \\ & + \theta_{n+1} \|x_{n+1} - x_n\| \pi_{n+1}. \end{aligned} \tag{33}$$

Now, we claim that

$$\frac{[k - (1 - \psi_n)](1 - \nu_n \theta_n)}{\psi_n} + \zeta_{n+1} \leq -\xi \tag{34}$$

for each $n \in \mathbb{N}$. In other words, we are claiming that

$$(\theta_n + \lambda \psi_n)(\zeta_{n+1} + \xi) + \lambda(k + \psi_n) \leq \lambda$$

holds taking into account the upper bounds of ζ_{n+1} and ψ_n, and after substituting the expression for ν_n. Indeed, upon substitution of the upper bounds of these sequences and employing (30), we get

$$\begin{aligned} & (\theta_n + \lambda \psi_n)(\zeta_{n+1} + \xi) + \lambda(k + \psi_n) \\ & \quad \leq (\theta_n + \lambda \psi_n)\big(\eta(1 + \eta) + \theta \lambda + \xi\big) + \lambda(k + \psi_n) \\ & \quad \leq \lambda. \end{aligned} \tag{35}$$

Combining inequalities (33) and (34), we get

$$\tau_{n+1} - \tau_n \leq -\xi \|x_{n+1} - x_n\|^2 + \theta_{n+1}\|x_{n+1} - x_n\|\pi_{n+1}. \tag{36}$$

Since π_n is bounded, there exists $M_1 > 0$ such that $\pi_n \leq M_1$ for all $n \geq 1$.

$$\begin{aligned} \tau_{n+1} - \tau_n \leq & -\xi \|x_{n+1} - x_n\|^2 + \theta_{n+1}\|x_{n+1} - x_n\|\pi_{n+1} \\ \leq & -\xi \|x_{n+1} - x_n\|^2 + \theta_{n+1}\|x_{n+1} - x_n\|M_1 \\ \leq & \, \theta_{n+1}\|x_{n+1} - x_n\|M_1. \end{aligned} \tag{37}$$

Taking the summation on both sides of inequality (37), we get

$$\tau_{n+1} - \tau_1 \leq M_2, \tag{38}$$

where $M_2 = M_1 \sum_{n=2}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$ using condition (E5).

Rearranging (28), we have

$$\begin{aligned} &\|x_{n+1} - q\|^2 - \Omega_n \|x_n - q\|^2 - \frac{[k - (1 - \psi_n)](\nu_n \theta_n - 1)}{\psi_n} \|x_{n+1} - x_n\|^2 \\ &\quad \leq \|x_n - q\|^2 - \Omega_n \|x_{n-1} - q\|^2 + \zeta_n \|x_n - x_{n-1}\|^2 \\ &\qquad + \theta_n \|x_n - x_{n-1}\|\pi_n = \tau_n. \end{aligned} \tag{39}$$

From (39), we get $\sigma_{n+1} - \Omega_n \sigma_n \leq \tau_n$.

Now, since Ω_n is bounded and nondecreasing, we have

$$\sigma_{n+1} - \eta \sigma_n \leq \tau_n. \tag{40}$$

Combining inequalities (38) and (40), we obtain

$$\sigma_{n+1} - \eta \sigma_n \leq \tau_n \leq M_2 + \tau_1, \tag{41}$$

which implies that

$$\sigma_n \leq M_2 + \tau_1 + \eta \sigma_{n-1}.$$

From this proceeding inductively, we derive that

$$\sigma_n \leq \eta^n \sigma_0 + \frac{M_2 + \tau_1}{1 - \eta} \tag{42}$$

for each $n \geq 1$, where $\tau_1 = \sigma_1 \geq 0$ (due to the relation $\Omega_1 = \theta_1 = \phi_1 = 0$).

Using (40), we have

$$-\tau_n \leq -\sigma_{n+1} + \eta \sigma_n \leq \eta \sigma_n,$$

which implies that

$$-\tau_{n+1} \le \eta\sigma_{n+1} \le \eta\left[\eta^{n+1}\sigma_0 + \frac{M_2 + \tau_1}{1-\eta}\right]. \tag{43}$$

Using (37), we see that

$$\xi\|x_{n+1} - x_n\|^2 \le \tau_n - \tau_{n+1} + \theta_{n+1}\|x_{n+1} - x_n\|M_1.$$

It follows that

$$\sum_{k=1}^{\infty}\xi\|x_{n+1} - x_n\|^2 \le \left(\sum_{k=1}^{\infty}(\tau_{n+1} - \tau_n)\right) + M_2 \le \tau_1 - \tau_{n+1} + M_2. \tag{44}$$

Now, using (43) and (44), we get

$$\begin{aligned}\xi\sum_{k=1}^{\infty}\|x_{n+1} - x_n\|^2 &\le \tau_1 - \tau_{n+1} + M_2\\ &\le \tau_1 + \eta\left[\eta^{n+1}\sigma_0 + \frac{M_2 + \tau_1}{1-\eta}\right] + M_2,\end{aligned} \tag{45}$$

which implies

$$\sum_{k=1}^{\infty}\|x_{n+1} - x_n\|^2 < \infty. \tag{46}$$

Thus, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{47}$$

From (20) and (47), we see that

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\| + \theta_n\|x_n - x_{n-1}\|,$$

which in turn implies that

$$\lim_{n\to\infty}\|y_n - x_{n+1}\| = 0. \tag{48}$$

Similarly, we have

$$\lim_{n\to\infty}\|z_n - x_{n+1}\| = 0. \tag{49}$$

Now, using (20), (48), and (49), we have

$$\begin{aligned}\|Sz_n - z_n\| &\le \|Sz_n - y_n\| + \|y_n - z_n\|\\ &\le \frac{1}{\psi}\|x_{n+1} - y_n\| + \big(\|y_n - x_{n+1}\| + \|x_{n+1} - z_n\|\big),\end{aligned} \tag{50}$$

which implies that

$$\lim_{n\to\infty} \|Sz_n - z_n\| = 0. \tag{51}$$

Using (E4), (28), and (30), we see that

$$\sigma_{n+1} \le \sigma_n + \Omega_n(\sigma_n - \sigma_{n-1}) + \Gamma_n,$$

where $\Omega_n \subset [0,\eta)$ is a nondecreasing sequence and $\Gamma_n = \theta_n\|x_n - x_{n-1}\|M_1 + [\eta(1+\eta) + \theta\lambda]\|x_{n+1} - x_n\|^2$.

Using (E5) and (46), we also see that $\sum_{n=1}^{\infty}\Gamma_n < \infty$. Hence all the conditions of Lemma 1 are satisfied and therefore $\lim_{n\to\infty}\|x_n - q\|$ exists which in turn implies that $\{x_n\}$ is bounded.

Now, let x be a sequential weak cluster point of $\{x_n\}$, that is, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to x. Since $\lim_{n\to\infty}\|z_n - x_n\| = 0$, it follows that $z_{n_k} \rightharpoonup x$ as $k \to \infty$. Furthermore, we obtained that $\|Sz_n - z_n\| \to 0$ as $k \to \infty$ and hence $x \in \mathrm{Fix}(S)$ by Lemma 2. Applying now Lemma 3, we conclude that the sequence $\{x_n\}$ converges weakly to a point x in $\mathrm{Fix}(S)$. □

Remark We can drive a corollary of Theorem 2 by putting $\theta_n = 0$ in (20). Consequently, some of the conditions imposed can be avoided.

Now, we provide an example in support of Theorem 2.

Example 3 The mapping $S:\mathbb{R}\to\mathbb{R}$ defined by $Sx = -2x$ is a $\frac{1}{3}$-strict pseudo-contractive mapping. Taking $\psi_n = 0.5$, $\theta_n = 0.9$, and $\phi_n = 0.45$, algorithm (3) becomes

$$x_{n+1} = \left(-\frac{1}{2}\right)^{n-1} x_1,$$

which goes to $0 = \mathrm{Fix}(S)$ swinging around it.

5 Conclusions

In this study, we established and proved a strong convergence theorem by combining the general inertial Mann algorithm [11] with the Halpern algorithm in the setting of Hilbert spaces. We also extended the works of Dong et al. [11] by using a more general mapping, that is, a k-strict pseudo-contractive mapping in the setting of a Hilbert space. We also verified the convergence of our algorithms by using examples.

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The authors declare no competing interests.

Author contributions

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