

RESEARCH

Open Access



# A new self-adaptive method for solving resolvent of sum of two monotone operators in Banach spaces

H.A. Abass<sup>1\*</sup>, M. Aphane<sup>1</sup> and O.K. Oyewole<sup>2</sup>

\*Correspondence:

[hammed.abass@smu.ac.za](mailto:hammed.abass@smu.ac.za);  
[hammedabass548@gmail.com](mailto:hammedabass548@gmail.com)

<sup>1</sup>Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, P.O. Box 94 Medunsa 0204, Pretoria, South Africa  
Full list of author information is available at the end of the article

## Abstract

We introduce a Tseng extragradient method for solving monotone inclusion problem in Banach space. A strong convergence result of an Halpern inertial extrapolation method for solving the resolvent of sum of two monotone operators without the knowledge of the Lipschitz constant was established. Lastly, we illustrate some numerical behavior of our iterative scheme to showcase the performance of the proposed method compared to other related results in the literature.

**Mathematics Subject Classification:** 47H06; 47H09; 47J05; 47J25

**Keywords:** Monotone inclusion problem; Bregman strongly nonexpansive mapping; Lipschitz continuous; Iterative method

## 1 Introduction

Let  $\mathcal{X}$  be a real Banach space equipped with its dual space  $\mathcal{X}^*$ . The monotone inclusion problem (MIP for short) is to find a point  $u^* \in \mathcal{X}$  such that

$$0 \in (\Phi + \Psi)u^*, \quad (1.1)$$

where  $\Phi : \mathcal{X} \rightarrow \mathcal{X}^*$  is a monotone operator and  $\Psi : \mathcal{X} \rightarrow 2^{\mathcal{X}^*}$  represent a maximal monotone operator. The solution set of the MIP is denoted by  $(\Phi + \Psi)^{-1}(0^*)$ . The MIP allows an elegant formulation for a wide range of problems, which involves finding an optimal solution for optimization related problems, such as mathematical programming, optimal control, variational inequalities, and many more (see [32]). The MIP have applications in various areas of real-life problems, such as image processing, statistical regression, and signal recovery (see [13, 14, 16, 22]).

Due to the variety of applications of MIP in fixed-point theory, over the years researchers working in this direction have proposed different iterative methods for solving (1.1) (see [2, 6, 11, 30]). One of these methods is the forward-backward splitting method introduced by Lions and Mercier [18] in the settings of real Hilbert space  $\mathcal{H}$ , this method is known to be efficient for solving MIP. The forward-backward splitting method is implemented as follows:

© The Author(s) 2023. **Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Given an arbitrary starting point  $q_0 \in \mathcal{H}$ , the sequence  $\{q_k\}$  subsequently generates the next iterates as follows:

$$q_{k+1} = R_{\lambda_k}^\Psi(q_k - \lambda_k \Phi q_k), \quad k \geq 1,$$

where  $R_{\lambda_k}^\Psi := (I + \lambda_k \Psi)^{-1}$  denotes the resolvent of maximal operator  $\Psi$ ,  $I$  is the identity operator and  $\{\lambda_k\}$  is a positive real sequence. They established a weak convergence result for solving MIP (1.1) by assuming that  $\Phi$  is  $\alpha$ -inverse strongly monotone ( $\alpha$ -ism). It is well-known that the inverse strong monotonicity of  $\Phi$  is a strict condition, so it is very important to dispense with the condition in solving MIP (1.1).

To dispense with the inverse strongly monotone assumption, Tseng [25] introduced the following splitting method, known as Tseng’s splitting method, which is computed using the following procedure:

$$\begin{cases} q_1 \in \mathcal{H} \\ y_k = R_{\lambda_k}^\Psi(q_k - \lambda_k \Phi q_k), \\ q_{k+1} = y_k - \lambda_k(\Phi y_k - \Phi q_k), \quad \forall k \geq 1, \end{cases}$$

where  $\Phi : \mathcal{H} \rightarrow \mathcal{H}$  is monotone and  $L$ -Lipschitz operator,  $\Psi : \mathcal{H} \rightarrow 2^{\mathcal{H}}$  is a multi-valued operator, and  $\{\lambda_k\}$  is a sequence in  $(0, \frac{1}{L})$ , where  $L$  is a Lipschitz constant.

*Remark 1.1* Though, Tseng [25] could dispense with the inverse strong monotonicity assumption on  $\Phi$ . The limitation of the Tseng’s method proposed above requires the prior knowledge of the Lipschitz constant of the underlying operator. However, from practical point of view, the Lipschitz constant in this case is very difficult to compute.

In 2019, Shehu [24] extended the Tseng’s [25] iterative method to the setting of 2-uniformly convex Banach space which is also uniformly smooth. He established a weak convergence result with the prior knowledge of the Lipschitz constant.

Very recently, Sunthrayuth et al. [23] extended the result of Shehu [24] to the setting of a reflexive Banach space. They [23] proposed two different iterative methods that do not require prior knowledge of the Lipschitz constants for solving MIP and fixed-point problem for a Bregman relatively nonexpansive mapping. One of their iterative methods uses the linesearch procedure, which would necessitate numerous additional computations and further lower the computational cost of their algorithm, while the other one uses a self-adaptive procedure, which looks more efficient. The method that uses the self-adaptive step-size is defined below:

---

**Algorithm 1.2** *Mann splitting algorithm for solving MIP.*

**Initialization:** Choose  $\lambda^1 > 0, \mu, \theta \in (0, \sigma)$ , where  $\sigma$  is a constant defined in (2.3). Let  $q^1 \in \mathcal{E}$  be arbitrary starting points.

**Iterative step:**

**Step 1:** Compute

$$w^k = R_{\lambda^k}^\Psi \nabla g^*(\nabla g(q^k) - \lambda^k \Phi(q^k)).$$

**Step 2:** Compute

$$z^k = \nabla g^*(\nabla g(w^k) - \lambda^k(\Phi(w^k) - \Phi(q^k))),$$

where  $\lambda^{k+1}$  is updated as follows:

$$\lambda^{k+1} = \begin{cases} \min\{\frac{\mu\|q^k - w^k\|}{\|\Phi q^k - \Phi w^k\|}, \lambda^k\} & \text{if } \Phi q^k \neq \Phi w^k, \\ \lambda^k & \text{otherwise.} \end{cases} \tag{1.2}$$

**Step 3:** Compute

$$q^{k+1} = \nabla g^*((1 - \alpha^k)\nabla g(z^k) + \alpha^k\nabla g(Tz^k))$$

**Stopping criterion** If  $q^{k+1} = z^k$  for some positive  $k$ , then stop. Otherwise set  $k := k + 1$  and return to **Iterative step**.

---

A weak convergence result was obtained using their iterative algorithm without any prior knowledge of the Lipschitz constant of the underlying operator.

Inspired by the heavy-ball methods of a two-order time dynamical system, Polyak [21] and Nesterov [20] proposed the following inertial method:

$$\begin{cases} u^k = q^k + \theta^k(q^k - q^{k-1}), \\ q^{k+1} = u^k - \lambda^k \nabla f(u^k), \quad \forall k \geq 1, \end{cases} \tag{1.3}$$

where  $\theta^k \in [0, 1)$  is simply the inertial and  $\lambda^k$  is a positive sequence (see [2–4, 19, 28, 31]).

Very recently, Abass et al. [1] proposed the following modified inertial method for approximating solution of systems of MIP and fixed-point problem for a finite family of multi-valued Bregman relatively nonexpansive as follows:

$$\begin{cases} u_k = \nabla h^*(\nabla h(q_k) + \theta_k(\nabla h(q_{k-1}) - \nabla h(q_k))) \\ w_k = \nabla h^*(R_\phi^N \circ R_\phi^{N-1} \circ \dots \circ R_\phi^1(u_k)) \\ y_k = \nabla h^*(\delta_{k,0}\nabla h(w_k) + \sum_{r=1}^N \delta_{k,r}\nabla h(z_{k,r})), \quad z_{k,r} \in S_r w_k \\ x_{k+1} = \nabla h^*(\alpha_k \nabla h(u) + \beta_k \nabla h(x_k) + \gamma_k \nabla h(y_k)), \end{cases}$$

where  $\{\theta_k\} \subset [0, \frac{1}{2}]$ ,  $\{\alpha_k\}$ ,  $\{\beta_k\}$ ,  $\{\phi_k\}$  and  $\{\delta_{k,r}\}$  are sequences in  $(0,1)$  such that  $\alpha_k + \beta_k + \delta_k = 1$ . They established a strong convergence result of their method for solving the aforementioned problems.

Motivated by the results in [23–25] and results from related literature in this direction, we develop a new self-adaptive method equipped with an inertial Halpern method for solving MIP in real Banach space. We establish a strong convergence result for solving MIP without the knowledge of the Lipschitz constant of the underlying operator. Lastly, we illustrate few numerical experiments in comparison with other related ones in the literature. Our result is a further contribution to related results in the literature.

We highlight some of the contributions in this study:

- (i) Results from [24, 25] were extended to a more general Banach spaces.

- (ii) Introduction of a self-adaptive procedure, which increase from iteration to iteration and is independent of the Lipschitz constant of underlying operator, is studied. This differs from the methods of Shehu [24] and Tseng [25], where the knowledge of Lipschitz constant is required.
- (iii) A strong convergence result desirable to weak convergence result was established (see [23]).
- (iv) The result discussed in [1] is a special of MIP (1.1) if  $\Phi \equiv 0$ .
- (v) We employ the inertial method as introduced by Polyak [21], which is quite different from the ones in [1, 29] (i.e.  $\theta_k(q_{k-1} - q_k)$ ) was changed to  $\theta_k(q_k - q_{k-1})$ .

## 2 Preliminaries

In this section, we denote strong and weak convergence by “ $\rightarrow$ ” and “ $\rightharpoonup$ ”, respectively.

Let  $K$  be a nonempty closed and convex subset of a real Banach space  $\mathcal{X}$ . Let  $h : \mathcal{X} \rightarrow (-\infty, +\infty]$  be a proper, lower semicontinuous and convex function, then the Fenchel conjugate of  $h$ , denoted by  $h^* : \mathcal{X}^* \rightarrow (-\infty, +\infty]$  is defined by

$$h^*(u^*) = \sup\{\langle u^*, u \rangle - h(u) : u \in \mathcal{X}\}, \quad u^* \in \mathcal{X}^*.$$

The domain of  $h$  be denoted as  $\text{dom}(h) = \{u \in \mathcal{X} : h(u) < +\infty\}$ , thus for any  $u \in \text{intdom}(h)$  and  $v \in \mathcal{X}$ , the right-hand derivative of  $h$  at  $x$  in the direction of  $v$  is defined by

$$h^0(u, v) = \lim_{t \rightarrow 0^+} \frac{h(u + tv) - h(u)}{t}.$$

The function  $h$  is said to be

- (i) Gâteaux differentiable at  $u$  if  $\lim_{t \rightarrow 0^+} \frac{h(u+tv) - h(u)}{t}$  exists for any  $v$ . In this case,  $h^0(u, v)$  coincides with  $\nabla h(u)$ ;
- (ii) Gâteaux differentiable, if it is Gâteaux differentiable for any  $u \in \text{intdom}(h)$ ;
- (iii) Fréchet differentiable at  $u$ , if its limit is attained uniformly in  $\|v\| = 1$ ;
- (iv) Uniformly Fréchet differentiable on a subset  $K$  of  $\mathcal{X}$ , if the above limit is attained uniformly for  $u \in K$  and  $\|v\| = 1$ .

Let  $h : \mathcal{X} \rightarrow (-\infty, +\infty]$  be a mapping, then  $h$  is said to be:

- (i) essentially smooth, if the subdifferential of  $h$  denoted as  $\partial h$  is both locally bounded and single-valued on its domain, where  $\partial h(u) = \{w \in \mathcal{X} : h(u) - h(v) \geq \langle w, v - u \rangle, v \in \mathcal{X}\}$ ;
- (ii) essentially strictly convex, if  $(\partial h)^{-1}$  is locally bounded on its domain and  $h$  is strictly convex on every convex subset of  $\text{dom } \partial h$ ;
- (iii) Legendre, if it is both essentially smooth and essentially strictly convex. See [8, 9] for more details on Legendre functions.

Alternatively, a function  $h$  is said to be Legendre if it satisfies the following conditions:

- (i) The  $\text{intdom } h$  is nonempty,  $h$  is Gâteaux differentiable on  $\text{intdom } h$  and  $\text{dom } \nabla h = \text{intdom } h$ ;
- (ii) The  $\text{intdom } h^*$  is nonempty,  $h^*$  is Gâteaux differentiable on  $\text{intdom } h^*$  and  $\text{dom } \nabla h^* = \text{intdom } h$ .

If  $B_s := \{z \in \mathcal{E} : \|z\| \leq s\}$  for all  $s > 0$ . Then, a function  $h : \mathcal{X} \rightarrow \mathbb{R}$  is called uniformly convex on bounded subsets of  $\mathcal{X}$ , [see pp. 203 and 221] [33] if  $\rho_s t > 0$  for all  $s, t > 0$ , where  $\rho_s :$

$[0, +\infty) \rightarrow [0, \infty]$  is defined by

$$\rho_s(t) = \inf_{x,y \in B_s, \|x-y\|=t, \alpha \in (0,1)} \frac{\alpha h(x) + (1-\alpha)h(y) - h(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)},$$

for all  $t \geq 0$ , with  $\rho_s$  denoting the gauge of uniform convexity of  $h$ . The function  $h$  is also said to be uniformly smooth on bounded subsets of  $\mathcal{X}$ , [33, see p. 221], if  $\lim_{t \downarrow 0} \frac{\sigma_s}{t}$  for all  $s > 0$ , where  $\sigma_s : [0, +\infty) \rightarrow [0, \infty]$  is defined by

$$\sigma_s(t) = \sup_{x \in B, y \in S_{\mathcal{X}}, \alpha \in (0,1)} \frac{\alpha h(x) + (1-\alpha)ty + (1-\alpha)h(x-\alpha ty) - h(x)}{\alpha(1-\alpha)},$$

for all  $t \geq 0$ . The function  $h$  is said to be uniformly convex if the function  $\delta h : [0, +\infty) \rightarrow [0, +\infty)$  defined by

$$\delta h(t) := \sup \left\{ \frac{1}{2}h(x) + \frac{1}{2}h(y) - h\left(\frac{x+y}{2}\right) : \|y-x\| = t \right\},$$

satisfies  $\lim_{t \downarrow 0} \frac{\delta h(t)}{t} = 0$ .

**Definition 2.1** [10] Let  $h : \mathcal{X} \rightarrow (-\infty, +\infty]$  be a convex and Gâteaux differentiable function. Then, the function  $G_h : \mathcal{X} \times \mathcal{X} \rightarrow [0, +\infty)$  defined by

$$G_h(u, v) := h(u) - h(v) - \langle h\nabla(v), u - v \rangle \tag{2.1}$$

is called the Bregman distance with respect to  $h$ , where  $u, v \in \mathcal{X}$ .

However, the Bregman distance satisfies the following three point identity: for any  $u \in \text{dom}(h)$  and  $v, z \in \text{int dom}(h)$ ,

$$G_h(u, v) + G_h(v, z) - G_h(u, z) = \langle \nabla h(z) - \nabla h(v), u - v \rangle. \tag{2.2}$$

Also the relationship between  $G_h$  and  $\|\cdot\|$  with strong convexity constant  $\sigma > 0$ , i.e.,

$$G_h(x, y) \geq \frac{\sigma}{2} \|x - y\|^2, \quad \forall x \in \text{dom}(h), y \in \text{int}(\text{dom}(h)). \tag{2.3}$$

Let  $\mathcal{U} : K \rightarrow \text{int}(\text{dom } h)$  be a nonlinear operator. An element  $p \in K$  is said to be a fixed point of  $\mathcal{U}$  if  $\mathcal{U}p = p$ . We denote by  $F(\mathcal{U})$  fixed point sets of the operator  $\mathcal{U}$ . In addition, a point  $p \in K$  is said to be an asymptotic fixed point of  $\mathcal{U}$  if  $K$  contains a sequence  $\{x^k\}$  such that  $\{x^k\} \rightarrow p$  and  $\lim_{k \rightarrow \infty} \|\mathcal{U}x^k - x^k\| = 0$ . We denote by  $\hat{F}(\mathcal{U})$  the asymptotic fixed-point sets of  $\mathcal{U}$ .

An operator  $M : K \rightarrow \mathcal{X}$  is said to be monotone if

- (i)  $\langle x - y, Mx - My \rangle \geq 0, \forall x, y \in K$ .
- (ii)  $\mathcal{L}$ -Lipschitz continuous if there exists a constant  $L > 0$  such that  $\|Mx - My\| \leq L\|x - y\|, \forall x, y \in K$ .
- (iii) Bregman quasi-nonexpansive, if  $F(M) \neq \emptyset$ , and

$$G_h(p, Mx) \leq G_h(p, x), \quad \forall p \in F(M), x \in K.$$

For a set-valued operator  $M: \mathcal{X} \rightarrow \mathcal{X}^*$ , the domain, range and graph are defined as follows:  $\text{dom}(M) := \{u \in \mathcal{X} : Mu \neq \emptyset\}$ ,  $\text{Ran } M := \bigcup \{Mu : u \in \text{dom}(M)\}$  and  $\text{Gra}(M) := \{(u, u^*) \in \mathcal{X} \times \mathcal{X}^* : u^* \in Mu\}$ , respectively. An operator  $M$  is said to be monotone if for each  $(u, u^*), (v, v^*) \in \text{Gra}(M)$ , we get  $\langle u - v, u^* - v^* \rangle \geq 0$ . A monotone operator  $M$  is said to be maximal, if its graph is not contained in the graph of any other monotone operator on  $\mathcal{X}$ . It is known that if  $h : \mathcal{X} \rightarrow \mathbb{R}$  is Gâteaux differentiable, strictly convex and co-finite, then  $M$  is maximal monotone if and only if  $\text{Ran}(\nabla h + \lambda M) = \mathcal{X}^*$ .

Let  $h : \mathcal{X} \rightarrow (-\infty, \infty]$  be Fréchet differentiable function, which is bounded on bounded subsets of  $\mathcal{X}$ , and  $M$  be a maximal monotone operator, then the resolvent of  $M$  for  $\lambda > 0$  defined by

$$R_\lambda^M(u) := (\nabla h + \nabla M)^{-1} \nabla h(u), \quad \forall u \in \mathcal{X}$$

is a single-valued Bregman quasi-nonexpansive mapping from  $\mathcal{X}$  onto  $\text{dom}(M)$  with  $F(J_\lambda^M) = M^{-1}(0)$ .

**Definition 2.2** A function  $h : \mathcal{X} \rightarrow \mathbb{R}$  is called strongly coercive if

$$\lim_{\|q^k\| \rightarrow \infty} \frac{h(q^k)}{\|q^k\|} = \infty.$$

**Lemma 2.3** [12] Let  $h : \mathcal{X} \rightarrow \mathbb{R}$  be a strongly coercive Bregman function and  $V$  be a function defined by

$$V(u, u^*) = h(u) - \langle u, u^* \rangle + h^*(u^*), \quad x \in \mathcal{X}, u^* \in \mathcal{X}^*.$$

Then the following holds:

$$G_h(u, \nabla h^*(u^*)) = V(u, u^*), \quad \text{for all } u \in \mathcal{X} \text{ and } u^* \in \mathcal{X}^*.$$

$$V(u, u^*) + \langle \nabla h^*(u^*) - u, v^* \rangle \leq V(u, u^* + v^*) \quad \text{for all } u \in \mathcal{X} \text{ and } u^*, v^* \in \mathcal{X}^*.$$

**Lemma 2.4** [12] Let  $h : \mathcal{X} \rightarrow \mathbb{R}$  a Gâteaux differentiable function which is uniformly convex on bounded subsets of  $\mathcal{X}$ . Suppose  $\{w^k\}_{k \in \mathbb{N}}$  and  $\{v^k\}_{k \in \mathbb{N}}$  are bounded sequences in  $\mathcal{X}$ . Then,

$$\lim_{k \rightarrow \infty} G_h(w^k, v^k) = 0 \Rightarrow \lim_{k \rightarrow \infty} \|v^k - w^k\| = 0.$$

**Lemma 2.5** [17] Suppose  $h : \mathcal{X} \rightarrow \mathbb{R}$  is a Gâteaux differentiable function which is uniformly convex on bounded subsets of  $E$ . If  $u_0 \in \mathcal{E}$  and the sequence  $\{G_h(u^k, u_0)\}$  is bounded, then the sequence  $\{u^k\}$  is also bounded.

**Definition 2.6** Let  $K$  be a nonempty closed and convex subset of  $\mathcal{X}$ . A Bregman projection of  $x \in \text{int}(\text{dom } h)$  onto  $K \subset \text{int}(\text{dom } h)$  is the unique vector  $\text{Proj}_K^h(x) \in K$  satisfying

$$G_h(\text{Proj}_K^h(u), u) = \inf \{G_h(v, u) : v \in K\}.$$

The Bregman projection is characterized by the following identities given in the following Lemma:

**Lemma 2.7** [26] *Let  $K$  be a nonempty closed and convex subset of a reflexive Banach space  $\mathcal{X}$  and  $x \in \mathcal{X}$ . Let  $h : \mathcal{X} \rightarrow \mathbb{R}$  be a Gâteaux differentiable and totally convex function. Then,*

- (i)  $q = \text{Proj}_K^h(u)$  if and only if  $\langle \nabla g(u) - \nabla g(q), v - q \rangle \leq 0, \forall v \in K$ .
- (ii)  $G_h(v, \text{Proj}_K^h(u)) + G_h(\text{Proj}_K^h(u), u) \leq G_h(v, u), \forall v \in K$ .

**Lemma 2.8** [7] *Let  $\mathcal{X}$  be a real Banach space and  $\Phi : \mathcal{X} \rightarrow \mathcal{X}^*$  be a monotone, hemicontinuous and bounded operator. Suppose  $\Psi : \mathcal{X} \rightarrow \mathcal{X}^*$  is a maximal monotone operator. Then  $\Phi + \Psi$  is maximal monotone.*

**Lemma 2.9** [5] *Let  $\{u_n\}, \{\beta_n\}$ , and  $\{\alpha_n\}$  be sequences in  $[0, \infty)$  such that*

$$u_{n+1} \leq u_n + \alpha_n(u_n - u_{n-1}) + \beta_n$$

for all  $n \geq 1, \sum_{n=1}^\infty \beta_n < \infty$  and there exists a real number  $\alpha$  with  $0 \leq \alpha_n \leq \alpha < 1$ , for all  $n \in \mathbb{N}$ . Then, the following hold:

- (i)  $\sum_{n \geq 1} [u_n - u_{n-1}]_+$  where  $t_+ = \max\{0, t\}$ ;
- (ii) there exists  $u^* \in [0, \infty)$  such that  $\lim_{n \rightarrow \infty} u_n = u^*$ .

**Lemma 2.10** [27] *Let  $\{u_n\}$  be a sequence of nonnegative real numbers,  $\{\alpha_n\}$  be a sequence of real numbers in  $(0, 1)$  such that  $\sum_{n=1}^\infty \alpha_n = \infty$ , and  $\{v_n\}$  be a sequence of real numbers. Assume that*

$$u_{n+1} \leq (1 - \alpha_n)u_n + \alpha_n v_n \quad \forall n \geq 1.$$

If  $\limsup_{k \rightarrow \infty} v_{n_k} \leq 0$  for every subsequence  $\{u_{n_k}\}$  of  $\{u_n\}$  satisfying the condition

$$\liminf_{k \rightarrow \infty} (u_{n_{k+1}} - u_{n_k}) \geq 0,$$

then  $\lim_{n \rightarrow \infty} u_n = 0$ .

### 3 Main result

Now, we present a new self-adaptive method for solving resolvent of sum of two monotone operators. Below are some important assumptions:

#### Assumption 3.1

- (R1) Let  $\mathcal{X}$  be a real Banach space equipped with its dual space  $\mathcal{X}^*$ . Let  $\Phi : \mathcal{X} \rightarrow \mathcal{X}^*$  be a monotone and  $\mathcal{L}$ -Lipschitz continuous mapping and  $\Psi : \mathcal{X} \rightarrow 2^{\mathcal{X}^*}$  be a maximal monotone mapping.
- (R2)  $h : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\}$  is a function which is Legendre, uniformly Fréchet differentiable, strongly coercive, uniformly convex,  $\varphi$ -strongly convex, and bounded on bounded subsets of  $\mathcal{X}$ .
- (R3) The solution set  $\Delta := (\Phi + \Psi)^{-1}(0)$  is nonempty.

---

**Algorithm 3.2** *MIP and its convergence analysis.*

**Initialization:** Given  $\kappa_1 > 0, \sigma > 0, \rho > 0$ , and  $\omega \in (0, 1)$ . Let  $v, q_0, q_1 \in \mathcal{X}$  and  $\psi_k$  be a sequence in  $(0, 1)$  such that  $\lim_{k \rightarrow \infty} \psi_k = 0$  and  $\sum_{k=1}^{\infty} \psi_k = \infty$  for all  $k \in \mathbb{N}$ .

**Step 1:** Given  $q_{k-1}, q_k$ , choose  $\sigma_k$  such that  $\sigma_k \in [0, \bar{\sigma}_k]$ , where

$$\bar{\sigma}_k = \begin{cases} \min\{\sigma, \frac{\pi_k}{\phi_r(\|\nabla h(q_k) - \nabla h(q_{k-1})\|)}, \frac{\pi_k}{G_h(q_k, q_{k-1})}\}, & \text{if } q_k \neq q_{k-1}, \\ \sigma, & \text{otherwise.} \end{cases} \tag{3.1}$$

and  $\{\pi_k\}$  is a sequence of nonnegative numbers such that  $\pi_k = o(\psi_k)$ , that is  $\lim_{k \rightarrow \infty} \frac{\pi_k}{\psi_k} = 0$ .

Compute

$$\begin{cases} r_k = \nabla h^*(\nabla h(q_k) + \sigma_k(\nabla h(q_k) - \nabla h(q_{k-1}))) \\ s_k = R_{\kappa_k}^{\Psi} \nabla h^*(\nabla h(r_k) - \kappa_k \Phi(r_k)) \end{cases} \tag{3.2}$$

**Step 2:** Compute

$$t_k = \nabla h^*(\nabla h(s_k) - \kappa_k(\Phi(s_k) - \Phi(r_k))), \tag{3.3}$$

and  $\kappa_{k+1}$  is defined as

$$\kappa_{k+1} = \begin{cases} \min\{\kappa_k, \frac{\mu(\rho G_h(s_k, r_k) + \frac{1}{\rho} G_h(s_k, t_k))}{\langle s_k - t_k, \Phi(s_k) - \Phi(r_k) \rangle}\}, & \text{if } \langle s_k - t_k, \Phi(s_k) - \Phi(r_k) \rangle \neq 0. \\ \kappa_k, & \text{otherwise.} \end{cases} \tag{3.4}$$

**Step 3:** Compute

$$q_{k+1} = \nabla h^*(\psi_k \nabla h(v) + (1 - \psi_k) \nabla h(t_k)) \tag{3.5}$$

**Stopping criterion** If  $q_{k+1} = r_k$  for  $k \geq 1$ , then stop. Otherwise set  $k := k + 1$  and return to **step 1**.

---

**Lemma 3.3** Let  $\{q_k\}$  be a sequence generated by Algorithm 3.2. Then the sequence  $\{\kappa_k\}$  is nonincreasing and

$$\lim_{k \rightarrow \infty} \kappa_k = \kappa \geq \min\left\{\kappa_1, \frac{\omega \varphi}{\mathcal{L}}\right\}.$$

*Proof* It is obvious from self-adaptive stepsize in step 2 of (3.2) that  $\kappa_{k+1} \leq \kappa_k$  for all  $k \geq 1$ . If  $\langle s_k - t_k, \Phi(s_k) - \Phi(r_k) \rangle = 0$ , then  $\{\kappa_k\}$  is a constant sequence and thus the conclusion holds. Otherwise, we have

$$\begin{aligned} \frac{\omega(\rho G_h(s_k, r_k) + \frac{1}{\rho} G_h(s_k, t_k))}{\langle s_k - t_k, \Phi(s_k) - \Phi(r_k) \rangle} &\geq \frac{\frac{\omega \varphi}{2}(\rho \|s_k - r_k\|^2 + \frac{1}{\rho} \|s_k - t_k\|^2)}{\|s_k - t_k\| \|\Phi(s_k) - \Phi(r_k)\|} \\ &\geq \frac{\omega \varphi \|s_k - r_k\| \|s_k - t_k\|}{\mathcal{L} \|s_k - t_k\| \|s_k - r_k\|} \end{aligned}$$



$$= \frac{\omega\varphi}{\mathcal{L}}.$$

Clearly

$$\kappa_{k+1} \geq \min \left\{ \kappa_k, \frac{\omega\varphi}{\mathcal{L}} \right\}.$$

By induction, we deduce that  $\kappa_k \geq \min\{\kappa_1, \frac{\omega\varphi}{\mathcal{L}}\}$ . Thus  $\lim_{k \rightarrow \infty} \kappa_k = \kappa \geq \min\{\kappa_1, \frac{\omega\varphi}{\mathcal{L}}\}$ . The proof completes.  $\square$

**Lemma 3.4** *Given that  $\kappa > 0$ , if  $s_k = r_k = q_{k+1}$  for some  $k > 0$ , then  $r_k \in \Delta$ .*

*Proof* Given that  $\kappa > 0$ , if  $s_k = r_k$ , then  $r_k = R_{\kappa_k}^{\Psi} \nabla h^*(\nabla h(r_k) - \kappa_k \Phi(r_k))$ . Thus,  $r_k = (\nabla h + \kappa_k \Psi)^{-1} \circ \nabla h^*(\nabla h - \kappa_k \Phi)r_k$ , that is  $\nabla h(r_k) - \kappa_k \Phi(r_k)$ , which implies that  $0 \in (\Phi + \Psi)r_k$ . Hence,  $r_k \in (\Phi + \Psi)^{-1}(0)$ . Since  $s_k = r_k$  and  $\nabla h$  is injective, we obtain from step 2 of Algorithm (3.2) that  $t_k = r_k$ . Therefore, we conclude that  $r_k \in \Delta := (\Phi + \Psi)^{-1}(0)$ .  $\square$

**Lemma 3.5** *Let  $\{r_k\}$  be a sequence generated iteratively by (3.2). Then*

$$G_h(u, t_k) \leq G_h(u, r_k) - \left(1 - \frac{\omega\rho\kappa_k}{\sigma_{k+1}}\right) G_h(s_k, r_k) - \left(1 - \frac{\omega\kappa_k}{\rho\sigma_{k+1}}\right) G_h(s_k, t_k).$$

*Proof* Let  $u \in \Delta$ , thus by applying the definition of Bregman distance, we deduce that

$$\begin{aligned} G_h(u, t_k) &= G_h(u, \nabla h^*(\nabla h(s_k) - \kappa_k(\Phi(s_k) - \Phi(r_k)))) \\ &= h(u) - h(t_k) - \langle u - t_k, \nabla h(s_k) - \kappa_k(\Phi(s_k) - \Phi(r_k)) \rangle \\ &= h(u) - h(t_k) - \langle u - t_k, \nabla h(s_k) \rangle + \kappa_k \langle u - t_k, \Phi(s_k) - \Phi(r_k) \rangle \\ &= h(u) - h(s_k) - \langle u - s_k, \nabla h(s_k) \rangle + \langle u - t_k, \nabla h(s_k) \rangle + h(s_k) - h(t_k) \\ &\quad - \langle u - t_k, \nabla h(s_k) \rangle + \kappa_k \langle u - t_k, \Phi(s_k) - \Phi(r_k) \rangle \\ &= h(u) - h(s_k) - \langle u - s_k, \nabla h(s_k) \rangle - h(t_k) + h(s_k) + \langle t_k - s_k, \nabla h(s_k) \rangle \\ &\quad + \kappa_k \langle u - t_k, \Phi(s_k) - \Phi(r_k) \rangle \\ &= G_h(u, s_k) - G_h(t_k, s_k) + \kappa_k \langle u - t_k, \Phi(s_k) - \Phi(r_k) \rangle. \end{aligned} \tag{3.6}$$

From (2.2), we get

$$G_h(u, s_k) = G_h(u, r_k) - G_h(s_k, r_k) + \langle u - s_k, \nabla h(r_k) - \nabla h(s_k) \rangle. \tag{3.7}$$

By combining (3.6) and (3.7), we obtain

$$\begin{aligned} G_h(u, t_k) &= G_h(u, r_k) - G_h(s_k, r_k) - G_h(t_k, s_k) + \langle u - s_k, \nabla h(r_k) - \nabla h(s_k) \rangle \\ &\quad - \kappa_k \langle u - t_k, \Phi(s_k) - \Phi(r_k) \rangle \\ &= G_h(u, r_k) - G_h(s_k, r_k) - G_h(t_k, s_k) + \langle u - s_k, \nabla h(r_k) - \nabla h(s_k) \rangle \\ &\quad + \kappa_k \langle s_t - K_t, \Phi(s_k) - \Phi(r_k) \rangle - \kappa_k \langle s_k - u, \Phi(s_k) - \Phi(r_k) \rangle \end{aligned}$$

$$\begin{aligned}
 &= G_h(u, r_k) - G_h(s_k, r_k) - G_h(t_k, s_k) + \kappa_k \langle s_k - t_k, \Phi(s_k) - \Phi(r_k) \rangle \\
 &\quad - \langle s_k - u, \nabla h(r_k) - \nabla h(s_k) - \kappa_k (\Phi(r_k) - \Phi(s_k)) \rangle.
 \end{aligned}
 \tag{3.8}$$

From the definition of  $s_k$ , we obtain  $\nabla h(r_k) - \kappa_k (\Phi(r_k)) \in \nabla h(s_k) + \kappa_k \Psi(s_k)$ . Since  $\Psi$  is maximal monotone, there exists  $d_k \in \Psi(s_k)$  such that  $\nabla h(r_k) - \kappa_k \Phi(r_k) = \nabla h(s_k) + \kappa_k d_k$ , thus it follows that

$$d_k = \frac{1}{\kappa_k} (\nabla h(r_k) - \nabla h(s_k) - \kappa_k \Phi(s_k)).
 \tag{3.9}$$

Using the fact that  $0 \in (\Phi + \Psi)u$  and  $\Phi(s_k) + d_k \in (\Phi + \Psi)s_k$ , it follows from Lemma 2.8 that

$$\langle s_k - u, \Phi(s_k) + d_k \rangle \geq 0.
 \tag{3.10}$$

By substituting (3.9) and (3.10), we get

$$\frac{1}{\kappa_k} \langle s_k - u, \nabla h(r_k) - \nabla h(s_k) - \kappa_k \Phi(r_k) + \kappa_k \Phi(s_k) \rangle \geq 0.$$

That is

$$\langle s_k - u, \nabla h(r_k) - \nabla h(s_k) - \kappa_k (\Phi(r_k) - \Phi(s_k)) \rangle \geq 0.
 \tag{3.11}$$

By combining (3.8) and (3.11), we get

$$\begin{aligned}
 G_h(u, t_k) &\leq G_h(u, r_k) - G_h(s_k, r_k) - G_h(t_k, s_k) \\
 &\quad + \kappa_k \langle s_k - t_k, \Phi(s_k) - \Phi(r_k) \rangle \\
 &\leq G_h(u, r_k) - G_h(s_k, r_k) - G_h(t_k, s_k) \\
 &\quad + \frac{\omega \kappa_k}{\kappa_{k+1}} \left( \rho G_h(s_k, r_k) + \frac{1}{\rho} G_h(s_k, t_k) \right) \\
 &= G_h(u, r_k) - \left( 1 - \frac{\omega \rho \kappa_k}{\kappa_{k+1}} \right) G_h(s_k, r_k) - \left( 1 - \frac{\omega \kappa_k}{\rho \kappa_{k+1}} \right) G_h(s_k, t_k).
 \end{aligned}
 \tag{3.12}$$

It is obvious from Lemma 3.3 that

$$\lim_{k \rightarrow \infty} \left( 1 - \frac{\omega \rho \kappa_k}{\kappa_{k+1}} \right) = 1 - \omega \rho > 0,$$

and

$$\lim_{k \rightarrow \infty} \left( 1 - \frac{\omega \kappa_k}{\rho \kappa_{k+1}} \right) = 1 - \frac{\omega}{\rho} > 0.$$

Hence, there exists a positive integer  $N_2 > 0$  such that

$$1 - \frac{\omega \rho \kappa_k}{\kappa_{k+1}} > 0, 1 - \frac{\omega \kappa_k}{\rho \kappa_{k+1}} > 0, \quad \forall k \geq N_2.$$

Therefore, we conclude from (3.12) that

$$G_h(u, t_k) \leq G_h(u, r_k). \tag{3.13}$$

□

**Lemma 3.6** *Suppose  $\{q_k\}$  is as defined in (3.2), then the sequences  $\{q_k\}, \{r_k\}, \{s_k\}$  and  $\{t_k\}$  are all bounded.*

*Proof* Let  $u \in \Delta$  and  $g_k = \nabla h(q_k) + \sigma_k(\nabla h(q_k) - \nabla h(q_{k-1}))$ . Then, we can write  $r_k$  as  $r_k = \nabla h^*(g_k)$ . Thus

$$\begin{aligned} G_h(u, r_k) &= G_h(u, \nabla h^*(g_k)) \\ &= h(u) - \langle u, g_k \rangle + h^*(g_k) \\ &= h(u) - \langle u, \nabla h(q_k) + \sigma_k(\nabla h(q_k) - \nabla h(q_{k-1})) \rangle \\ &\quad + h^*(\nabla h(q_k) + \sigma_k(\nabla h(q_k) - \nabla h(q_{k-1}))) \\ &= h(u) - \langle u, \nabla h(q_k) \rangle + (1 + \sigma_k)h^*(\nabla h(q_k)) - \sigma_k h^*(\nabla h(q_{k-1})) \\ &\quad + \sigma_k(1 + \sigma_k)\phi_r(\|\nabla h(q_k) - \nabla h(q_{k-1})\|) \\ &\quad - \langle u, \sigma_k(\nabla h(q_k) - \nabla h(q_{k-1})) \rangle \\ &= h(u) - \langle u, \nabla h(q_k) \rangle + h^*(\nabla h(q_k)) + \sigma_k(h^*(\nabla h(q_k)) - h^*(\nabla h(q_{k-1}))) \\ &\quad - \langle u, \sigma_k(\nabla h(q_k) - \nabla h(q_{k-1})) \rangle \\ &\quad + \sigma_k(1 + \sigma_k)\phi_r(\|\nabla h(q_k) - \nabla h(q_{k-1})\|) \\ &= G_h(u, q_k) + \sigma_k(h^*(\nabla h(q_k)) - h^*(\nabla h(q_{k-1}))) - \langle u, \sigma_k(\nabla h(q_k) - \nabla h(q_{k-1})) \rangle \\ &\quad + \sigma_k(1 + \sigma_k)\phi_r(\|\nabla h(q_k) - \nabla h(q_{k-1})\|) \\ &\leq G_h(u, q_k) + \sigma_k\langle q_k, \nabla h(q_k) - \nabla h(q_{k-1}) \rangle - \langle u, \sigma_k(\nabla h(q_k) - \nabla h(q_{k-1})) \rangle \\ &\quad + \sigma_k(1 + \sigma_k)\phi_r(\|\nabla h(q_k) - \nabla h(q_{k-1})\|) \\ &= G_h(u, q_k) + \sigma_k\langle u - q_k, \nabla h(q_{k-1}) - \nabla h(q_k) \rangle \\ &\quad + \sigma_k(1 + \sigma_k)\phi_r(\|\nabla h(q_k) - \nabla h(q_{k-1})\|) \\ &= G_h(u, q_k) + \sigma_k(G_h(u, q_k) - G_h(u, q_{k-1})) + \sigma_k G_h(q_k, q_{k-1}) \\ &\quad + \sigma_k(1 + \sigma_k)\phi_r(\|\nabla h(q_k) - \nabla h(q_{k-1})\|). \end{aligned} \tag{3.14}$$

We deduce from step 3 of (3.2), (3.13), and (3.14) that

$$\begin{aligned} G_h(u, q_{k+1}) &= G_h(u, \nabla h^*(\psi_k \nabla h(v) + (1 - \psi_k) \nabla h(t_k))) \\ &\leq \psi_k G_h(u, v) + (1 - \psi_k) G_h(u, t_k) \\ &\leq \psi_k G_h(u, v) + (1 - \psi_k) G_h(u, r_k) \\ &\leq \psi G_h(u, v) + (1 - \psi_k)(G_h(u, q_k) \\ &\quad + \sigma_k(G_h(u, q_k) - G_h(u, q_{k-1})) + \sigma_k G_h(q_k, q_{k-1})) \end{aligned}$$

$$\begin{aligned}
 & + \sigma_k(1 + \sigma_k)\phi_r(\|\nabla h(q_k) - \nabla h(q_{k-1})\|) \\
 \leq & \max\{G_h(u, v), G_h(u, q_k) + \sigma_k(G_h(u, q_k) - G_h(u, q_{k-1})) + \sigma_k G_h(q_k, q_{k-1}) \\
 & + \sigma_k(1 + \sigma_k)\phi_r(\|\nabla h(q_k) - \nabla h(q_{k-1})\|)\}.
 \end{aligned}$$

Suppose  $G_h(u, v)$  is the maximum, then the conclusion follows trivially. Otherwise, there exists  $k_0 \in \mathbb{N}$  such that for all  $k \geq k_0$ , we have

$$\begin{aligned}
 G_h(u, q_{k+1}) \leq & G_h(u, q_k) + \sigma_k(G_h(u, q_k) - G_h(u, q_{k-1})) + \sigma_k G_h(q_k, q_{k-1}) \\
 & + \sigma_k(1 + \sigma_k)\phi_r(\|\nabla h(q_k) - \nabla h(q_{k-1})\|).
 \end{aligned}$$

Then by Lemma 2.9 we have that  $\{G_h(u, q_k)\}$  convergent. Hence,  $\{G_h(u, q_k)\}$  is bounded. It follows from Lemma 2.5 that  $\{q_k\}$  is bounded. Consequently, the sequences  $\{r_k\}, \{s_k\}$  and  $\{t_k\}$  are bounded. □

**Theorem 3.7** *Assume that  $\{\psi_k\}$  is a sequence in  $(0, 1)$ ,  $\nabla h$  is weakly sequentially continuous on  $\mathcal{X}$ , and Assumptions (3.2) holds. Then the sequence  $\{q_k\}$  generated by (3.2) converges strongly to an element in  $\Delta$ .*

*Proof* Let  $u \in \Delta$ , then we have from Lemma 2.3, (3.14), and (3.12) that

$$\begin{aligned}
 G_h(u, q_{k+1}) & = V_h(u, \psi_k \nabla h(v) + (1 - \psi_k)\nabla h(t_k)) \\
 & \leq V_h(u, \psi_k \nabla h(v) + (1 - \psi_k)\nabla h(t_k) - \psi_k(\nabla h(t_k) - \nabla h(u))) \\
 & \quad + \psi_k \langle \nabla h(v) - \nabla h(u), q_{k+1} - u \rangle \\
 & = V_h(u, \psi_k \nabla h(u) + (1 - \psi_k)\nabla h(t_k)) + \psi_k \langle \nabla h(v) - \nabla h(u), q_{k+1} - u \rangle \\
 & \leq \psi_k G_h(u, u) + (1 - \psi_k)G_h(u, t_k) + \psi_k \langle \nabla h(v) - \nabla h(u), q_{k+1} - u \rangle \\
 & = (1 - \psi_k)G_h(u, t_k) + \psi_k \langle \nabla h(v) - \nabla h(u), q_{k+1} - u \rangle \\
 & \leq (1 - \psi_k)G_h(u, r_k) - (1 - \psi_k) \left(1 - \frac{\omega \rho \kappa_k}{\kappa_{k+1}}\right) G_h(s_k, r_k) \\
 & \quad - (1 - \psi_k) \left(1 - \frac{\omega \kappa_k}{\rho \kappa_{k+1}}\right) G_h(s_k, t_k) + \psi_k \langle \nabla h(v) - \nabla h(u), q_{k+1} - u \rangle \\
 & \leq (1 - \psi_k)G_h(u, q_k) - (1 - \psi_k) \left(1 - \frac{\omega \rho \kappa_k}{\kappa_{k+1}}\right) G_h(s_k, r_k) \\
 & \quad - (1 - \psi_k) \left(1 - \frac{\omega \kappa_k}{\rho \kappa_{k+1}}\right) G_h(s_k, t_k) \\
 & \quad + \psi_k \frac{\sigma_k}{\psi_k} [(G_h(u, q_k) - G_h(u, q_{k-1})) + G_h(q_k, q_{k-1}) \\
 & \quad + (1 + \sigma_k)\phi_r(\|\nabla h(q_k) - \nabla h(q_{k-1})\|)] \\
 & \quad + \psi_k \langle \nabla h(v) - \nabla h(u), q_{k+1} - u \rangle \tag{3.15} \\
 & \leq (1 - \psi_k)G_h(u, q_k) + \psi_k \langle \nabla h(v) - \nabla h(u), q_{k+1} - u \rangle \\
 & \quad + \psi_k \frac{\sigma_k}{\psi_k} [(G_h(u, q_k) - G_h(u, q_{k-1})) + G_h(q_k, q_{k-1})]
 \end{aligned}$$

$$+ (1 + \sigma_k)\phi_r(\|\nabla h(q_k) - \nabla h(q_{k-1})\|), \tag{3.16}$$

with  $Z_k = ((\nabla h(v) - \nabla h(u), q_{k+1} - u) + \frac{\sigma_k}{\psi_k} [(G_h(u, q_k) - G_h(u, q_{k-1})) + G_h(q_k, q_{k-1}) + (1 + \sigma_k)\phi_r(\|\nabla h(q_k) - \nabla h(q_{k-1})\|)])$ .

Thus

$$G_h(u, q_{k+1}) \leq (1 - \psi_k)G_h(u, q_k) + \psi_k Z_k. \tag{3.17}$$

Let  $a_k = G_h(u, q_{k+1})$ , then (3.16) becomes

$$q_{k+1} \leq (1 - \psi_k)a_k + \psi_k Z_k. \tag{3.18}$$

To establish a strong convergence result, we claim that  $\limsup_{l \rightarrow \infty} Z_{k_l} \leq 0$  whenever exists a subsequence  $\{a_{k_l}\}$  of  $\{a_k\}$  satisfying

$$\liminf_{l \rightarrow \infty} (a_{k_{l+1}} - a_{k_l}) \geq 0,$$

Indeed, assume there exists such subsequence, in view of (3.15), we obtain

$$\begin{aligned} & \limsup_{l \rightarrow \infty} \left[ (1 - \psi_{k_l}) \left( 1 - \frac{\omega \rho \kappa_{k_l}}{\kappa_{k_{l+1}}} \right) G_h(s_{k_l}, r_{k_l}) + \left( 1 - \frac{\omega \kappa_{k_l}}{\rho \kappa_{k_{l+1}}} \right) G_h(s_{k_l}, t_{k_l}) \right] \\ & \leq \limsup_{l \rightarrow \infty} [(1 - \psi_{k_l})G_h(u, q_{k_l}) \\ & \quad - G_h(u, q_{k_{l+1}}) + M_3 \psi_{k_l}] \\ & = -\liminf_{l \rightarrow \infty} (a_{k_{l+1}} - a_{k_l}) \\ & \leq 0, \end{aligned} \tag{3.19}$$

where  $M_3 = \sup_{l \in \mathbb{N}} Z_{k_l}$  and thus

$$\lim_{l \rightarrow \infty} G_h(s_{k_l}, r_{k_l}) = 0 = \lim_{l \rightarrow \infty} G_h(s_{k_l}, t_{k_l}). \tag{3.20}$$

By Lemma 2.4, we get

$$\lim_{l \rightarrow \infty} \|s_{k_l} - r_{k_l}\| = 0 = \lim_{l \rightarrow \infty} \|s_{k_l} - t_{k_l}\|. \tag{3.21}$$

Observe from step 1 of (3.2) that

$$\begin{aligned} \|\nabla h(r_{k_l}) - \nabla h(q_{k_l})\| &= \sigma_{k_l} \|\nabla h(q_{k_l}) - \nabla h(q_{k_{l-1}})\| \\ &\leq \psi_{k_l} \frac{\sigma_{k_l}}{\psi_{k_l}} \|\nabla h(q_{k_l}) - \nabla h(q_{k_{l-1}})\| \rightarrow 0, \quad \text{as } l \rightarrow \infty. \end{aligned} \tag{3.22}$$

Since  $\nabla h^*$  is continuous on bounded subsets of  $\mathcal{X}^*$ , we obtain

$$\lim_{l \rightarrow \infty} \|r_{k_l} - q_{k_l}\| = 0. \tag{3.23}$$

It is obvious from (3.21) that

$$\lim_{l \rightarrow \infty} \|t_{k_l} - r_{k_l}\| = 0. \tag{3.24}$$

Using (3.21), (3.23), and (3.24), we obtain that

$$\lim_{l \rightarrow \infty} \|s_{k_l} - q_{k_l}\| = 0 = \lim_{l \rightarrow \infty} \|t_{k_l} - q_{k_l}\|. \tag{3.25}$$

We deduce from step 3 of (3.2) that

$$\|\nabla h(q_{k_{l+1}}) - \nabla h(t_{k_l})\| \leq \psi_{k_l} \|\nabla h(v) - \nabla h(t_{k_l})\| \rightarrow 0 \text{ as } l \rightarrow \infty. \tag{3.26}$$

Using the fact that  $\nabla h^*$  is continuous on bounded subsets of  $\mathcal{X}^*$ , we get

$$\lim_{l \rightarrow \infty} \|q_{k_{l+1}} - t_{k_l}\| = 0, \tag{3.27}$$

which implies from (3.25) that

$$\lim_{l \rightarrow \infty} \|q_{k_{l+1}} - q_{k_l}\| = 0. \tag{3.28}$$

We next show that  $\limsup_{l \rightarrow \infty} Z_{k_l} \leq 0$ . Clearly, it suffices to show that

$$\limsup_{l \rightarrow \infty} \langle \nabla h(v) - \nabla h(u), q_{k_{l+1}} - u \rangle \leq 0.$$

Let  $\{q_{k_{l_i}}\}$  be a s subsequence of  $\{q_{k_l}\}$  such that

$$\lim_{j \rightarrow \infty} \langle \nabla h(v) - \nabla h(u), q_{k_{l_j+1}} - u \rangle = \limsup_{l \rightarrow \infty} \langle \nabla h(v) - \nabla h(u), q_{k_{l+1}} - u \rangle.$$

From the boundedness of the sequence  $\{q_{k_l}\}$  is the existence of  $\{q_{k_{l_j}}\}$  such that  $q_{k_{l_j}} \rightharpoonup x^* \in \mathcal{X}$ . In view of (3.25), there exist subsequences  $\{s_{k_{l_j}}\}$  and  $\{t_{k_{l_j}}\}$  which converge weakly to  $x^*$ , respectively. Now, to establish that  $x^* \in (\Phi + \Psi)^{-1}(0)$ , let  $(\mu_1, \mu_2) \in \text{Gra}(\Phi + \Psi)$ , we have  $\mu_2 - \Phi\mu_1 \in \Psi\mu_1$ . From the definition of  $s_{k_l}$ , we observe that

$$\nabla h(r_{k_l}) - \kappa_{k_l} \Phi r_{k_l} \in \nabla h(s_{k_{l_j}}) + \kappa_{k_{l_j}} \Psi s_{k_{l_j}},$$

which implies

$$\frac{1}{\kappa_{k_{l_j}}} (\nabla h(r_{k_{l_j}}) - \nabla h(s_{k_{l_j}}) - \kappa_{k_{l_j}} \Phi(r_{k_{l_j}})) \in \Psi(s_{k_{l_j}}).$$

Applying the maximal monotonicity of  $\Psi$ , we obtain

$$\left\langle \mu_1 - s_{k_{l_j}}, \mu_2 - \Phi\mu_1 + \frac{1}{\kappa_{k_{l_j}}} (\nabla h(r_{k_{l_j}}) - \nabla h(s_{k_{l_j}}) - \kappa_{k_{l_j}} \Phi(r_{k_{l_j}})) \right\rangle \geq 0.$$

In addition, by the monotonicity of  $\Phi$ , we obtain

$$\begin{aligned} \langle \mu_1 - s_{k_j}, \mu_2 \rangle &\geq \left\langle \mu_1 - s_{k_j}, \Phi \mu_1 + \frac{1}{\kappa_{k_j}} (\nabla h(r_{k_j}) - \nabla h(s_{k_l}) - \kappa_{k_j} \Phi(r_{k_j})) \right\rangle \\ &= \langle \mu_1 - s_{k_j}, \Phi \mu_1 - \Phi(r_{k_j}) \rangle + \frac{1}{\kappa_{k_j}} \langle \mu_1 - s_{k_j}, \nabla h(r_{k_j}) - \nabla h(s_{k_l}) \rangle \\ &= \langle \mu_1 - s_{k_j}, \Phi(\mu_1) - \Phi(s_{k_l}) \rangle + \langle \mu_1 - s_{k-l_j}, \Phi(s_{k_l}) - \Phi(r_{k_j}) \rangle \\ &\quad + \frac{1}{\kappa_{k_j}} \langle \mu_1 - s_{k_j}, \nabla h(r_{k_j}) - \nabla h(s_{k_l}) \rangle \\ &\geq \langle \mu_1 - s_{k_j}, \Phi(s_{k_l}) - \Phi(r_{k_j}) \rangle + \frac{1}{\kappa_{k_j}} \langle \mu_1 - s_{k_j}, \nabla h(r_{k_l}) - \nabla h(s_{k_l}) \rangle. \end{aligned} \tag{3.29}$$

From the fact that  $\Phi$  is Lipschitz continuous and  $s_{k_j} \rightharpoonup x^*$ , it follows from (3.21) that

$$\langle \mu_1 - x^*, \mu_2 \rangle \geq 0.$$

From the monotonicity of  $\Phi + \Psi$ , we obtain that  $0 \in (\Phi + \Psi)x^*$ , thus  $x^* \in \Delta$ . It follows from Lemma 2.7 and (3.28), that

$$\begin{aligned} &\limsup_{l \rightarrow \infty} \langle \nabla h(v) - \nabla h(u), q_{k_{l+1}} - u \rangle \\ &\leq \limsup_{l \rightarrow \infty} \langle \nabla h(v) - \nabla h(u), q_{k_{l+1}} - q_{k_l} \rangle + \limsup_{l \rightarrow \infty} \langle \nabla h(v) - \nabla h(u), q_{k_l} - u \rangle \\ &= \lim_{j \rightarrow \infty} \langle \nabla h(v) - \nabla h(u), q_{k_j} - u \rangle \\ &= \langle \nabla h(v) - \nabla h(u), x^* - u \rangle \\ &\leq 0. \end{aligned} \tag{3.30}$$

By applying (3.30) and Lemma 2.10, we obtain that  $\lim_{l \rightarrow \infty} G_h(u, q_{k_l}) \rightarrow 0$  and  $l \rightarrow \infty$ , thus  $\lim_{l \rightarrow \infty} \|q_{k_l} - u\| = 0$ . Therefore, we conclude that  $\{q_k\}$  converges strongly to  $u \in \Omega$ , where  $u = \text{Proj}_\Omega v$ .  $\square$

If  $\mathcal{X}$  is a 2-uniformly convex and uniformly smooth Banach space, and  $h(u) = \frac{1}{2} \|u\|^2$ , then Algorithm 3.2 becomes

**Algorithm 3.8** *MIP and its convergence analysis.*

**Initialization:** Given  $\kappa_1 > 0, \sigma > 0, \rho > 0$ , and  $\omega \in (0, 1)$ . Let  $v, q_0, q_1 \in \mathcal{X}$  and  $\psi_k$  be a sequence in  $(0, 1)$  such that  $\lim_{k \rightarrow \infty} \psi_k = 0$  and  $\sum_{k=1}^\infty \psi_k = \infty$  for all  $k \in \mathbb{N}$ .

**Step 1:** Given  $q_{k-1}, q_k$ , choose  $\sigma_k$  such that  $\sigma_k \in [0, \bar{\sigma}_k]$ , where

$$\bar{\sigma}_k = \begin{cases} \min\{\sigma, \frac{\pi_k}{\|(q_k) - (q_{k-1})\|}\} & \text{if } \|(q_k) - (q_{k-1})\| \neq 0, \\ \sigma & \text{otherwise.} \end{cases} \tag{3.31}$$

and  $\{\pi_k\}$  is a sequence of nonnegative numbers such that  $\pi_k = o(\psi_k)$ , that is  $\lim_{k \rightarrow \infty} \frac{\pi_k}{\psi_k} = 0$ .

Compute

$$\begin{cases} r_k = J^{-1}(J(q_k) + \sigma_k(J(q_k) - J(q_{k-1}))) \\ s_k = R_{\kappa_k}^\Psi J^{-1}(J^{-1}(r_k) - \kappa_k \Phi(r_k)) \end{cases} \tag{3.32}$$

**Step 2:** Compute

$$t_k = J^{-1}(J(s_k) - \kappa_k(\Phi(s_k) - \Phi(r_k))), \tag{3.33}$$

and  $\kappa_{k+1}$  is defined as

$$\kappa_{k+1} = \begin{cases} \min\{\kappa_k, \{\frac{\mu((\rho\phi(s_k, r_k)) + \frac{1}{\rho}\phi(s_k, t_k))}{(Js_k - Jt_k, \Phi(s_k) - \Phi(r_k))}\}, & \text{if } \langle Js_k - Jt_k, \Phi(s_k) - \Phi(r_k) \rangle \neq 0. \\ \kappa_k, & \text{otherwise.} \end{cases} \tag{3.34}$$

**Step 3:** Compute

$$q_{k+1} = J^{-1}(\psi_k J(v) + (1 - \psi_k)J(t_k)) \tag{3.35}$$

**Stopping criterion** If  $q_{k+1} = r_k$  for  $k \geq 1$  then stop. Otherwise set  $k := k + 1$  and return to step 1.

---

#### 4 Numerical example

In this section, we provide the numerical experiments to illustrate the convergence of the proposed algorithms and compare with other algorithms.

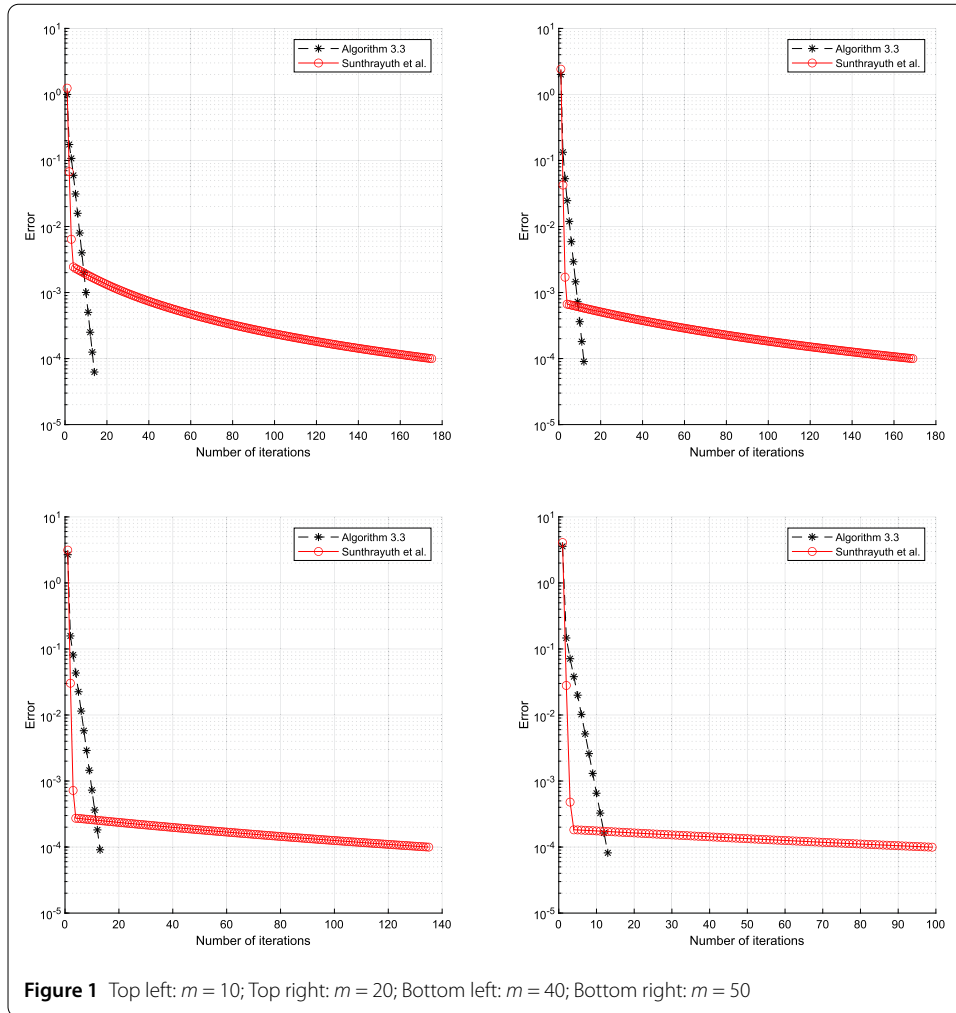
*Example 4.1* This example is taken from [23]. Let  $\mathcal{X} = \mathbb{R}^m (m = 2k)$ , and  $K := \{x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^m x_i = 1\}$ . Let  $h : K \rightarrow \mathbb{R}$  be defined by  $h(x) = \sum_{i=1}^m x_i \ln x_i$ , then  $h$  satisfies the condition of Theorem 3.7 and strongly convex with  $\sigma = 1$  with respect to  $\ell_1$ -norm on  $K$ . It follows that  $\nabla h(x) = (1 + \ln x_1, 1 + \ln x_2, \dots, 1 + \ln x_m)$  and  $\nabla h^*(y) = (e^{x_1-1}, e^{x_2-1}, \dots, e^{x_m-1})$ . Now, define the mapping  $\Phi : \mathcal{X} \rightarrow \mathcal{X}^*$  by  $\Phi(x) = (2x_1 + 1, 0, 2x_3 + 1, 0, \dots, 0, 2x_{2k-1} + 1, 0)$  and  $\Psi : \mathcal{X} \rightarrow \mathcal{X}^*$  by  $\Psi(x) = N_K(x)$ , where  $N_K$  is the normal cone defined by  $N_K(x) = \{p \in \mathbb{R}^m : \langle p, y - x \rangle \geq 0, \forall y \in K\}$ . Then, the mapping  $\Phi$  is monotone and Lipschitz continuous with a constant 2. The mapping  $\Psi$  is maximal monotone. Thus,  $R_{\kappa_k}^{\Phi, \Psi}(x) = \text{Proj}_K^h x$ . From [15, Remark 4] the Bregman projection onto  $K$  is defined as

$$\text{Proj}_K^h(x) = \left( \frac{x_1 e^{a_1}}{\sum_{i=1}^m x_i e^{a_i}}, \frac{x_2 e^{a_2}}{\sum_{i=1}^m x_i e^{a_i}}, \dots, \frac{x_m e^{a_m}}{\sum_{i=1}^m x_i e^{a_i}} \right) a \in \mathbb{R}^m \quad \text{and} \quad x \in \text{int}(K).$$

For this example, we choose  $\pi_k = \frac{1}{k^{1.2}}, \psi_k = \frac{1}{k+1}, \mu = 0.5, \rho = 0.4, \sigma = 0.7$  and  $\kappa_1 = 1.7$ . The starting points  $x_0$  and  $x_1$  are chosen randomly in  $\mathbb{R}^m$ . We compare our Algorithm 3.2 with Algorithm 2 in Sunthrayuth et al. [23]. The result of this experiment is given in Fig. 1 with the  $m = 10, 20, 40, 50$ .

The next example is given in  $\ell_p$  spaces ( $1 \leq p < \infty$ ) with  $p \neq 2$ . It is known that  $\ell_p^*$  is isomorphic to  $\ell_q$  if  $\frac{1}{p} + \frac{1}{q} = 1$ . It is known that  $\ell_p$  is a reflexive Banach space. In this case, we set  $h(x) = \frac{1}{2} \|x\|^2$ . Thus, we obtain that  $\nabla h(x) = x = \nabla h^*(x)$ .





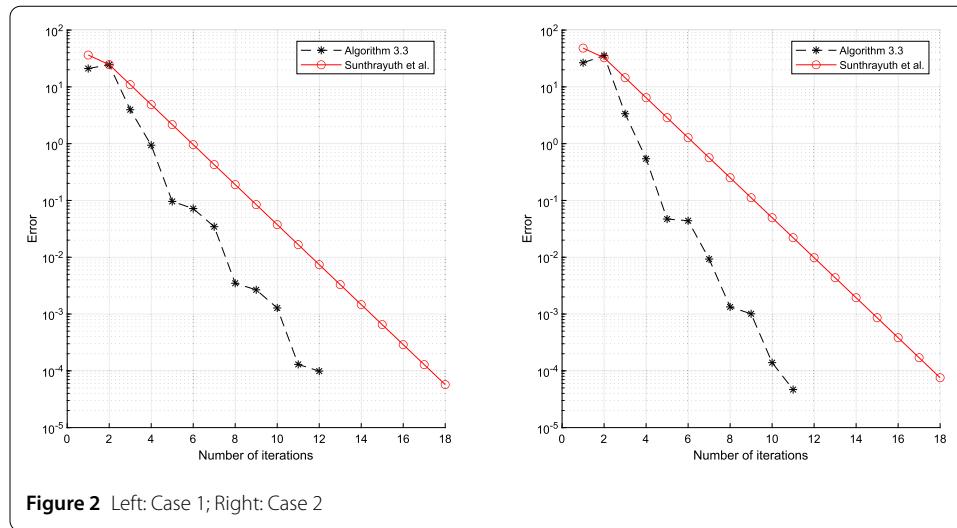
**Example 4.2** Let  $\mathcal{X} = \ell_3(\mathbb{R})$  be a Banach space defined by  $\ell_3(\mathbb{R}) = \{x = (x_1, x_2, x_3, \dots), x_i \in \mathbb{R} : (\sum_{i=1}^{\infty} |x_i|^3)^{\frac{1}{3}}\}$ , with norm  $\|\cdot\|_{\ell_3} : \ell_3 \rightarrow \mathbb{N}$  defined by  $\ell_3(x) = (\sum_{i=1}^{\infty} |x_i|^3)^{\frac{1}{3}}$  for all  $x = (x_1, x_2, x_3, \dots) \in \ell_3$ . Let  $\Phi : \ell_3 \rightarrow \ell_3$  be given by  $\Phi(x) = 3x + (1, 1, 1, 0, 0, 0, \dots)$ . It is easy to see that  $\Phi$  is monotone. Also, define the mapping  $\Psi : \ell_3 \rightarrow \ell_3$  by  $\Psi(x) = 7x$ . By direct calculation, we obtain for  $\kappa_k > 0$ , that

$$\begin{aligned}
 s_k &= (\nabla h(r_k) - \kappa_k \Psi)^{-1} \nabla h \circ \nabla h^* (\nabla h(r_k) - \kappa_k \Phi(r_k)) \\
 &= \frac{1 - 3\kappa_k}{1 + 7\kappa_k} r_k - \frac{\kappa_k}{1 + 7\kappa_k} (1, 1, 1, 0, 0, 0, \dots).
 \end{aligned}$$

We choose the parameters as in the previous example with comparison to Algorithm 2 of Sunthrayuth et al. [23]. The report of this example is given in Fig. 2, which shows the competitive advantage of our method in terms of number of iterations to convergence. The stopping criterion in both examples is given as  $\|x_{n+1} - x_n\| \leq \epsilon$ , where  $\epsilon = 10^{-4}$ .

(Case 1)  $x_0 = [1, 1, 1, 0, 0, \dots]$  and  $x_1 = [2, 2, 2, 1, 1, \dots]$ ,

(Case 2)  $x_0 = [1, 1, 0, 0, \dots]$  and  $x_1 = [3, 3, 0, 0, 0, \dots]$ .



## 5 Conclusions

In our article, we propose a new self-adaptive step-size and an inertial Tseng method for solving monotone inclusion problem using the Bregman distance approach in reflexive Banach space. Using an inertial Halpern method, we proved that our proposed method converges strongly to an element in the solution set. Lastly, we show through our experiment that our new step size for the proposed algorithm is more efficient than the result of [23].

### Funding

Not applicable.

### Availability of data and materials

Not applicable.

## Declarations

### Ethics approval and consent to participate

Not applicable.

### Competing interests

The authors declare no competing interests.

### Author contributions

Conceptualization: H. A. Abass, Methodology: H. A. Abass, O. K. Oyewole and M. Aphane, Software: O. K. Oyewole, validation: H. A. Abass and M. Aphane. All authors have read and agreed to the submission of the article.

### Author details

<sup>1</sup>Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, P.O. Box 94 Medunsa 0204, Pretoria, South Africa. <sup>2</sup>Department of Mathematics, The Technion-Israel Institute of Technology, Haifa, Israel.

Received: 11 July 2023 Accepted: 26 October 2023 Published online: 11 December 2023

## References

1. Abass, H.A., Narain, O.K., Onifade, O.M.: Inertial extrapolation method for solving system of monotone variational inclusion and fixed point problems using Bregman distance approach. *Nonlinear Funct. Anal. Appl.* **28**(2), 497–520 (2023)
2. Abass, H.A., Aremu, K.O., Jolaoso, L.O., Mewomo, O.T.: An inertial forward-backward splitting method for approximating solutions of certain optimization problem. *J. Nonlinear Funct. Anal.* **2020**, Article ID 6 (2020)
3. Abass, H.A., Oyewole, O.K., Jolaoso, L.O., Aremu, K.O.: Modified inertial Tseng for solving variational inclusion and fixed point problems on Hadamard manifolds. *Appl. Anal.* (2023). <https://doi.org/10.1080/00036811.2023.2256357>

4. Alvarez, F., Attouch, H.: An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. *Set-Valued Anal.* **9**, 3–11 (2001)
5. Alvarez, F.: Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space. *SIAM J. Optim.* **14**(3), 773–782 (2004)
6. Attouch, H., Peyrouquet, J., Redont, P.: Backward-forward algorithms for structured monotone inclusion in Hilbert space. *J. Math. Anal. Appl.* **457**, 1095–1117 (2018)
7. Barbu, V., Precupanu, T.: *Convexity and Optimization in Banach Spaces*. Springer, New York (2010)
8. Bauschke, H.H., Borwein, J.M.: Legendre functions and method of random Bregman functions. *J. Convex Anal.* **4**, 27–67 (1997)
9. Bauschke, H.H., Borwein, J.M., Combettes, P.L.: Essentially smoothness, essentially strict convexity and Legendre functions in Banach spaces. *Commun. Contemp. Math.* **3**, 615–647 (2001)
10. Bregman, L.M.: The relaxation method for finding the common point of convex sets and its application to solution of problems in convex programming. *USSR Comput. Math. Math. Phys.* **7**, 200–217 (1967)
11. Bruck, R.: On the weak convergence of an ergodic iteration for the solution of variational inequalities for monotone operators in Hilbert space. *J. Math. Anal. Appl.* **61**, 159–164 (1977)
12. Butnairu, D., Resmerita, E.: Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces. *Abstr. Appl. Anal.* **2006**, Article ID 84919 (2006)
13. Combettes, P.L., Wajs, V.R.: Signal recovery by proximal forward-backward splitting. *SIAM J. Multiscale Model. Simul.* **4**, 1168–1200 (2008)
14. Daubechies, I., Defrise, M., De Mol, C.: An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. *Commun. Pure Appl. Math.* **57**, 1413–1457 (2004)
15. Denisov, S.V., Semenov, V.V., Stetsynk, P.I.: Bregman extragradient method with monotone rule of step adjustment. *Cybern. Syst. Anal.* **55**(3), 377–383 (2019)
16. Duchi, J., Singer, Y.: Efficient online and batch learning using forward-backward splitting. *J. Mach. Learn. Res.* **10**, 2899–2934 (2009)
17. Eskandani, G.Z., Raeisi, M., Rassias, T.M.: A hybrid extragradient method for solving pseudomonotone equilibrium problem using Bregman distance. *J. Fixed Point Theory Appl.* **20**, 132 (2018)
18. Lions, P.L., Mercier, B.: Splitting algorithms for the sum of two nonlinear operators. *SIAM J. Numer. Anal.* **16**, 964–979 (1979)
19. Naraghirad, E., Yao, J.C.: Bregman weak relatively nonexpansive mappings in Banach spaces. *Fixed Point Theory Appl.* **2013**, Article ID 141 (2013)
20. Nesterov, Y.: A method for solving the convex programming problem with convergence rate  $O(\frac{1}{k^2})$ . *Dokl. Akad. Nauk SSSR* **269**, 543–547 (1983)
21. Polyak, B.T.: Some methods of speeding up the convergence of iteration methods. *USSR Comput. Math. Math. Phys.* **4**, 1–77 (1964)
22. Raguét, H., Fadili, J., Peyre, G.: A generalized forward-backward splitting. *SIAM J. Imaging Sci.* **6**, 1199–1226 (2013)
23. Sunthrayuth, P., Pholasa, N., Cholamjiak, P.: Mann-type algorithms for solving the monotone inclusion problem and the fixed point problem in reflexive Banach spaces. *Ric. Mat.* **72**(1), 63–90 (2023)
24. Shehu, Y.: Convergence results of forward-backward algorithms for sum of monotone operators in Banach spaces. *Results Math.* **74**, Article ID 138 (2019)
25. Tseng, P.: A modified forward-backward splitting method for maximal monotone mappings. *SIAM J. Control Optim.* **38**, 431–446 (2000)
26. Reich, S., Sabach, S.: A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. *J. Nonlinear Convex Anal.* **10**, 471–485 (2009)
27. Saejung, S., Yotkaew, P.: Approximation of zeros of inverse strongly monotone operator in Banach spaces. *Nonlinear Anal.* **75**, 742–750 (2012)
28. Suantai, S., Pholasa, N., Cholamjiak, P.: The modified inertial relaxed CQ algorithm for solving split feasibility problems. *J. Ind. Manag. Optim.* **14**, Article ID 4 (2018)
29. Suantai, S., Pholasa, N., Cholamjiak, P.: Relaxed CQ algorithms involving the inertial technique for multiple-sets split feasibility problems. *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat.* **113**, 1081–1099 (2019)
30. Tie, J.V.: *Convex Analysis: An Introductory Text*. Wiley, New York (1984)
31. Ugwunnadi, G.C., Abass, H.A., Aphane, M., Oyewole, O.K.: Inertial Halpern-typemethod for solving split feasibility and fixed point problems via dynamical stepsize in real Banach spaces. *Ann. Univ. Ferrara* (2023). <https://doi.org/10.1007/s11565-023-00473-6>
32. Wang, Z.B., Sunthrayuth, P., Adamu, A., Cholamjiak, P.: Modified accelerated Bregman projection methods for solving quasi-monotone variational inequalities. *Optimization*, 1–35 (2023)
33. Zalinescu, C.: *Convex Analysis in General Vector Spaces*. World Scientific, River Edge (2002)

## Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.