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On a generalization of a relatively nonexpansive mapping and best proximity pair

Karim Chaira[1] and Belkassem Seddoug[1*]

*Correspondence:
bseddoug@gmail.com
[1]Department of Mathematics, CRMEF Rabat-Salé-Kénitra, Avenue Allal El Fassi, B.P 6210, Rabat, 10000, Morocco

Abstract

Let A and B be two nonempty subsets of a normed space X, and let $T : A \cup B \to A \cup B$ be a cyclic (resp., noncyclic) mapping. The objective of this paper is to establish weak conditions on T that ensure its relative nonexpansiveness.

The idea is to recover the results mentioned in two papers by Matkowski (Banach J. Math. Anal. 2:237–244, 2007; J. Fixed Point Theory Appl. 24:70, 2022), by replacing the nonexpansive mapping $f : C \to C$ with a cyclic (resp., noncyclic) relatively nonexpansive mapping to obtain the best proximity pair. Additionally, we provide an application to a functional equation.

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1 Introduction and preliminaries

Let A and B be two nonempty subsets of a normed space $(X, \|\cdot\|)$. A self-mapping $T : A \cup B \to A \cup B$ is said to be cyclic (resp., noncyclic) if $T(A) \subseteq B$ and $T(B) \subseteq A$ (resp., $T(A) \subseteq A$ and $T(B) \subseteq B$). For such a mapping, we can consider the minimization problem of finding a best proximity pair of the mapping T, that is, a pair $(p,q) \in A \times B$ such that

$$\|p - T(p)\| = \|q - T(q)\| = \operatorname{dist}(A,B)$$
$$\big(\text{resp., } T(p) = p, T(q) = q, \text{ and } \|p - q\| = \operatorname{dist}(A,B)\big),$$

where $\operatorname{dist}(A,B) = \inf\{d(x,y) : (x,y) \in A \times B\}$.

A cyclic (resp., noncyclic) mapping $T : A \cup B \to A \cup B$ is said to be relatively nonexpansive if $\|T(x) - T(y)\| \le \|x - y\|$ for all $x \in A$ and $y \in B$ (notice that in general a relatively nonexpansive mapping need not be continuous).

Recall that a real normed vector space $(X, \|\cdot\|)$ is called uniformly convex (see Clarkson [4]) if for every $\varepsilon \in (0,2]$, there is $\delta > 0$ such that for any two vectors $x, y \in X$ with $\|x\| =$

$\|y\| = 1$, the condition $\|x - y\| \geq \varepsilon$ implies that

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

The existence of a best proximity pair was first considered and studied in 2005 by Eldred et al. [5]:

- if (A, B) is a nonempty closed bounded convex pair of a uniformly convex Banach space X, then every cyclic relatively nonexpansive mapping defined on $A \cup B$ has a best proximity pair.
- if (A, B) is a nonempty closed bounded convex pair of a uniformly convex Banach space X, then every noncyclic relatively nonexpansive mapping defined on $A \cup B$ has a best proximity pair.

The relevance of best proximity points is that they provide optimal solutions for the problem of best approximation between two sets. Some references concerning best proximity points are given in [3, 6, 8, 11–14].

Let us recall the definitions of the lower and upper bounds of a function $f : [0, +\infty) \to [0, +\infty)$ at a point t_0:

$$\liminf_{t \to t_0^+} f(t) = \sup_{\eta > 0} \Big(\inf_{t_0 < t < t_0 + \eta} f(t) \Big) \quad \text{and} \quad \limsup_{t \to t_0^+} f(t) = \inf_{\eta > 0} \Big(\sup_{t_0 < t < t_0 + \eta} f(t) \Big).$$

Remark 1 If $\liminf_{t \to 0^+} \frac{f(t+a)}{t+a} = \ell$ with $a, \ell \in [0, +\infty)$, then for all $\varepsilon > 0$ and $\eta > 0$, there exists $t_\varepsilon \in]0, \eta[$ such that

$$f(t_\varepsilon + a) < (\ell + \varepsilon)(t_\varepsilon + a) \quad \text{and} \quad \lim_{\varepsilon \to 0^+} t_\varepsilon = 0.$$

For the reader's convenience, we recall the main results in [10].

Theorem 1 ([10], Theorem 1) *Let X be a uniformly convex Banach space, let C be a nonempty bounded convex closed subset of X, and let T be a self-mapping of C. If there is a function $\beta : [0, +\infty) \to [0, +\infty)$ such that*

$$\big\| T(x) - T(y) \big\| \leq \beta\big(\|x - y\|\big), \quad x, y \in C, x \neq y, \tag{1}$$

$$\limsup_{t \to 0^+} \frac{\beta(t)}{t} < +\infty, \quad \textit{and} \quad \liminf_{t \to 0^+} \frac{\beta(t)}{t} = 1, \tag{2}$$

then T has a fixed point in C.

Proposition 2 ([10], Proposition 1) *Let X be a uniformly convex Banach space, and let C be a nonempty bounded convex closed subset of X. Suppose that $T : C \to C$ is continuous. If there exist two positive sequences $(t_n)_{n \geq 0}$ and $(c_n)_{n \geq 0}$, $\lim_{n \to +\infty} t_n = 0$, $\lim_{n \to +\infty} c_n = 1$, such that for every $n \in \mathbb{N}$ and for all $x, y \in C$,*

$$\|x - y\| = t_n \quad \Rightarrow \quad \big\| T(x) - T(y) \big\| \leq c_n t_n, \tag{3}$$

then T has a fixed point.

The idea of this paper is to present weak conditions under which a cyclic (resp., noncyclic) mapping $T : A \cup B \to A \cup B$, where A and B are two subsets of a normed space X, is relatively nonexpansive. As a result, we establish the existence of the best proximity pair for this mapping.

On the other hand, Let $\mathcal{B}_{L,d}$ be the family consisting of functions $\beta : [0,+\infty) \to [0,+\infty)$ that satisfy the following conditions:

(i) β increases on $[0,+\infty)$,

(ii) $\beta(kt) \le k\beta(t)$ for all $t \in [d,+\infty)$ and $k \in \mathbb{N} \setminus \{0\}$,

(iii) $\liminf_{t\to 0^+} \frac{\beta(t+d)}{t+d} = L$.

Then we establish the following result: Any cyclic (resp., noncyclic) mapping $T : A \cup B \to A \cup B$ such that

$$\bigl\| T(u) - T(v) \bigr\| \le \beta\bigl(\|u - v\|\bigr) \tag{4}$$

for all $(u,v) \in A \times B$ is relatively (L,d)-mapping (see Sect. 2.2.), where $d \in]0,+\infty)$ and $L \ge \frac{1}{2}$.

Note that the set $\mathcal{B}_{L,d}$ is not empty. For example, the function β defined as $\beta(t) = \frac{2t}{t+1}$ for $t \in [0,+\infty)$ satisfies all three conditions (i), (ii), and (iii) with $L = d = 1$.

We denote by $\mathcal{B}_{L,0}$ ($d = 0$) the family of functions $\beta : [0,+\infty) \to [0,+\infty)$ that satisfy conditions (i) and (iii).

The paper is organized as follows. Our main results are presented in Sect. 2. Theorem 3 is a modification of Theorem 1 by Matkowski [10], in which the hypothesis $\limsup_{t\to 0^+} \frac{\beta(t)}{t} < +\infty$ is replaced with $\beta \in \mathcal{B}_{1,0}$. In Lemma 4, we show that any cyclic (resp., noncyclic) mapping $T : A \cup B \to A \cup B$ satisfying

$$\bigl\| T \circ g(u) - T \circ h(v) \bigr\| \le c_n\bigl(t_n + \|u - v\|\bigr) \tag{5}$$

for all $(u,v) \in (\operatorname{cov}(A \cup B))^2$ such that $d \le \|u - v\| \le 3d$, where $d = \operatorname{dist}(A,B) > 0$, and $\operatorname{cov}(A \cup B)$ is the convex hull of two parts A and B, is relatively nonexpansive. Using this lemma, under certain conditions on the parts A and B of a uniformly convex space X, we present Theorem 5 on the existence of the best proximity pair. Corollary 6 describes two cases $d = 0$ and $d > 0$ for a mapping $T : A \cup B \to A \cup B$ satisfying

$$\bigl(d \le \|u - v\| \le t_n + 3d \Rightarrow \bigl\| T \circ g(u) - T \circ h(v) \bigr\| \le c_n(t_n + d)\bigr), \tag{6}$$

$(u,v) \in (\operatorname{cov}(A \cup B))^2$, where $d = \operatorname{dist}(A,B)$.

Proposition 7 in Sect. 2.2 says that any cyclic (resp., noncyclic) mapping $T : A \cup B \to A \cup B$ that satisfies the condition

$$\bigl\| T(u) - T(v) \bigr\| \le \beta\bigl(\|u - v\|\bigr) \tag{7}$$

for all $(u,v) \in A \times B$ is relatively (L,d)-mapping on $A \cup B$ (see Definition 2).

In Sect. 2.3, we use Proposition 7 to get a result on the existence of the best proximity pair of a functional equation in $L^2(\mathcal{U})$, where $\mathcal{U}$ is a nonempty open subset of $\mathbb{R}^m$.

2 Mains results

The following result is a useful reformulation of Theorem 1 in [10].

Theorem 3 *Let A be a nonempty bounded closed convex subset in a uniformly convex Banach space X. Let $\beta \in \mathcal{B}_{1,0}$, and let $T : A \to A$ be a mapping satisfying the inequality*

$$\left\| T(x) - T(y) \right\| \le \beta\left(\|x - y\|\right) \tag{8}$$

for all $(x, y) \in A^2$ such that $x \neq y$. Then there exists $x^* \in A$ such that $Tx^* = x^*$.

Proof Take $(x, y) \in A$ such that $x \neq y$. For $\varepsilon > 0$, as $\liminf_{t\to 0^+} \frac{\beta(t)}{t} = 1$, there exists $t_\varepsilon > 0$ such that

$$\beta(t_\varepsilon) < (1 + \varepsilon)t_\varepsilon \tag{9}$$

$$\text{and} \quad \lim_{\varepsilon\to 0^+} t_\varepsilon = 0. \tag{10}$$

Let $n_\varepsilon \in \mathbb{N}$ be such that

$$n_\varepsilon t_\varepsilon \le \|x - y\| < (n_\varepsilon + 1)t_\varepsilon, \tag{11}$$

Put

$$z_k = \left(1 - \frac{k}{n_\varepsilon + 1}\right) \cdot x + \frac{k}{n_\varepsilon + 1} \cdot y \quad \text{for } k = 0, 1, \dots, n_\varepsilon + 1.$$

By the convexity of C, $z_k \in C$ for all $k \in \{0, 1, \dots, n_\varepsilon\}$; moreover,

$$\|z_k - z_{k+1}\| = \frac{\|x - y\|}{n_\varepsilon + 1} < t_\varepsilon. \tag{12}$$

Applying the triangle inequality, condition (8), inequalities (9), (11), and (12), and the monotony of β, we get

$$\begin{aligned} \left\| T(x) - T(y) \right\| &\le \sum_{j=0}^{n_\varepsilon} \left\| T(z_j) - T(z_{j+1}) \right\| \\ &\le \sum_{j=0}^{n_\varepsilon} \beta\left(\|z_j - z_{j+1}\|\right) \\ &\le \sum_{j=0}^{n_\varepsilon} \beta(t_\varepsilon) \\ &\le (n_\varepsilon + 1)(1 + \varepsilon)t_\varepsilon \\ &\le \left(t_\varepsilon + \|x - y\|\right)(1 + \varepsilon). \end{aligned}$$

Letting ε tend to 0^+ and using (10), we obtain

$$\|Tx - Ty\| \le \|x - y\|.$$

The mapping T is relatively nonexpansive in a bounded closed convex nonempty subset of the uniformly convex Banach space X; so the result follows from the original version of the Browder–Göhde–Kirk theorem (Browder [1], Göhde [7], Kirk [8]). □

2.1 Some auxiliary results on relatively nonexpansive and best proximity pairs

We denote by $\operatorname{cov}(A \cup B)$ the convex hull of two parts A and B of a normed vector space and $d = \operatorname{dist}(A, B)$.

Lemma 4 *Let (A, B) be a nonempty pair in a normed space $(X, \|\cdot\|)$. Let $g : \operatorname{cov}(A \cup B) \to A$ and $h : \operatorname{cov}(A \cup B) \to B$ be two mappings such that $g_{|A} = Id_A$ and $h_{|B} = Id_B$. Let $T : A \cup B \to A \cup B$ be a cyclic (resp., noncyclic) mapping, and let $(t_n)_n$ and $(c_n)_n$ be two positive sequences, $\lim_{n\to+\infty} t_n = 0$, $\lim_{n\to+\infty} c_n = 1$, such that for every $n \in \mathbb{N}$ and for all $(u, v) \in (\operatorname{cov}(A \cup B))^2$ such that $d \le \|u - v\| \le 3d$, where $d = \operatorname{dist}(A, B) > 0$,*

$$\bigl\| T \circ g(u) - T \circ h(v) \bigr\| \le c_n \bigl(t_n + \|u - v\| \bigr). \tag{13}$$

Then, for all $(x, y) \in A \times B$,

$$\bigl\| T(x) - T(y) \bigr\| \le \|x - y\|.$$

Proof Let $(u, v) \in (\operatorname{cov}(A \cup B))^2$ be such that $d \le \|u - v\| \le 3d$. Then for all $n \in \mathbb{N}$,

$$\bigl\| T \circ g(u) - T \circ h(v) \bigr\| \le c_n \bigl(t_n + \|u - v\| \bigr).$$

Taking the limit as n goes to $+\infty$, we have

$$\bigl\| T \circ g(u) - T \circ h(v) \bigr\| \le \|u - v\|; \tag{14}$$

in particular, if $(x, y) \in A \times B$ and $\|x - y\| \le 3d$, then we have

$$\bigl\| T(x) - T(y) \bigr\| \le \|x - y\|.$$

Now let $(x, y) \in A \times B$ be such that $\|x - y\| > 3d$; in this case, $\operatorname{diam}(A, B) > d$. There is $p \in \mathbb{N}$ such that

$$2p + 1 < \frac{\|x - y\|}{3d} \le 2p + 3.$$

For $k = 0, 1, \ldots, 2p + 3$, let $x_k = x + \frac{k}{2p+3}(y - x)$. We have $x_0 = x$, $x_{2p+3} = y$, $x_k \in \operatorname{cov}(A \cup B)$ for every k in $\{0, 1, \ldots, 2p + 2\}$, and

$$\|x_{k+1} - x_k\| = \frac{\|x - y\|}{2p + 3} \in \,]d, 3d].$$

Applying the triangle inequality and (14), we have

$$\begin{aligned} \bigl\| T(x) - T(y) \bigr\| \le{}& \sum_{k=0}^{p} \bigl\| T \circ g(x_{2k}) - T \circ h(x_{2k+1}) \bigr\| \\ &+ \sum_{k=0}^{p} \bigl\| T \circ h(x_{2k+1}) - T \circ g(x_{2k+2}) \bigr\| \end{aligned}$$

$$\begin{aligned} &+ \left\| T \circ g(x_{2p+2}) - T \circ h(x_{2p+3}) \right\| \\ &\leq \sum_{k=0}^{p} \left(\|x_{2k} - x_{2k+1}\| + \|x_{2k+1} - x_{2k+2}\| \right) + \|x_{2p+2} - x_{2p+3}\| \\ &= \|x - y\|. \end{aligned}$$

This finishes the proof. □

Theorem 5 *Let (A,B) be a nonempty closed bounded convex pair in a uniformly convex Banach space X. Let $g : \operatorname{cov}(A \cup B) \to A$ and $h : \operatorname{cov}(A \cup B) \to B$ be mappings such that $g_{|A} = Id_A$ and $h_{|B} = Id_B$. Let $T : A \cup B \to A \cup B$ be a cyclic (resp., noncyclic) mapping, and let $(t_n)_n$ and $(c_n)_n$ be positive sequences, $\lim_{n\to+\infty} t_n = 0$, $\lim_{n\to+\infty} c_n = 1$, such that for every $n \in \mathbb{N}$ and for all $(u,v) \in (\operatorname{cov}(A \cup B))^2$ such that $d \leq \|u - v\| \leq 3d$, where $d = \operatorname{dist}(A,B) > 0$,*

$$\left\| T \circ g(u) - T \circ h(v) \right\| \leq c_n \left(t_n + \|u - v\| \right).$$

Then there exists $(x^*, y^*) \in A \times B$ such that

$$\left\| x^* - Tx^* \right\| = \operatorname{dist}(A,B) = \left\| y^* - Ty^* \right\| \tag{15}$$

$$\left(\textit{resp.}, Tx^* = x^*, Ty^* = y^*, \text{ and } \left\| x^* - y^* \right\| = \operatorname{dist}(A,B) \right). \tag{16}$$

Proof According to Lemma 4, the mapping T is cyclic (resp., noncyclic) relatively nonexpansive in $A \cup B$, where (A,B) is a nonempty closed bounded convex pair of the uniformly convex Banach space X; so the result follows from the paper of Eldred et al. [5]. □

Corollary 6 *Let (A,B) be a nonempty closed bounded convex pair in a uniformly convex Banach space X. Let $g : \operatorname{cov}(A \cup B) \to A$ and $h : \operatorname{cov}(A \cup B) \to B$ be mappings such that $g_{|A} = Id_A$ and $h_{|B} = Id_B$. Let $T : A \cup B \to A \cup B$ be a cyclic (resp., noncyclic) mapping, and let $(t_n)_n$ and $(c_n)_n$ be strictly positive sequences, $\lim_{n\to+\infty} t_n = 0$, $\lim_{n\to+\infty} c_n = 1$, such that for every $n \in \mathbb{N}$ and for all $(u,v) \in (\operatorname{cov}(A \cup B))^2$,*

$$\left(d \leq \|u - v\| \leq t_n + 3d \Rightarrow \left\| T \circ g(u) - T \circ h(v) \right\| \leq c_n (t_n + d) \right), \tag{17}$$

where $d = \operatorname{dist}(A,B)$. Then there exists $(x^*, y^*) \in A \times B$ such that

$$\left\| x^* - Tx^* \right\| = \operatorname{dist}(A,B) = \left\| y^* - Ty^* \right\|$$

$$\left(\textit{resp.}, Tx^* = x^*, Ty^* = y^*, \text{ and } \left\| x^* - y^* \right\| = \operatorname{dist}(A,B) \right).$$

Proof We distinguish two cases $d > 0$ and $d = 0$.

Case 1: $d > 0$.

Let $(u,v) \in (\operatorname{cov}(A \cup B))^2$ be such that $d \leq \|u - v\| \leq 3d$, so for each $n \in \mathbb{N}$, $d \leq \|u - v\| \leq t_n + 3d$, and according to implication (17), we get

$$\left\| T \circ g(u) - T \circ h(v) \right\| \leq c_n (t_n + d) \quad \text{for all } n \in \mathbb{N}.$$

We thus obtain the result according to Theorem 5.

Case 2: $d = 0$.

In this case, we claim that $A \cap B \neq \emptyset$. Indeed, since $\operatorname{dist}(A,B) = 0$, there exists a sequence $((x_m, y_m))_{m\geq 0}$ in $A \times B$ such that $\lim_{m\to+\infty} d(x_m, y_m) = 0$. Since the space X is a uniformly convex Banach space, it is therefore reflexive, and since A and B are closed and bounded, the sequence $((x_m, y_m))_{m\geq 0}$ admits a subsequence $((x_{\phi(m)}, y_{\phi(m)}))_{m\geq 0}$ that converges weakly to $(a,b) \in A \times B$. By the weak lower semicontinuity of the norm $\|\cdot\|$ we have

$$\|a-b\| \leq \lim_{m\to+\infty} \|x_{\phi(m)} - y_{\phi(m)}\| = 0.$$

Thus $a = b$, which shows that $A \cap B \neq \emptyset$.

Take $x, y \in A \cap B$ such that $x \neq y$ and $n \in \mathbb{N}$. There is a unique $p_n \in \mathbb{N}$ such that

$$p_n \leq \frac{\|x-y\|}{t_n} < p_n + 1. \tag{18}$$

Put

$$z_k = \left(1 - \frac{k}{p_n+1}\right) \cdot x + \frac{k}{p_n+1} \cdot y \quad \text{for } k = 0, 1, \dots, p_n + 1.$$

Then $z_k \in A \cap B$ for all $k \in \{0, 1, \dots, p_n + 1\}$, because $A \cap B$ is convex; moreover,

$$\|z_k - z_{k+1}\| = \frac{\|x-y\|}{p_n+1} < t_n \quad \text{for all } k \in \{0, 1, \dots, p_n\}. \tag{19}$$

Applying implication (17) and inequalities (18) and (19), we obtain

$$\begin{aligned} \|Tx - Ty\| &\leq \sum_{j=0}^{p_n} \bigl\| T(z_k) - T(z_{k+1}) \bigr\| \\ &\leq \sum_{j=0}^{p_n} c_n t_n \\ &\leq (p_n + 1) c_n t_n \\ &\leq c_n \|x-y\| + c_n t_n. \end{aligned}$$

Letting n tend to $+\infty$ in the previous inequality, since $\lim_{n\to+\infty} c_n = 1$ and $\lim_{n\to+\infty} t_n = 0$, we obtain

$$\|Tx - Ty\| \leq \|x - y\|.$$

In this case, $A \cap B \neq \emptyset$, the restriction of T to $A \cap B$ is nonexpansive, and the result follows from the Browder–Göhde–Kirk result. □

Remark 2 Under the hypotheses of the corollary, if we take $A = B$, then $\operatorname{cov}(A \cup A) = A$, $d = \operatorname{dist}(A,A) = 0$, and $h = g = Id_A$, and there is a fixed point of T. The difference between this corollary (for the case $d = 0$) and Proposition 1 in [10] is that the corollary uses the implication

$$\bigl(\|u - v\| \leq t_n \Rightarrow \bigl\| T(u) - T(v) \bigr\| \leq c_n t_n\bigr), \tag{20}$$

whereas Matkowski's proposition uses (3) and the continuity of T.

We will provide an example of Corollary 6, which justifies that for elements u and v in $\operatorname{cov}(A \cup B)$, where $|u-v| > 3d$, we can infer the following:

$$\bigl(|u-v| \leq t_n + 3d \Rightarrow \bigl|T \circ g(u) - T \circ h(v)\bigr| \leq c_n(t_n + d)\bigr). \tag{21}$$

Furthermore, we can obtain the result of this example using Theorem 5 in a straightforward manner.

Example 1 Let $A = [-6,-1]$ and $B = [0,1]$ be two parts of $\mathbb{R}$. We denote by pr_A and pr_B the projections on A and B, respectively. Let T be the mapping defined on $A \cup B$ by

$$T(x) = \begin{cases} \frac{1}{n} & \text{if } x \in]\frac{1}{n+1}, \frac{1}{n}] \text{ and } n \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x \in A. \end{cases}$$

It is clear that (A,B) is a nonempty bounded closed convex pair in a uniformly convex Banach space $\mathbb{R}$, $\operatorname{cov}(A \cup B) = [-6,1]$, $d = \operatorname{dist}(A,B) = 1$, $\operatorname{diam}(A,B) = 7$, and $TA \subset A$, $TB \subset B$. Consider the sequences $(t_n)_{\geq 1}$ and $(c_n)_{\geq 1}$ defined by $t_n = \frac{1}{n}$ and $c_n = 1 + \frac{1}{n}$ for $n \in \mathbb{N} \setminus \{0\}$.

Let $(u,v) \in (\operatorname{cov}(A \cup B))^2$ and $n \in \mathbb{N} \setminus \{0\}$ be such that $1 \leq |u-v| \leq t_n + 3$ (we can have elements u and v such that $3d = 3 < |u-v| \leq t_n + 3$; for example, $x = t_n$ and $y = -3$). We have

$$\bigl|T \circ pr_A(u) - T \circ pr_B(v)\bigr| = \begin{cases} |T(u) - T(0)| & \text{if } (u,v) \in A \times (\operatorname{cov}(A \cup B) \setminus B), \\ |T(u) - T(v)| & \text{if } (u,v) \in A \times B \text{ or } (u,v) \in B \times A, \\ |T(-1) - T(0)| & \text{if } u \in (\operatorname{cov}(A \cup B) \setminus A) \\ & \text{and } v \in (\operatorname{cov}(A \cup B) \setminus B), \\ |T(-1) - T(v)| & \text{if } u \in (\operatorname{cov}(A \cup B) \setminus A) \\ & \text{and } v \in B. \end{cases}$$

In the case where $v \in B \setminus \{0\} =]0,1]$ and $u \in A$, there exists $m \in \mathbb{N} \setminus \{0\}$ such that $v \in]t_{m+1}, t_m]$,

$$1 \leq v - u \leq t_n + 3 \quad \text{if and only if} \quad 0 < v \leq t_n + 3 + u.$$

We must have $-3 \leq u \leq -1$. In particular, for the element $u = -3$, since $1 \leq v - u \leq t_n + 3$, we have $v \leq t_n$, so $t_m \leq t_n$, and, consequently,

$$\bigl|T \circ pr_A(u) - T \circ pr_B(v)\bigr| = t_m + 1 \leq t_n + 1 \leq c_n(t_n + 1).$$

This inequality is also true for the other cases,

$$\bigl|T \circ pr_A(u) - T \circ pr_B(v)\bigr| = 1 < t_n + 1 \leq c_n(t_n + 1).$$

Hence

$$\bigl(1 \leq |u-v| \leq t_n + 3 \Rightarrow \bigl|T \circ pr_A(u) - T \circ pr_B(v)\bigr| \leq c_n(t_n + 1)\bigr).$$

Since $\lim_{n\to+\infty} t_n = 0$ and $\lim_{n\to+\infty} c_n = 1$, by Corollary 6 there exists a pair (x^*, y^*) in $A \times B$ such that

$$Tx^* = x^*, \qquad Ty^* = y^*, \quad \text{and} \quad \left|x^* - y^*\right| = \operatorname{dist}(A,B)$$

with $x^* = -1$ and $y^* = 0$.

2.2 Some auxiliary results on relatively (L,d)-mappings

Definition 1 Let (A,B) be a nonempty pair in a normed space $(X, \|\cdot\|)$, $d := \operatorname{dist}(A,B)$, and $L \geq \frac{1}{2}$. A mapping $T : A \cup B \to A \cup B$ is said to be a relatively (L,d)-mapping on $A \cup B$ if for all $(x,y) \in A \times B$,

$$\left\|T(x) - T(y)\right\| \leq L\bigl(\|x - y\| + d\bigr). \tag{22}$$

Proposition 7 *Let (A,B) be a nonempty pair in a normed space X. Let $\beta \in \mathcal{B}_{L,d}$ with $d = \operatorname{dist}(A,B) > 0$ and $L \geq \frac{1}{2}$. Let $T : A \cup B \to A \cup B$ be a cyclic (resp., noncyclic) mapping satisfying the following condition*:

$$\left\|T(u) - T(v)\right\| \leq \beta\bigl(\|u - v\|\bigr) \tag{23}$$

for all $(u,v) \in A \times B$. Then T is a relatively (L,d)-mapping on $A \cup B$.

Proof

- As $\liminf_{t\to 0^+} \frac{\beta(t+d)}{t+d} = L$, for every $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$\beta(t_\varepsilon + d) < (L + \varepsilon)(t_\varepsilon + d) \tag{24}$$

$$\text{and} \quad \lim_{\varepsilon\to 0^+} t_\varepsilon = 0. \tag{25}$$

  By the monotony of β and inequalities(24) and (25) we have

$$\beta(d) \leq \liminf_{\varepsilon\to 0^+} \beta(t_\varepsilon + d) \leq \lim_{\varepsilon\to 0^+} (L + \varepsilon)(t_\varepsilon + d) = Ld.$$

  Thus $\beta(d) \leq Ld$. Moreover, by (ii), $\beta(nd) \leq ndL$ for all $n \in \mathbb{N} \setminus \{0\}$.
- Let $t \in [d, +\infty[$. Then there is an integer $n \geq 1$ such that

$$nd \leq t < (n+1)d \leq t + d,$$

  and then

$$\beta(t) \leq \beta\bigl((n+1)d\bigr) \leq (n+1)\beta(d) \leq (n+1)Ld \leq L(t + d). \tag{26}$$

- Let $(u,v) \in A \times B$. Then $\|u - v\| \geq d$, and from (26) we have

$$\left\|T(u) - T(v)\right\| \leq \beta\bigl(\|u - v\|\bigr) \leq L\bigl(\|u - v\| + d\bigr).$$

This finishes the proof. □

We give the following simple example of a relatively (L,d)-mapping T that is not relatively nonexpansive.

Example 2 Consider the space $\ell^p(\mathbb{R})$, $1 \leq p < \infty$, endowed with the norm $\|\cdot\| := \|\cdot\|_p$. Let

$$A = \left\{ x = \lambda.e_1 + \mu.e_2 + \sum_{n=3}^{+\infty} \frac{1}{2^n}.e_n \in \ell^p(\mathbb{R}) : 0 \leq \lambda \leq 2 \text{ and } 1 \leq \mu \leq 2 \right\}$$

and

$$B = \left\{ y = \lambda'.e_1 + \mu'.e_2 + \sum_{n=3}^{+\infty} \frac{1}{2^n}.e_n \in \ell^p(\mathbb{R}) : 0 \leq \lambda' \leq 2 \text{ and } -2 \leq \mu' \leq -1 \right\},$$

where e_n the sequence consisting of 1*s* at the *n*th place and 0s elsewhere.

For all $(x,y) \in A \times B$,

$$\|x-y\|_p = \left((\lambda - \lambda')^p + (\mu - \mu')^p \right)^{\frac{1}{p}} \geq 2,$$

where $2 = \|a-b\|_p$ with $a = e_2 + \sum_{n=3}^{+\infty} \frac{1}{2^n} \cdot e_n \in A$ and $b = -e_2 + \sum_{n=3}^{+\infty} \frac{1}{2^n} \cdot e_n \in B$. Hence $d = \operatorname{dist}(A,B) = 2$.

Consider the mapping $T : A \cup B \to A \cup B$ defined by

$$T(x) = 2.e_1 + \frac{\mu+1}{2} \cdot e_2 + \sum_{n=3}^{+\infty} \frac{1}{2^n} \cdot e_n \quad \text{if } x \in A,$$

$$T(y) = \frac{\mu'-1}{2} \cdot e_2 + \sum_{n=3}^{+\infty} \frac{1}{2^n} \cdot e_n \quad \text{if } y \in B.$$

We have $T(A) \subset A$ and $T(B) \subset B$.

– Letting $x \in A$ and $y \in B$,

$$\begin{aligned} \left\| T(x) - T(y) \right\|_p &= \left(2^p + \left(\frac{\mu - \mu'}{2} + 1 \right)^p \right)^{\frac{1}{p}} \\ &\leq \left(2^p + (\mu - \mu')^p \right)^{\frac{1}{p}} \\ &\leq \left(2^p + \|x-y\|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Hence

$$\left\| T(x) - T(y) \right\|_p \leq \beta\left(\|x-y\|_p \right)$$

with $\beta(t) = (2^p + t^p)^{\frac{1}{p}}$ for $t \in [0, +\infty[$. We have

$$\liminf_{t\to 0^+} \frac{\beta(t+2)}{t+2} = \lim_{t\to 0^+} \frac{(2^p + (t+2)^p)^{\frac{1}{p}}}{t+2} = 2^{\frac{1}{p}} > \frac{1}{2}.$$

Moreover, β satisfies conditions (i) and (ii), and so $\beta \in \mathcal{B}_{2^{\frac{1}{p}},2}$.

By Proposition 7, for all $(x,y) \in A \times B$,

$$\left\| T(x) - T(y) \right\|_p \le 2^{\frac{1}{p}} \left(\|x-y\|_p + \operatorname{dist}(A,B) \right).$$

Then T is a noncyclic relatively $(2^{\frac{1}{p}},2)$-mapping on $A \cup B$.

- The mapping T from the previous example is not relatively nonexpansive on $A \cup B$/ For example, taking $x = e_1 + e_2 + \sum_{n=3}^{+\infty} \frac{1}{2^n} \cdot e_n$ and $y = -e_2 + \sum_{n=3}^{+\infty} \frac{1}{2^n} \cdot e_n$, we get

$$\begin{aligned} 2 = \operatorname{dist}(A,B) &< \|x-y\|_p = \left(1 + 2^p\right)^{\frac{1}{p}} \\ &< \left\| T(x) - T(y) \right\|_p = \left(2^p + 2^p\right)^{\frac{1}{p}} = 2^{1+\frac{1}{p}}. \end{aligned}$$

2.3 An application to functional equations

Let $\mathcal{U}$ be a nonempty open subset of $\mathbb{R}^m$ such that $\mu(\mathcal{U}) = 1$, where $m \in \mathbb{N} \setminus \{0\}$, and μ is the Lebesgue measure on $\mathbb{R}^m$. We denote by $L^2(\mathcal{U})$ the space of measurable functions $f : \mathcal{U} \to \mathbb{R}$ for which $|f|^2$ is integrable with respect to μ. We equip $L^2(\mathcal{U})$ with the norm

$$\|f\|_2 = \left(\int_{\mathcal{U}} |f|^2 \, d\mu \right)^{\frac{1}{2}}.$$

It is known that $(L^2(\mathcal{U}), \| \cdot \|_2)$ is a uniformly convex Banach space (see Clarkson [4]).

We assume the following conditions:

(H_1) Let $\mathcal{M} : \mathcal{U} \to \mathcal{U}$ be a locally Lipschitzian homeomorphic mapping, and let $\gamma : \mathcal{U} \times \mathcal{U} \to [0,+\infty)$ be a measurable Lebesgue function such that

$$\left(\gamma\left(x, \mathcal{M}(x)\right)\right)^2 = \left|J_{\mathcal{M}}(x)\right| \text{ a.e. in } \mathcal{U} \text{ and } \mu\left(\mathcal{M}(\mathcal{U})\right) = 1, \tag{27}$$

where for $\mathcal{M} = (\mathcal{M}_1, \dots, \mathcal{M}_m)$ and $x = (x_1, \dots, x_m)$.

$$J_{\mathcal{M}}(x) := \frac{\partial(\mathcal{M}_1, \dots, \mathcal{M}_m)}{\partial(x_1, \dots, x_m)}$$

is the Jacobian of $\mathcal{M}$.

(H_2) $h, k : \mathcal{U} \times \mathbb{R} \to \mathbb{R}$ are functions such that for all $y \in \mathbb{R}$, the functions $x \mapsto h(x,y)$ and $x \mapsto k(x,y)$ are Lebesgue measurable on $\mathcal{U}$, and for almost all $x \in \mathcal{U}$ (with respect to μ), the functions $y \mapsto h(x,y)$ and $y \mapsto k(x,y)$ are continuous on $\mathbb{R}$,

(H_3) Let $g_1, g_2 \in L^2(\mathcal{U})$ be such that $0 < g_1 \le g_2$ a.e. in $\mathcal{U}$ and for all $(x,y) \in \mathcal{U} \times \mathbb{R}$, we have the following implications:

$$g_1\left(\mathcal{M}(x)\right) \le y \le g_2\left(\mathcal{M}(x)\right) \Rightarrow \begin{cases} g_1(x) \le \gamma(x, \mathcal{M}(x)) h(x,y) \le g_2(x), \\ g_1(x) \le \gamma(x, \mathcal{M}(x)) k(x,y) \le g_2(x), \end{cases} \tag{28}$$

$$h_1\left(\mathcal{M}(x)\right) \le y \le h_2\left(\mathcal{M}(x)\right) \Rightarrow \begin{cases} h_1(x) \le \gamma(x, \mathcal{M}(x)) h(x,y) \le h_2(x), \\ h_1(x) \le \gamma(x, \mathcal{M}(x)) k(x,y) \le h_2(x), \end{cases} \tag{29}$$

where $h_1 = -g_2$ and $h_2 = -g_1$

We set

$$A = \{\phi \in L^2(\mathcal{U}) : g_1 \le \phi \le g_2 a.e. \text{ in } \mathcal{U}\}$$
$$\text{and} \quad B = \{\phi \in L^2(\mathcal{U}) : h_1 \le \phi \le h_2 a.e. \text{ in } \mathcal{U}\}.$$

We have $d = \operatorname{dist}(A,B) = \|g_1 - h_2\|_2 > 0$ and $A \cap B = \emptyset$.

We define the maps T and S on $A \cup B$ by

$$\begin{cases} T(\phi)(x) = \gamma(x,\mathcal{M}(x))h(x,\phi(\mathcal{M}(x))), \\ S(\phi)(x) = \gamma(x,\mathcal{M}(x))k(x,\phi(\mathcal{M}(x))), \end{cases} \quad \phi \in A \cup B \text{ and } x \in \mathcal{U}. \tag{30}$$

Theorem 8 *Assume that hypotheses* (H_1)*–*(H_3) *hold. Suppose*

$$\sqrt{2}\big(\big|k(x,y_1) - k(x,y_2)\big| + d\big) \le \big|h(x,y_1) - h(x,y_2)\big| \le \beta\big(|y_1 - y_2|\big) \tag{31}$$

for all $x \in \mathcal{U}$ *and* $y_1, y_2 \in \mathbb{R}$, *where* $\beta \in \mathcal{B}_{1,d}$, *and* $t \mapsto [\beta(t^{\frac{1}{2}})]^2$ *is concave on* $[0,+\infty)$. *Then there exists* $(\phi_0,\psi_0) \in A \times B$ *such that*

$$S(\phi_0) = \phi_0, \qquad S(\psi_0) = \psi_0 \quad \textit{and} \quad \operatorname{dist}(A,B) = \|\psi_0 - \psi_0)\|_2.$$

Proof First, we verify without difficulty that (A,B) is nonempty bounded closed and convex in the Hilbert space $(L^2(\mathcal{U}), \|\cdot\|_2)$ equipped with the real scalar product

$$\langle \phi, \psi \rangle = \int_{\mathcal{U}} \phi(x)\psi(x)\,dx \quad \text{for } \phi, \psi \in L^2(\mathcal{U}).$$

Take an arbitrary $\phi \in A \cup B$. Then, in view of the Carathéodory theorem [2], conditions (H_1) and (H_2) imply that the functions $T(\phi)$ and $S(\phi)$ are Lebesgue measurable.

Note that $T(A) \subseteq A$. Indeed, for $\phi \in A$, we have $g_1 \le \phi \le g_2$ a.e. in $\mathcal{U}$, so from implication (28) we have

$$g_1\big(\mathcal{M}(x)\big) \le \phi\big(\mathcal{M}(x)\big) \le g_2\big(\mathcal{M}(x)\big) \quad \text{a.e. in } \Omega,$$

and thus, in view of condition (H_3),

$$g_1(x) \le \gamma\big(x,\mathcal{M}(x)\big)h\big(x,\phi\big(\mathcal{M}(x)\big)\big) \le g_2(x) \quad \text{a.e. in } \mathcal{U},$$

that is, $T(\phi) \in A$.

Similarly, we justify that $T(B) \subseteq B$, $S(A) \subseteq A$, and $S(B) \subseteq B$ using condition (H_3) and implications (28) and (29).

Step 1: Let $(\phi,\psi) \in A \times B$. Using the assumptions $\mu(\mathcal{M}(\mathcal{U})) = 1$, $(\gamma(x,\mathcal{M}(x)))^2 = |J_{\mathcal{M}}(x)|$ a.e. in $\mathcal{U}$ (27), and $\beta \in \mathcal{B}_{1,d}$, we obtain the following inequalities:

$$\begin{aligned} \|T(\phi) - T(\psi)\|_2^2 &= \int_{\mathcal{U}} \big|T(\phi)(x) - T(\psi)(x)\big|^2\,dx \\ &= \int_{\mathcal{U}} \big(\gamma\big(x,\mathcal{M}(x)\big)\big)^2 \big|h\big(x,\phi\big(\mathcal{M}(x)\big) - h\big(x,\psi\big(\mathcal{M}(x)\big)\big)\big|^2\,dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathcal{U}} \big(\gamma(x), \mathcal{M}(x)\big)^2 \big(\beta\big(\phi\big(\mathcal{M}(x)\big) - \psi\big(\mathcal{M}(x)|\big)\big)^2 \, dx \\
&= \int_{\mathcal{U}} \big|J_{\mathcal{M}}(x)\big| \big(\beta\big(\big|\phi\big(\mathcal{M}(x)\big) - \psi\big(\mathcal{M}(x)\big|\big)\big)^2 \, dx \\
&= \int_{\mathcal{M}(\mathcal{U})} \big(\beta\big(\big|\phi(x) - \psi(x)\big|\big)\big)^2 \, dx \\
&\leq \int_{\mathcal{U}} \big(\beta\big(\big|\phi(x) - \psi(x)\big|\big)\big)^2 \, dx.
\end{aligned}
$$

Since the function $t \mapsto [\beta(t^{\frac{1}{2}})]^2$ is concave on $[0, +\infty)$, we have

$$
\begin{aligned}
\big\|T(\phi) - T(\psi)\big\|_2 &\leq \left(\int_{\mathcal{U}} \big[\beta\big(\big|\phi(x) - \psi(x)\big|\big)\big]^2 \, dx\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathcal{U}} \big[\beta\big(\big(\big|\phi(x) - \psi(x)\big|^2\big)^{\frac{1}{2}}\big)\big]^2 \, dx\right)^{\frac{1}{2}} \\
&\leq \left[\left(\beta\left(\int_{\mathcal{U}} \big|\phi(x) - \psi(x)\big|^2 \, dx\right)^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} \\
&\leq \beta\big(\|\phi - \psi\|_2\big),
\end{aligned}
$$

Thus all the assumptions of Proposition 7 are satisfied. Consequently, for $(\phi, \psi) \in A \times B$,

$$
\big\|T(\phi) - T(\psi)\big\|_2 \leq \|\phi - \psi\|_2 + d. \tag{32}
$$

Step 2: Let $(\phi, \psi) \in A \times B$. By inequality (31), for each $x \in \mathcal{U}$, we have

$$
2\big(\big|k\big(x, \phi\big(\mathcal{M}(x)\big) - k\big(x, \psi\big(\mathcal{M}(x)\big)\big| + d\big)^2 \leq \big|h\big(x, \phi\big(\mathcal{M}(x)\big) - h\big(x, \psi\big(\mathcal{M}(x)\big)\big)\big|^2,
$$

so that

$$
\begin{aligned}
&2\big(\gamma\big(x, \mathcal{M}(x)\big)\big)^2 \big(\big|k\big(x, \phi\big(\mathcal{M}(x)\big) - k\big(x, \psi\big(\mathcal{M}(x)\big)\big|^2 + d^2\big) \\
&\quad \leq \big(\gamma\big(x, \mathcal{M}(x)\big)\big)^2 \big(\big|h\big(x, \phi\big(\mathcal{M}(x)\big) - h\big(x, \psi\big(\mathcal{M}(x)\big)\big)\big|^2\big).
\end{aligned}
$$

Integrating both sides, we get

$$
\begin{aligned}
&2\int_{\mathcal{U}} \big(\gamma\big(x, \mathcal{M}(x)\big)\big)^2 \big(\big|k\big(x, \phi\big(\mathcal{M}(x)\big) - k\big(x, \psi\big(\mathcal{M}(x)\big)\big|^2 + d^2\big) \, dx \\
&\quad \leq \int_{\mathcal{U}} \big(\gamma\big(x, \mathcal{M}(x)\big)\big)^2 \big(\big|h\big(x, \phi\big(\mathcal{M}(x)\big) - h\big(x, \psi\big(\mathcal{M}(x)\big)\big)\big|^2\big) \, dx,
\end{aligned}
$$

whence

$$
\begin{aligned}
&2\left(\int_{\mathcal{U}} \big|S(\phi)(x) - S(\psi)(x)\big|^2 \, dx + d^2\right) \leq \int_{\mathcal{U}} \big|T(\phi)(x) - T(\psi)(x)\big|^2 \, dx, \\
&\sqrt{2}\sqrt{\int_{\mathcal{U}} \big|S(\phi)(x) - S(\psi)(x)\big|^2 \, dx + d^2} \leq \sqrt{\int_{\mathcal{U}} \big|T(\phi)(x) - T(\psi)(x)\big|^2 \, dx}.
\end{aligned}
$$

Since

$$\sqrt{\int_{\mathcal{U}} \left|S(\phi)(x) - S(\psi)(x)\right|^2 dx} + d \le \sqrt{2}\sqrt{\int_{\mathcal{U}} \left|S(\phi)(x) - S(\psi)(x)\right|^2 dx + d^2},$$

we have

$$\left\|S(\phi) - S(\psi)\right\|_2 + d \le \left\|T(\phi) - T(\psi)\right\|_2$$

and, according to inequality (32),

$$\left\|S(\phi) - S(\psi)\right\|_2 + d \le \left\|T(\phi) - T(\psi)\right\|_2 \le \|\phi - \psi\|_2 + d.$$

Hence, for all $(\phi, \psi) \in A \times B$,

$$\left\|S(\phi) - S(\psi)\right\|_2 \le \|\phi - \psi\|_2.$$

Thus S is relatively nonexpansive on $A \cup B$. The hypotheses of the result of Eldred et al. [5] for a noncyclic mapping hold for S, so there exists $(\phi_0, \psi_0) \in A \times B$ such that

$$S(\phi_0) = \phi_0, \qquad S(\psi_0) = \psi_0, \quad \text{and} \quad \operatorname{dist}(A,B) = \|\phi_0 - \psi_0)\|_2,$$

and necessarily $\phi_0 = g_1$ and $\psi_0 = h_2$. □

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