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Extending Snow's algorithm for computations in the finite Weyl groups

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Abstract

In 1990, D. Snow proposed an effective algorithm for computing the orbits of finite Weyl groups. Snow's algorithm is designed for computation of weights, *W*-orbits, and elements of the Weyl group. An extension of Snow's algorithm is proposed, which allows to find pairs of mutually inverse elements together with the calculation of *W*-orbits in the same runtime cycle. This simplifies the calculation of conjugacy classes in the Weyl group. As an example, the complete list of elements of the Weyl group $W(D_4)$ obtained using the extended Snow's algorithm. The elements of $W(D_4)$ are specified in two ways: as reduced expressions and as matrices of the faithful representation. Then we give a partition of this group into conjugacy classes with elements specified as reduced expressions. Various forms are given for representatives of the conjugacy classes of $W(D_4)$: with Carter diagrams, with reduced expressions, and with signed cycle-types. In the Appendix, we provide an implementation of the algorithm in Python.

1 Introduction

1.1 Snow's algorithm: finding *W*-orbits

In 1990, D. Snow in [8] proposed an effective algorithm for computing the orbits of the finite Weyl groups. The algorithm starts with a certain dominant weight and acts on it by all simple reflections. This operation produces the complete list of weights of level 1 and the complete list of all elements of length 1 in the Weyl group W. In the next step, we again use reflections to obtain a list of level 2 weights and all elements of length 2, and so on. This approach has a repetition problem: the same weight can be obtained in several ways, and the list of elements of the Weyl group lying in some level contains duplicate elements. Snow presented a solution showing which weight ν should be taken on the level L_k and which reflection s_i should be applied to ν to get the given weight ξ at the level L_{k+1}. Using Snow's algorithm, the choice of ν and s_i can be done in the unique way. This solution avoids duplicate elements, see Sect. 2.4.

The computation of the elements of the Weyl group in Snow's algorithm is based on the following fact: there is a one-to-one correspondence between the Weyl chambers and the elements of the Weyl group, and the Weyl group acts transitively on the set of Weyl chambers. Each element from the closure of the fundamental Weyl chamber generates a Weyl group orbit (W-orbit) whose length coincides with the order of the Weyl group. The

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W-orbit is constructed under the action of the Weyl group on some dominant weight. The weights of the W-orbit are constructed together with the elements of the Weyl group W.

Let $\varPhi$ be the root system associated with a certain semisimple Lie algebra $\mathcal{L}$, W be the Weyl group associated to $\varPhi$, and $\mathcal{E}$ be a real space spanned by the roots of $\varPhi$. A *weight* is an element $\xi \in \mathcal{E}$ such that $\langle \xi, \alpha \rangle \in \mathbb{Z}$ for all roots $\alpha \in \varPhi$. The set of weights Λ forms a subgroup of $\mathcal{E}$, i.e., $\varPhi \subset \Lambda \subset \mathcal{E}$. The significance of the weights theory is largely determined by the highest weight theorem in the representation theory of semisimple Lie algebras.[1]

Snow's algorithm produces the weights of W-orbits and elements of the Weyl group by *levels*. For any $\xi \in \mathcal{E}$ there exist $w \in W$ and v from the closure $\overline{C}$ of the fundamental Weyl chamber such that $\xi = w(v)$, see Theorem A.1. The level of ξ is as follows:

$$\operatorname{level}(\xi) = l(w), \tag{1.1}$$

where $l(w)$ is the smallest length of w given as a reduced expression, [2, Ch. IV, §1, n°1]. The *level of weight* ξ is equal to the number of reflections needed to move ξ to some dominant weight lying in the closure of the fundamental chamber $\overline{C}$, see Proposition A.3. Following Snow, [8], the level of $w \in W$ is also defined as $l(w)$:

$$\operatorname{level}(w) = l(w).$$

Using Snow's algorithm, searching for elements of the Weyl group and their partitioning is carried out in accordance with the level of the element, see tables in Sect. 5.

In Sect. 2, we will look at some details of Snow's algorithm. The sizes of all levels and the total computation time for cases B_7, D_8, E_7, B_8 are gathered in Table 1.

1.2 Extended Snow's algorithm: finding inverse elements

To construct conjugacy classes of a group, one must first find all pairs of mutually inverse elements of the group. In the case of the Weyl group, each element and its inverse belong to the same level. However, even searching within a level can be quite an expensive task, especially for very large levels, see Table 1, where the length of the levels is several hundred thousand elements. Let

$$w = s_{i_1} s_{i_2} \cdots s_{i_{k-1}} s_{i_k} \tag{1.2}$$

be an element of the level L_k. We can find the inverse element w^{-1} by reversing the order of the reduced expression w:

$$w^{-1} = s_{i_k} s_{i_{k-1}} \cdots s_{i_2} s_{i_1}. \tag{1.3}$$

However, the inverse element must be found in accordance to the repetition prevention mechanism from Theorem 2.1. Then the reduced expression may differ from the reverse order of w.

An extension of Snow's algorithm is designed to get around this obstacle: for any element $w \in W$, one must obtain the inverse element w^{-1}, but this must be done in the order

---

[1] The highest weight theorem was proved by E. Cartan in 1913, [3], see Sect. A.5.

Table 1 The Weyl groups B_7, D_8, E_7, B_8: level sizes and total runtime of the extended Snow's algorithm

B_7		D_8		E_7		B_8	
Level	Size	Level	Size	Level	Size	Level	Size
0, 49	1	0, 56	1	0, 63	1	0, 64	1
1, 48	7	1, 55	8	1, 62	7	1, 63	8
2, 47	27	2, 54	35	2, 61	27	2, 62	35
3, 46	77	3, 53	112	3, 60	77	3, 61	112
4, 45	181	4, 52	293	4, 59	182	4, 60	293
5, 44	371	5, 51	664	5, 58	378	5, 59	664
6, 43	686	6, 50	1350	6, 57	713	6, 58	1350
7, 42	1170	7, 49	2520	7, 56	1247	7, 57	2520
8, 41	1869	8, 48	4388	8, 55	2051	8, 56	4389
9, 40	2827	9, 47	7208	9, 54	3205	9, 55	7216
10, 39	4082	10, 46	11,263	10, 53	4975	10, 54	11,298
11, 38	5662	11, 45	16,848	11, 52	6909	11, 53	16,960
12, 37	7581	12, 44	24,248	12, 51	9632	12, 52	24,541
13, 36	9835	13, 43	33,712	13, 50	13,040	13, 51	34,376
14, 35	12,399	14, 42	45,425	14, 49	17,194	14, 50	46,775
15, 34	15,225	15, 41	59,480	15, 48	22,134	15, 49	62,000
16, 33	18,242	16, 40	75,853	16, 47	27,874	16, 48	80,241
17, 32	21,358	17, 39	94,384	17, 46	34,398	17, 47	101,592
18, 31	24,464	18, 38	114,766	18, 45	41,657	18, 46	126,029
19, 30	27,440	19, 37	136,544	19, 44	49,567	19, 45	153,392
20, 29	30,162	20, 36	159,125	20, 43	58,009	20, 44	183,373
21, 28	32,150	21, 35	181,800	21, 42	66,831	21, 43	215,512
22, 27	34,376	22, 34	203,777	22, 41	75,852	22, 42	249,201
23, 26	35,672	23, 33	224,224	23, 40	84,868	23, 41	283,704
24, 25	36,336	24, 32	242,318	24, 39	93,659	24, 40	318,171
		25, 31	257,295	25, 38	101,997	25, 39	351,680
		26, 30	268,504	26, 37	109,655	26, 38	383,270
		27, 29	275,440	27, 36	116,417	27, 37	411,984
		28	277,788	28, 35	122,087	28, 36	436,913
				29, 34	126,497	29, 35	457,240
				30, 33	129,514	30, 34	472,281
				31, 32	131,046	31, 33	481,520
						32	484,636
total 645,120		total 5,169,960		total 2,903,040		total 10,321,920	
time 59 sec		time 570 sec		time 269 sec		time 1153 sec	

specified by Theorem 2.1. The reduced expression of the calculated inverse element will not necessarily be of the form (1.3). Bypassing the specified obstacle achieved through the exchange of information between any element and its inverse during the traversal performed by Snow's algorithm. This information exchange is carried out using the dictionary mechanism described in Sect. 3.

The Weyl group $W(D_4)$ contains 192 elements. In Sect. 4, Carter diagrams and signed cycle-types are used to study of conjugacy classes in $W(D_4)$. In Sect. 5, all elements of $W(D_4)$ are divided into 12 levels. The elements of $W(D_4)$ are specified in two ways: as matrices and as reduced expressions, see Tables 6–24. For each element w, we provide also the reduced expression of the inverse element and its location.

The partition of the group $W(D_4)$ into conjugate classes is given in Sect. 6. There are 13 conjugacy classes including the trivial class containing only identity element e, see Tables 26–37. For each element w of the conjugacy class, we provide the level number k such that $w \in L_k$ and the position of w in the level L_k. With this information, the element w can be found in the tables of levels of Sect. 5.

The execution time of the extended Snow's algorithm for Weyl groups B_7, D_8, E_7, B_8 on CPU 3.7 GHz/Python 3.7.3 are as follows:

B_7	645,120 elements	59 sec
E_7	2,903,040 elements	269 sec
D_8	5,169,960 elements	570 sec
B_8	10,321,920 elements	1153 sec

For the execution time for each level, see Table 1.

Appendix A lists some properties of weights related to Lie algebras and Weyl groups. An implementation of the extended Snow's algorithm in Python is given in Appendix B. An example of procedure for obtaining conjugacy classes is presented in Appendix C.

2 Snow's algorithm: computation of *W*-orbits and levels

2.1 Computation of the *W*-orbits

Snow's algorithm starts with a certain dominant weight and acts on it with all simple reflections. This produces all the weights of level 1 and a list of all elements of length 1 in W. If we apply this procedure again, ignoring duplicates, we obtain the weights of level 2 and a required list of elements of length 2 in W. By repeating this procedure, we compute a list of weights of any level, and the entire group W can be generated if an appropriate initial weight is chosen.

2.2 Computation of level(ξ)

The algorithm provides a simple criterion for adding an orbit element to the list of weights. Let $\xi = (x_1, \dots, x_n)$ be any weight in the basis consisting of fundamental dominant weights, see Sect. A.2.2. What is the level of $s_i(\xi)$ for any simple reflection s_i?

Let w be the element in W such that $\xi = w(\nu)$ for some ν from the fundamental domain $\overline{C}$ with level$(\xi) = l(w)$. By definition of the fundamental weights (A.7), we have

$$\xi = \sum_i x_i \bar{\omega}_i, \quad \text{and} \quad x_i = \langle \xi, \alpha_i \rangle = \big\langle w(\nu), \alpha_i \big\rangle. \tag{2.1}$$

By (A.3) the sign of x_i coincides with the sign of $(w(\nu), \alpha_i)$, then

$$\begin{cases} x_i = 0 & \Longrightarrow \quad s_i(\xi) = \xi, \\ x_i > 0 & \Longrightarrow \quad (w(\nu), \alpha_i) > 0, \\ x_i < 0 & \Longrightarrow \quad (w(\nu), \alpha_i) < 0. \end{cases} \tag{2.2}$$

Here, the first line in (2.2) follows from (A.11). Thus, in the case of $x_i = 0$, the reflection s_i does not change the level:

$$x_i = 0 \quad \Longrightarrow \quad \text{level}\big(s_i(\xi)\big) = \text{level}(\xi). \tag{2.3}$$

Further, since the Cartan–Killing form is invariant under the Weyl group W, we have

$$\begin{aligned} x_i > 0 \quad &\Longrightarrow \quad \big(\nu, w^{-1}(\alpha_i)\big) > 0, \\ x_i < 0 \quad &\Longrightarrow \quad \big(\nu, w^{-1}(\alpha_i)\big) < 0. \end{aligned} \tag{2.4}$$

Since ν is a dominant weight, we have $\langle \nu, \alpha \rangle \geq 0$ for all $\alpha \in \Phi$, see Sect. A.2. Then by Theorem A.5, we have

$$\begin{aligned} x_i > 0 \quad &\Longrightarrow \quad w^{-1}(\alpha_i) \in \Phi^+ \quad \Longrightarrow \quad l(s_i w) = l(w) + 1, \\ x_i < 0 \quad &\Longrightarrow \quad w^{-1}(\alpha_i) \in \Phi^- \quad \Longrightarrow \quad l(s_i w) = l(w) - 1. \end{aligned} \tag{2.5}$$

Thus the level is updated as follows:

$$\operatorname{level}\bigl(s_i(\xi)\bigr) = \begin{cases} \operatorname{level}(\xi) + 1 & \text{if } x_i > 0, \\ \operatorname{level}(\xi) & \text{if } x_i = 0, \\ \operatorname{level}(\xi) - 1 & \text{if } x_i < 0. \end{cases} \tag{2.6}$$

2.3 Arranging the weights by levels

We start from a dominant weight $\mu \in \Lambda^+$, see Eq. (A.6). Let L_k be the kth level of $W \cdot \mu$, i.e.,

$$L_k = \bigl\{\text{weights } \xi \in W \cdot \mu \mid \operatorname{level}(\xi) = k\bigr\}.$$

Then, the orbit $W \cdot \mu$ is the disjoint union of all levels:

$$W \cdot \mu = \bigsqcup_{i=0}^{N} L_i,$$

where N is the maximal possible level in $W \cdot \mu$. By Proposition A.6, the number N is the number of positive roots in C, since this is the maximal length of a Weyl group element.

To construct level L_{k+1} from the previously computed level L_k, we apply reflections s_i. By (2.6), if $x_i > 0$ only reflection s_i move ξ from L_k to L_{k+1}:

$$L_{k+1} = \bigl\{s_i(\xi) \mid i = 1, \dots, l, \xi = (x_1, \dots, x_l) \in L_k, x_i > 0\bigr\}. \tag{2.7}$$

2.4 Snow's solution to the repetition problem

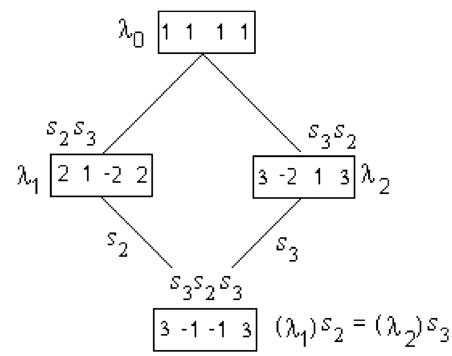
2.4.1 An example of the repetition problem

For explanations about bases $\{\alpha\}$ of simple roots and $\{\bar{\omega}\}$ of fundamental weights, see Sect. A.3.5 and Sect. A.3.6. The main formulas used in calculation are (A.11) and (A.16).

We start with the dominant weight $\lambda_0 = (1, 1, 1, 1)$ and act on this weight by two different elements of level 2 of the Weyl group $W(D_4)$:

$$\begin{aligned} w_1 = s_2 s_3 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(Table 6, elm. 6)}, \\ w_2 = s_3 s_2 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(Table 6, elm. 4)}. \end{aligned} \tag{2.8}$$

Figure 1 Two different reduced expressions that are equal



By Eq. (A.16), we apply w_1 and w_2 to the row vector λ_0 as follows:

$$\begin{aligned} \lambda_1 &= \lambda_0 w_1 = \lambda_0 s_2 s_3 = (2, 1, -2, 2), \\ \lambda_2 &= \lambda_0 w_2 = \lambda_0 s_3 s_2 = (3, -2, 1, 3). \end{aligned} \tag{2.9}$$

Using (A.11), we act by reflection s_2 onto weight λ_1 (one could also use (A.16) as in (2.9)). Here, $m_2 = 1$, $\bar{c}_2 = (-1, 2, -1, -1)$. Similarly, we act by reflection s_3 on λ_2, where $m_3 = 1$, $\bar{c}_3 = (0, -1, 2, 0)$.

$$\begin{aligned} \lambda_1 s_2 &= \lambda_1 - m_2 \bar{c}_2 = (2, 1, -2, 2) - (-1, 2, -1, -1) = (3, -1, -1, 3), \\ \lambda_2 s_3 &= \lambda_2 - m_3 \bar{c}_3 = (3, -2, 1, 3) - (0, -1, 2, 0) = (3, -1, -1, 3). \end{aligned}$$

So, $(\lambda_0)s_3 s_2 s_3 = (\lambda_0)s_2 s_3 s_2$.[2] Thus, weight $(3, -1, -1, 3)$ can be obtained in different ways. This means that both $s_3 s_2 s_3$ and $s_2 s_3 s_2$ must be included in the list of level 3, even though they are two different reduced expressions for the same element.

This is an example of the repetition problem, see Fig. 1. Snow's algorithm solves this problem with the following statement.

Theorem 2.1 (Snow, [8]) *Let L_k be the kth level in the orbit $W \cdot \mu$ of a dominant weight $\mu \in \overline{C}$. Then, for each $\xi = (x_1, \dots, x_l) \in L_{k+1}$, there exists a unique $\nu \in L_k$ and a unique simple reflection s_i such that $s_i(\nu) = \xi$ and $x_i \geq 0$ for $j > i$. In particular, the next level L_{k+1} can be constructed without repetitions from the weights $\nu \in L_k$ by adding $s_i(\nu)$ to L_{k+1} if and only if the ith coordinate of ν is positive and the coordinates of $s_i(\nu)$ after the ith are nonnegative*:

$$\begin{aligned} L_{k+1} = \Big\{ & s_i(\nu) = (x_1, \dots, x_l \mid i = 1, \dots, l, \\ & \nu = (y_1, \dots, y_l) \in L_k, y_i > 0, x_k \geq 0, j > i \Big\}. \end{aligned} \tag{2.10}$$

2.4.2 Application of Theorem 2.1 to Example 2.4.1

Here $\xi = (3, -1, -1, 3)$. For $\nu = \lambda_2$ and reflection s_3, we have $i = 3$ and $x_4 > 0$. By Theorem 2.1, the element $s_3 s_2 s_3$ is added to level 3, see Table 7, element 10. On the other hand, for $\nu = \lambda_1$ and reflection s_2, we have $i = 2$ and $x_3 < 0$. Then, the element $s_2 s_3 s_2$, which is essentially another reduced expression for $s_3 s_2 s_3$, is not added to level 3.

[2] The last relation also follows from the well-known braid relation $s_3 s_2 s_3 = s_2 s_3 s_2$.

3 Extended Snow's algorithm: computation of inverse elements

3.1 Double identification

Because the reduced expression is not unique, we must use another element identification w to recognize the inverse element. The matrix of w in the faithful representation can be chosen as such a requested identifier. We store the following information (*class Element*) about each element w:

```
weight       ''' w(v), where v is dominant integral weight '''
name         ''' reduced expression  w like s3.s4.s2.s3.s1 '''
matr         ''' matrix  w, two—dimensional list '''
name_inv     ''' reduced expression of inverse '''
matr_inv     ''' inverse matrix '''
n_in_lvl     ''' location of w in Level '''
n_inv_in_lvl ''' location of inverse in Level '''
```

For a complete description of this class, see Sect. B.8. The pair (*name, matr*) forms the double identification of the element. The question is why not use a weight that is simply a $1D$-array instead of a matrix that is $2D$-array. The reason is that at the time of calculating the new element given the element w, we do not know the weight of the inverse element w^{-1}. However, we know the inverse matrix w^{-1} and at the same time do not perform a very expensive matrix inversion procedure. Let i the index of the desired reflection in the list of reflections *refl*. Then *refl*[i] (resp. s_i) is the matrix (resp. the symbol) of this reflection. All we have to do is

- multiply the given matrix w on the left by *refl*[i] and the inverse matrix w^{-1} on the right by the same reflection,
- add the symbol s_i on the left to the reduced expression w, and for the reduced expression w^{-1} add the symbol s_i on the right.

When implemented in Python, it looks like this:

```
new_name = 's' + str(i) + '.' + name
new_matr =  np.mathmul(refl[i], matr)
new_name_inv = name_inv + '.s' + str(i)
new_matr = np.mathmul(matr_inv, refl[i])
```

See function *newElem* in Sect. B.9. Here, *np.mathmul* is a function from the *Numpy* package for multiplying two matrices. The dot " . " is used as delimiter between generators in string fields *name*, *name_inv* and *new_name_inv*.

3.2 Dictionary whose key is a matrix

The dictionary *dictElemsOfLevel* is used to exchange information between any element w and its inverse w^{-1}. The dictionary key is the matrix from *class Element*. The matrix is presented as a two-dimensional list. Since a list cannot be a dictionary key in Python, we convert the matrix to a string as follows:

```
key   = ''.join(str(i) for row in self.matr for i in row)
```

The dictionary value corresponding to this *key* is the location *n_in_lvl* of the matrix in *level*(ξ). See function *keyValAndKeyInv()* in Sect. B.8. Let *key* (resp. *key_inv*) be the key corresponding to the *new_matr* (resp. *new_matr_inv*). In the calculation cycle new level L_{k+1} by the level L_k, there are 3 cases, see function *findAllLevels_to_LvlK()* in Sect. B.9. Each record of the dictionary is the pair (*key*, *value*), where *key* is the matrix converted to string, and *value* is the location of w in L_{k+1}.

It should be noted that the dictionary mechanism in Python is realized very efficiently [11].

3.3 Exchange information between w and w^{-1}

The element w leaves in the dictionary record about its location in L_{k+1}. The inverse element w^{-1} will read this record later. There are three typical cases:

Case 1. If the computed matrix *new_matr* is of order 2, i.e., the matrix is inverse to itself, then no message should be left in the dictionary. This is the simplest case. Here,

```
new_elm.n_inv_in_lvl = new_elm.n_in_lvl        #   (1)
```

Case 2: Suppose, after checking the key of the element w, it turned out that *the key is not in the dictionary*. This means that the inverse element will appear later in the calculation loop. Then, the record about the location of w is recorded in the dictionary.

```
dictElemsOfLevel[key_inv] = n_in_lvl          #   (2)
```

The inverse element w^{-1} will read this record later, see (3).

Case 3: Suppose *the key is in the dictionary*. This means that the inverse element left an exact record about its location, see (2):

```
n_in_lvl = dictElemsOfLevel[key]              #   (3)
```

Then, there is no need to write any information in the dictionary, because both *new_elem* and *new_elem_inv* are already informed about each other's location:

```
new_elem_inv = new_level[n_in_lvl]
new_elem_inv.n_inv_in_lvl = new_elm.n_in_lvl
new_elm.n_inv_in_lvl = new_elem_inv.n_in_lvl
```

The keys will be recorded into the dictionary only for *Case 2*. Let ν be number of records of some level L_k, let ω_2 be the number of elements of order 2 in L_k. Then, the number of elements of L_k in the dictionary at the end of the run cycle is $(\nu - \omega_2)/2$. The number of elements of any level in the dictionary will always be less than half of all elements of this level.

Extended Snow's algorithm (ESA) has comparable complexity to the original Snow's algorithm and is, in practice, very efficient in providing information about inverse elements.

A possible strategy for computing conjugacy classes in a Weyl group using the obtained information on inverse elements is presented in Appendix C.

4 Conjugacy classes in $W(D_4)$

In this section, we consider different representations of the conjugacy classes in $W(D_4)$. An algorithm for obtaining conjugacy classes based on a priori information about inverse elements is presented in Appendix C.

4.1 Conjugacy classes of $W(D_4)$ represented by Carter diagrams

First, we will see why, in Table 2, the representative element

$$s_1s_2s_3s_4s_2s_1s_2s_3s_4s_2s_3s_4 \tag{4.1}$$

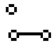
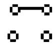
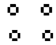
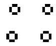
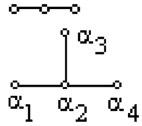
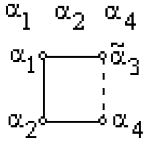
of the conjugacy class 12 is represented as 4 unconnected vertices (root subset $4A_1$), and the representative element

$$s_3s_2s_4s_3s_2s_1 \tag{4.2}$$

of the conjugacy class 11 is represented by the Carter diagram $D_4(a_1)$.

For more convenient work with roots of the root system D_4, we change the notation of vertices from i to α_i. We use the Bourbaki numbering of the vertices of the Dynkin diagram D_4: The reflection s_{α_2} does not commute with reflections s_{α_i}, $i = 1,3,4$, while the reflections s_{α_1}, s_{α_3}, s_{α_4} commute with each other, see Carter diagram in Table 2, line 10.

Table 2 Conjugacy classes in the Weyl groups D_4, see Tables 26–37 and Table 3

N°	Carter diagram[a]	Representative element	Elms	Root subset	Order	Signed cycle-type[b]
0	–	e	1	$\emptyset$	1	[1111]
1		s_1	12	A_1	2	[211]
2		s_1s_2	32	A_2	3	[31]
3		s_1s_3	6	$2A_1$	2	[22]
4		s_1s_4	6	$2A_1$	2	[22]
5		s_3s_4	6	D_2	2	$[\bar{1}\bar{1}11]$
6		$s_1s_2s_3$	24	A_3	4	[4]
7		$s_1s_2s_4$	24	A_3	4	[4]
8		$s_1s_3s_4$	12	$3A_1$	2	$[2\bar{1}\bar{1}]$
9		$s_3s_2s_4$	24	D_3	4	$[\bar{2}\bar{1}1]$
10		$s_1s_4s_2s_3$	32	D_4	6	$[\bar{3}\bar{1}]$
11		$s_3s_2s_4s_3s_2s_1$	12	$D_4(a_1)$	4	$[\bar{2}\bar{2}]$
12		$s_1s_2s_3s_4s_2s_1s_2s_3s_4s_2s_3s_4$	1	$4A_1$	2	$[\bar{1}\bar{1}\bar{1}\bar{1}]$

[a]For an explanation of the Carter diagram $D_4(a_1)$ with a dotted edge (in the 11th conjugacy class), see [9, §1.1.1].
[b]For the definition of signed cycle-types, see [5, §7].

Table 3 Weyl groups D_4. Partitioning by element orders

Order	1	2	3	4	6
Elements	1	43	32	84	32

For any pair of non-orthogonal roots α and β, such that $(\alpha,\beta) = -1$, the following relations hold:

$$\begin{aligned} &s_\beta s_\alpha s_\beta = s_{s_\beta(\alpha)} = s_{\alpha+\beta}, \quad \text{and} \quad s_\beta s_\alpha = s_{\alpha+\beta} s_\beta, \qquad s_\alpha s_\beta = s_\beta s_{\alpha+\beta}, \\ &(s_\beta s_\alpha)^3 = 1, \quad \text{since } (s_\beta s_\alpha)^3 = (s_\beta s_\alpha s_\beta)(s_\alpha s_\beta s_\alpha) = s^2_{\alpha+\beta} = 1. \end{aligned} \tag{4.3}$$

4.1.1 Conjugacy class 11, *Carter diagram* $D_4(a_1)$

The representative element $w = s_3 s_2 s_4 s_3 s_2 s_1$ is the first element of conjugacy class 11, see Table 36. Using the roots from the root system as indices, we get the following expression for w:

$$w = s_{\alpha_3} s_{\alpha_2} s_{\alpha_4} s_{\alpha_3} s_{\alpha_2} s_{\alpha_1} = s_{\alpha_3} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_1} = s_{\alpha_2+\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_1}.$$

Further,

$$w \overset{s_{\alpha_1}}{\simeq} s_{\alpha_1} s_{\alpha_2+\alpha_3} s_{\alpha_4} s_{\alpha_2}, \tag{4.4}$$

where, the notaion $\overset{A}{\simeq}$ means conjugacy by the element A. The element (4.4) can be transformed as follows:

$$w = s_{\alpha_1} s_{\alpha_2+\alpha_3} s_{\alpha_4} s_{\alpha_2} = s_{\alpha_2+\alpha_3+\alpha_1} s_{\alpha_1} s_{\alpha_4} s_{\alpha_2} = s_{\widetilde{\alpha}_3} s_{\alpha_1} s_{\alpha_4} s_{\alpha_2}, \tag{4.5}$$

where $\widetilde{\alpha}_3 = -(\alpha_1 + \alpha_2 + \alpha_3)$.

The element w is represented by the Carter diagram $D_4(a_1)$, where the dotted edge $\{\widetilde{\alpha}_3, \alpha_4\}$ corresponds to the inner product $(\widetilde{\alpha}_3, \alpha_4) = 1$, see [9, 10].

4.1.2 Conjugacy class 12, *four unconnected vertices*

The element (4.1) looks like this:

$$w = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4}.$$

First of all, according to (4.3), we change $s_{\alpha_2} s_{\alpha_1} s_{\alpha_2}$ to $s_{\alpha_2+\alpha_1}$, and $s_{\alpha_4} s_{\alpha_2} s_{\alpha_4}$ to $s_{\alpha_2+\alpha_4}$. Then

$$w = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2+\alpha_1} s_{\alpha_3} s_{\alpha_2+\alpha_4} s_{\alpha_3}.$$

Further, by (4.3), we change $s_{\alpha_3} s_{\alpha_2+\alpha_4} s_{\alpha_3}$ to $s_{\alpha_2+\alpha_4+\alpha_3}$, and $s_{\alpha_4} s_{\alpha_2+\alpha_1}$ to $s_{\alpha_2+\alpha_1} s_{\alpha_2+\alpha_1+\alpha_4}$. Thus,

$$w = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_2+\alpha_1} s_{\alpha_2+\alpha_1+\alpha_4} s_{\alpha_2+\alpha_4+\alpha_3}.$$

Similarly, we replace $s_{\alpha_3} s_{\alpha_2+\alpha_1}$ with $s_{\alpha_2+\alpha_1} s_{\alpha_2+\alpha_1+\alpha_3}$, we get

$$w = s_{\alpha_1} s_{\alpha_2} s_{\alpha_2+\alpha_1} s_{\alpha_2+\alpha_1+\alpha_3} s_{\alpha_2+\alpha_1+\alpha_4} s_{\alpha_2+\alpha_4+\alpha_3}.$$

Finally, since $s_{\alpha_2}s_{\alpha_2+\alpha_1} = s_{\alpha_1}s_{\alpha_2}$, we have $s_{\alpha_1}s_{\alpha_2}s_{\alpha_2+\alpha_1} = s_{\alpha_2}$ and

$$w = s_{\alpha_2}s_{\alpha_2+\alpha_1+\alpha_3}s_{\alpha_2+\alpha_1+\alpha_4}s_{\alpha_2+\alpha_4+\alpha_3}. \tag{4.6}$$

Note that in Eq. (4.6), there are four mutually orthogonal roots:

$$\alpha_2, \qquad \alpha_2+\alpha_1+\alpha_3, \qquad \alpha_2+\alpha_1+\alpha_4, \qquad \alpha_2+\alpha_4+\alpha_3. \tag{4.7}$$

The subset (4.7) is represented by 4 unconnected vertices, i.e., $4A_1$.

4.2 Conjugacy classes of $W(D_4)$ represented by signed cycle-types

In this section, we consider the representation of conjugacy classes 8–12 of Table 2 using the signed cycle-types. According to Bourbaki's notaion:

$$s_{\alpha_1} = s_{e_1-e_2}, \qquad s_{\alpha_2} = s_{e_2-e_3}, \qquad s_{\alpha_3} = s_{e_3-e_4}, \qquad s_{\alpha_4} = s_{e_3+e_4}.$$

We will use the following mappings:

$$s_{e_i-e_j} : \begin{cases} e_i \longmapsto e_j, \\ e_j \longmapsto e_i, \end{cases} \qquad s_{e_i+e_j} : \begin{cases} e_i \longmapsto -e_j, \\ e_j \longmapsto -e_i, \end{cases} \qquad s_{e_i-e_j}s_{e_i+e_j} : \begin{cases} e_i \longmapsto -e_i, \\ e_j \longmapsto -e_j, \end{cases} \tag{4.8}$$

see [2, Ch. VI, §4, n°8].

4.2.1 Conjugacy class 8*, signed cycle-type* $[2\bar{1}\bar{1}]$

Consider representative element $s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}$. Let us find the signed cycle-type of this element. By (4.8), $s_{e_1-e_2}$ permutes e_1 and e_2, i.e., $s_{e_1-e_2}$ acts as permutation (12). Further, the product $s_{e_3-e_4}s_{e_3+e_4}$ maps e_3 to $-e_3$ and e_4 to $-e_4$, i.e., acts as the pair of negative cycles $[\bar{1}\bar{1}]$. All together gives $[2\bar{1}\bar{1}]$.

4.2.2 Conjugacy class 9*, signed cycle-type* $[\bar{2}\bar{1}1]$

Here, the representative element is $s_{\alpha_3}s_{\alpha_2}s_{\alpha_4}$. By (4.8) s_{α_2} permutes e_2 and e_3; s_{α_3} permutes e_3 and e_4. At last, s_{α_4} maps e_4 to $-e_3$ and e_3 to $-e_4$. Then,

$$s_{\alpha_3}s_{\alpha_2}s_{\alpha_4} : \begin{cases} e_2 \longmapsto e_4, \\ e_3 \longmapsto -e_3, \\ e_4 \longmapsto -e_2. \end{cases}$$

The second mapping corresponds to the negative cycle $[\bar{1}]$. The first and third mappings form the cycle $e_2 \longmapsto e_4 \longmapsto -e_2$, i.e., the negative cycle $[\bar{2}]$. Thus, we get the signed cycle-type $[\bar{2}\bar{1}]$, or, that is the same, $[\bar{2}\bar{1}1]$. By [5, Prop. 25], $[\bar{i}\bar{1}]$ corresponds to the Carter diagram D_{i+1}. In our case, we get D_3.

4.2.3 Conjugacy class 10, *signed cycle-type* $[\bar{3}\bar{1}]$

The representative element

$$s_{\alpha_1}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3} = s_{e_1-e_2}s_{e_3+e_4}s_{e_2-e_3}s_{e_3-e_4} \tag{4.9}$$

acts as follows:

$$s_{\alpha_1}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3} : \begin{cases} e_1 \longmapsto e_2, & e_3 \longmapsto -e_3, \\ e_2 \longmapsto -e_4, & e_4 \longmapsto e_1. \end{cases}$$

The mapping $e_3 \longmapsto -e_3$ corresponds to the negative cycle $[\bar{1}]$. The remaining mappings form the cycle $e_1 \longmapsto e_2 \longmapsto -e_4 \longmapsto -e_1$, i.e., the negative cycle $[\bar{3}]$. So, we get the signed cycle-type $[\bar{3}\bar{1}]$. As above, by [5, Prop. 25], the signed cycle-type $[\bar{3}\bar{1}]$ corresponds to D_4.

4.2.4 Conjugacy class 11, *signed cycle-type* $[\bar{2}\bar{2}]$

By (4.5), the representative element

$$s_{\alpha_2+\alpha_3+\alpha_1}s_{\alpha_1}s_{\alpha_4}s_{\alpha_2} = s_{e_1-e_4}s_{e_1-e_2}s_{e_3+e_4}e_{e_2-e_3}$$

acts as follows:

$$s_{\alpha_2+\alpha_3+\alpha_1}s_{\alpha_1}s_{\alpha_4}s_{\alpha_2} : \begin{cases} e_1 \longmapsto e_2, \\ e_2 \longmapsto -e_4 \longmapsto -e_1, \\ e_3 \longmapsto e_4, \\ e_4 \longmapsto -e_3. \end{cases}$$

The first and second mappings form the cycle $e_1 \longmapsto e_2 \longmapsto -e_1$, i.e., the negative cycle $[\bar{2}]$. The third and fourth mappings form the cycle $e_3 \longmapsto e_4 \longmapsto -e_3$, which is also the negative cycle $[\bar{2}]$. Thus, we get the signed cycle-type $[\bar{2}\bar{2}]$.

4.2.5 Conjugacy class 12, *signed cycle-type* $[\bar{1}\bar{1}\bar{1}\bar{1}]$

By (4.6) the representative element is as follows

$$s_{\alpha_2}s_{\alpha_2+\alpha_1+\alpha_3}s_{\alpha_2+\alpha_1+\alpha_4}s_{\alpha_2+\alpha_4+\alpha_3} = e_{e_2-e_3}e_{e_1-e_4}e_{e_1+e_4}e_{e_2+e_3}.$$

Since $s_{e_i-e_j}s_{e_i+e_j}$ maps e_i to $-e_i$ and e_j to $-e_j$, we get

$$s_{\alpha_2}s_{\alpha_2+\alpha_1+\alpha_3}s_{\alpha_2+\alpha_1+\alpha_4}s_{\alpha_2+\alpha_4+\alpha_3} : e_i \longmapsto -e_i \quad \text{for } i = 1, 2, 3, 4.$$

This corresponds to the signed cycle-type $[\bar{1}\bar{1}\bar{1}\bar{1}]$.

5 Weyl group D_4. Partitioning by levels

See Tables 4–24.

Table 4 Weyl group D_4, level 0, element e

N^o	Weight	Element	Matrix
0	1, 1, 1, 1	e	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Table 5 Weyl group D_4, level 1, elements 0–3

N^o	Weight	Element	Matrix	N^o	Weight	Element	Matrix
0	−1, 2, 1, 1	s_1	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	1	2, −1, 2, 2	s_2	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
2	1, 2, −1, 1	s_3	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	3	1, 2, 1, −1	s_4	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

Table 6 Weyl group D_4, level 2, elements 0–8

N^o	Weight	Elem	Matrix	N^o	Weight	Elem	Matrix
3 (0)[a]	−2, 1, 2, 2	s_1s_2	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	6 (4)	3, −2, 1, 3	s_2s_3	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
1 (1)	−1, 3, −1, 1	s_1s_3	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	8 (5)	3, −2, 3, 1	s_2s_4	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$
2 (2)	−1, 3, 1, −1	s_1s_4	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	0 (3)	1, −2, 3, 3	s_2s_1	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
4 (6)	2, 1, −2, 2	s_3s_2	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	5 (8)	2, 1, 2, −2	s_4s_2	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$
7 (7)	1, 3, −1, −1	s_3s_4	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$				

[a] Hereinafter, the number in this column without parentheses (resp. in parentheses) means the ordinal number of element (resp. inverse element) in Tables 6–24.

Table 7 Weyl group D_4, level 3, elements 0–13

N^o	Weight	Element	Matrix	Inverse
0 (9)	1, 1, −3, 3	$s_3s_2s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$s_1s_2s_3$
1 (13)	1, 1, 3, −3	$s_4s_2s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_1s_2s_4$
2 (6)	2, −3, 2, 4	$s_2s_3s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$s_3s_1s_2$
3 (3)	−1, 4, −1, −1	$s_4s_3s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_3s_1$
4 (7)	2, −3, 4, 2	$s_2s_4s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_1s_2$
5 (5)	−1, −1, 3, 3	$s_2s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$s_2s_1s_2$
6 (2)	−2, 3, −2, 2	$s_3s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$s_2s_3s_1$
7 (4)	−2, 3, 2, −2	$s_4s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_2s_4s_1$
8 (12)	2, 3, −2, −2	$s_4s_3s_2$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_2s_4s_3$
9 (0)	−3, 1, 1, 3	$s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$s_3s_2s_1$
10 (10)	3, −1, −1, 3	$s_3s_2s_3$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$s_3s_2s_3$
11 (14)	3, 1, 1, −3	$s_4s_2s_3$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4$
12 (8)	4, −3, 2, 2	$s_2s_4s_3$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2$
13 (1)	−3, 1, 3, 1	$s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_2s_1$

Table 8 Weyl group D_4, level 3, elements 14–15

N^o	Weight	Element	Matrix	Inverse
14 (11)	3, 1, −3, 1	$s_3s_2s_4$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_2s_3$
15 (15)	3, −1, 3, −1	$s_4s_2s_4$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4$

Table 9 Weyl group D_4, level 4, elements 0–13

N^o	Weight	Element	Matrix	Inverse
0 (16)	1, 4, −3, −3	$s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_1s_2s_4s_3$
1 (13)	2, −1, −2, 4	$s_3s_2s_3s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$s_3s_1s_2s_3$
2 (20)	2, 1, 2, −4	$s_4s_2s_3s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_3s_1s_2s_4$
3 (9)	3, −4, 3, 3	$s_2s_4s_3s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_3s_1s_2$
4 (14)	2, 1, −4, 2	$s_3s_2s_4s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_1s_2s_3$
5 (21)	2, −1, 4, −2	$s_4s_2s_4s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_4s_1s_2s_4$
6 (12)	−1, 2, −3, 3	$s_3s_2s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$s_2s_1s_2s_3$
7 (19)	−1, 2, 3, −3	$s_4s_2s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_2s_1s_2s_4$
8 (8)	1, −3, 1, 5	$s_2s_3s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 2 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$s_2s_3s_1s_2$
9 (3)	−2, 5, −2, −2	$s_4s_3s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_2s_4s_3s_1$
10 (10)	1, −3, 5, 1	$s_2s_4s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_2s_4s_1s_2$
11 (11)	5, −3, 1, 1	$s_2s_4s_3s_2$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_2s_4s_3s_2$

Table 9 (*Continued*)

N^o	Weight	Element	Matrix	Inverse
12 (6)	−2, −1, 2, 4	$s_2s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$s_3s_2s_1s_2$
13 (1)	−3, 2, −1, 3	$s_3s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$s_3s_2s_3s_1$

Table 10 Weyl group D_4, level 4, elements 14–22

N^o	Weight	Element	Matrix	Inverse
14 (4)	−3, 4, 1, −3	$s_4s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4s_1$
15 (17)	3, 2, −1, −3	$s_4s_3s_2s_3$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4s_3$
16 (0)	−4, 1, 2, 2	$s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_1$
17 (15)	4, −1, −2, 2	$s_3s_2s_4s_3$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_3$
18 (22)	4, −1, 2, −2	$s_4s_2s_4s_3$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4$
19 (7)	−2, −1, 4, 2	$s_2s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_2s_1s_2$
20 (2)	−3, 4, −3, 1	$s_3s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_2s_3s_1$
21 (5)	−3, 2, 3, −1	$s_4s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4s_1$
22 (18)	3, 2, −3, −1	$s_4s_3s_2s_4$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4s_3$

Table 11 Weyl group D_4, level 5, elements 0–13

N^o	Weight	Element	Matrix	Inverse
0 (11)	5, −4, 1, 1	$s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_1s_2s_4s_3s_2$
1 (20)	2, 3, −2, −4	$s_4s_3s_2s_3s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_3s_1s_2s_4s_3$
2 (17)	3, −1, −3, 3	$s_3s_2s_4s_3s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_3s_1s_2s_3$
3 (26)	3, −1, 3, −3	$s_4s_2s_4s_3s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_4s_3s_1s_2s_4$
4 (21)	2, 3, −4, −2	$s_4s_3s_2s_4s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_4s_1s_2s_4s_3$
5 (19)	−1, 5, −3, −3	$s_4s_3s_2s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_2s_1s_2s_4s_3$
6 (16)	1, −2, −1, 5	$s_3s_2s_3s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$s_2s_3s_1s_2s_3$
7 (25)	1, 2, 1, −5	$s_4s_2s_3s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 2 \\ 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_2s_3s_1s_2s_4$
8 (8)	3, −5, 3, 3	$s_2s_4s_3s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_2s_4s_3s_1s_2$
9 (18)	1, 2, −5, 1	$s_3s_2s_4s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_2s_4s_1s_2s_3$
10 (27)	1, −2, 5, −1	$s_4s_2s_4s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_2s_4s_1s_2s_4$
11 (0)	−5, 2, 1, 1	$s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_2s_4s_3s_2s_1$
12 (13)	5, −2, −1, 1	$s_3s_2s_4s_3s_2$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4s_3s_2$
13 (12)	5, −2, 1, −1	$s_4s_2s_4s_3s_2$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4s_3s_2$

Table 12 Weyl group D_4, level 5, elements 14–27

N^o	Weight	Element	Matrix	Inverse
14 (14)	−2, 1, −2, 4	$s_3s_2s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$s_3s_2s_1s_2s_3$
15 (23)	−2, 3, 2, −4	$s_4s_2s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_3s_2s_1s_2s_4$
16 (6)	−1, −2, 1, 5	$s_2s_3s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 2 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$s_3s_2s_3s_1s_2$
17 (2)	−3, 5, −1, −3	$s_4s_3s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4s_3s_1$
18 (9)	1, −4, 5, 1	$s_2s_4s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4s_1s_2$
19 (5)	−3, −1, 3, 3	$s_2s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_1s_2$
20 (1)	−4, 3, −2, 2	$s_3s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_3s_1$
21 (4)	−4, 3, 2, −2	$s_4s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4s_1$
22 (22)	4, 1, −2, −2	$s_4s_3s_2s_4s_3$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4s_3$
23 (15)	−2, 3, −4, 2	$s_3s_2s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_2s_1s_2s_3$
24 (24)	−2, 1, 4, −2	$s_4s_2s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_4s_2s_1s_2s_4$
25 (7)	1, −4, 1, 5	$s_2s_3s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_2s_3s_1s_2$
26 (3)	−3, 5, −3, −1	$s_4s_3s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4s_3s_1$
27 (10)	−1, −2, 5, 1	$s_2s_4s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4s_1s_2$

Table 13 Weyl group D_4, level 6, elements 0–13

N^o	Weight	Element	Matrix	Inverse
0 (0)	−5, 1, 1, 1	$s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_1s_2s_4s_3s_2s_1$
1 (11)	5, −3, −1, 1	$s_3s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_4s_1s_2s_4s_3s_2$
2 (10)	5, −3, 1, −1	$s_4s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_3s_1s_2s_4s_3s_2$
3 (22)	3, 2, −3, −3	$s_4s_3s_2s_4s_3s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_4s_3s_1s_2s_4s_3$
4 (9)	4, −5, 2, 2	$s_2s_4s_3s_2s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_2s_1s_2s_4s_3s_2$
5 (21)	1, 3, −1, −5	$s_4s_3s_2s_3s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_2s_3s_1s_2s_4s_3$
6 (16)	3, −2, −3, 3	$s_3s_2s_4s_3s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_2s_4s_3s_1s_2s_3$
7 (27)	3, −2, 3, −3	$s_4s_2s_4s_3s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_2s_4s_3s_1s_2s_4$
8 (23)	1, 3, −5, −1	$s_4s_3s_2s_4s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_2s_4s_1s_2s_4s_3$
9 (4)	−3, −2, 3, 3	$s_2s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_2s_4s_3s_2s_1s_2$
10 (2)	−5, 3, −1, 1	$s_3s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4s_3s_2s_1$
11 (1)	−5, 3, 1, −1	$s_4s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4s_3s_2s_1$
12 (12)	5, −1, −1, −1	$s_4s_3s_2s_4s_3s_2$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4s_3s_2$
13 (19)	−2, 5, −2, −4	$s_4s_3s_2s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_3s_2s_1s_2s_4s_3$

Table 14 Weyl group D_4, level 6, elements 14–27

N^o	Weight	Element	Matrix	Inverse
14 (14)	−1, −1, −1, 5	$s_3s_2s_3s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$s_3s_2s_3s_1s_2s_3$
15 (25)	−1, 3, 1, −5	$s_4s_2s_3s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 2 \\ 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_3s_2s_3s_1s_2s_4$
16 (6)	2, −5, 4, 2	$s_2s_4s_3s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4s_3s_1s_2$
17 (17)	1, 1, −5, 1	$s_3s_2s_4s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4s_1s_2s_3$
18 (28)	1, −3, 5, −1	$s_4s_2s_4s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4s_1s_2s_4$
19 (13)	−3, 2, −3, 3	$s_3s_2s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_1s_2s_3$
20 (24)	−3, 2, 3, −3	$s_4s_2s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_1s_2s_4$
21 (5)	−1, −3, 1, 5	$s_2s_3s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_3s_1s_2$
22 (3)	−4, 5, −2, −2	$s_4s_3s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4s_3s_1$
23 (8)	−1, −3, 5, 1	$s_2s_4s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4s_1s_2$
24 (20)	−2, 5, −4, −2	$s_4s_3s_2s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_4s_2s_1s_2s_4s_3$
25 (15)	1, −3, −1, 5	$s_3s_2s_3s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_2s_3s_1s_2s_3$
26 (26)	1, 1, 1, −5	$s_4s_2s_3s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_2s_3s_1s_2s_4$
27 (7)	2, −5, 2, 4	$s_2s_4s_3s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4s_3s_1s_2$

Table 15 Weyl group D_4, level 6, elements 28–29

N^o	Weight	Element	Matrix	Inverse
28 (18)	−1, 3, −5, 1	$s_3s_2s_4s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4s_1s_2s_3$
29 (29)	−1, −1, 5, −1	$s_4s_2s_4s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4s_1s_2s_4$

Table 16 Weyl group D_4, level 7, elements 0–13

N^o	Weight	Element	Matrix	Inverse
0 (0)	−4, −1, 2, 2	$s_2s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_2s_1s_2s_4s_3s_2s_1$
1 (2)	−5, 2, −1, 1	$s_3s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_4s_1s_2s_4s_3s_2s_1$
2 (1)	−5, 2, 1, −1	$s_4s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_3s_1s_2s_4s_3s_2s_1$
3 (10)	5, −2, −1, −1	$s_4s_3s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_4s_3s_1s_2s_4s_3s_2$
4 (11)	4, −3, −2, 2	$s_3s_2s_4s_3s_2s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_2s_4s_1s_2s_4s_3s_2$
5 (9)	4, −3, 2, −2	$s_4s_2s_4s_3s_2s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_2s_3s_1s_2s_4s_3s_2$
6 (20)	3, 1, −3, −3	$s_4s_3s_2s_4s_3s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_2s_4s_3s_1s_2s_4s_3$
7 (12)	−3, 1, −3, 3	$s_3s_2s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_2s_4s_3s_2s_1s_2s_3$
8 (23)	−3, 1, 3, −3	$s_4s_2s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_2s_4s_3s_2s_1s_2s_4$
9 (5)	−2, −3, 2, 4	$s_2s_3s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4s_3s_2s_1s_2$
10 (3)	−5, 4, −1, −1	$s_4s_3s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4s_3s_2s_1$
11 (4)	−2, −3, 4, 2	$s_2s_4s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4s_3s_2s_1s_2$

Table 16 (*Continued*)

N^o	Weight	Element	Matrix	Inverse
12 (7)	3, −5, 3, 1	$s_2s_4s_3s_2s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_3s_2s_1s_2s_4s_3s_2$
13 (18)	−1, 4, −1, −5	$s_4s_3s_2s_3s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_3s_2s_3s_1s_2s_4s_3$

Table 17 Weyl group D_4, level 7, elements 14–27

N^o	Weight	Element	Matrix	Inverse
14 (14)	2, −1, −4, 2	$s_3s_2s_4s_3s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4s_3s_1s_2s_3$
15 (25)	2, −3, 4, −2	$s_4s_2s_4s_3s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_3s_2s_4s_3s_1s_2s_4$
16 (21)	1, 2, −5, −1	$s_4s_3s_2s_4s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4s_1s_2s_4s_3$
17 (17)	−3, 5, −3, −3	$s_4s_3s_2s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_1s_2s_4s_3$
18 (13)	−1, −2, −1, 5	$s_3s_2s_3s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & -2 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_3s_1s_2s_3$
19 (24)	−1, 2, 1, −5	$s_4s_2s_3s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_3s_1s_2s_4$
20 (6)	1, −5, 3, 3	$s_2s_4s_3s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4s_3s_1s_2$
21 (16)	−1, 2, −5, 1	$s_3s_2s_4s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4s_1s_2s_3$
22 (27)	−1, −2, 5, −1	$s_4s_2s_4s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4s_1s_2s_4$
23 (8)	3, −5, 1, 3	$s_2s_4s_3s_2s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_4s_2s_1s_2s_4s_3s_2$
24 (19)	1, 2, −1, −5	$s_4s_3s_2s_3s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_2s_3s_1s_2s_4s_3$
25 (15)	2, −3, −2, 4	$s_3s_2s_4s_3s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4s_3s_1s_2s_3$

Table 17 (*Continued*)

N^o	Weight	Element	Matrix	Inverse
26 (26)	2, −1, 2, −4	$s_4s_2s_4s_3s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_2s_4s_3s_1s_2s_4$
27 (22)	−1, 4, −5, −1	$s_4s_3s_2s_4s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4s_1s_2s_4s_3$

Table 18 Weyl group D_4, level 8, elements 0–13

N^o	Weight	Element	Matrix	Inverse
0 (4)	−4, 1, −2, 2	$s_3s_2s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_2s_4s_1s_2s_4s_3s_2s_1$
1 (2)	−4, 1, 2, −2	$s_4s_2s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_2s_3s_1s_2s_4s_3s_2s_1$
2 (1)	−3, −2, 1, 3	$s_2s_3s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_4s_2s_1s_2s_4s_3s_2s_1$
3 (3)	−5, 3, −1, −1	$s_4s_3s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_4s_3s_1s_2s_4s_3s_2s_1$
4 (0)	−3, −2, 3, 1	$s_2s_4s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_3s_2s_1s_2s_4s_3s_2s_1$
5 (9)	4, −1, −2, −2	$s_4s_3s_2s_4s_3s_2s_1s_2$	$\begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_2s_4s_3s_1s_2s_4s_3s_2$
6 (15)	−3, 4, −3, −3	$s_4s_3s_2s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_2s_4s_3s_2s_1s_2s_4s_3$
7 (13)	−2, −1, −2, 4	$s_3s_2s_3s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4s_3s_2s_1s_2s_3$
8 (21)	−2, 1, 2, −4	$s_4s_2s_3s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_2s_4s_3s_2s_1s_2s_4$
9 (5)	−1, −4, 3, 3	$s_2s_4s_3s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4s_3s_2s_1s_2$
10 (12)	−2, 1, −4, 2	$s_3s_2s_4s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4s_3s_2s_1s_2s_3$
11 (20)	−2, −1, 4, −2	$s_4s_2s_4s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_3s_2s_4s_3s_2s_1s_2s_4$

Table 18 (*Continued*)

N^o	Weight	Element	Matrix	Inverse
12 (10)	3, −2, −3, 1	$s_3s_2s_4s_3s_2s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4s_1s_2s_4s_3s_2$
13 (7)	3, −4, 3, −1	$s_4s_2s_4s_3s_2s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_3s_2s_3s_1s_2s_4s_3s_2$

Table 19 Weyl group D_4, level 8, elements 14–22

N^o	Weight	Element	Matrix	Inverse
14 (17)	2, 1, −4, −2	$s_4s_3s_2s_4s_3s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_3s_2s_4s_3s_1s_2s_4s_3$
15 (6)	2, −5, 2, 2	$s_2s_4s_3s_2s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_1s_2s_4s_3s_2$
16 (16)	−1, 3, −1, −5	$s_4s_3s_2s_3s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & -2 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_3s_1s_2s_4s_3$
17 (14)	1, −2, −3, 3	$s_3s_2s_4s_3s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4s_3s_1s_2s_3$
18 (22)	1, −2, 3, −3	$s_4s_2s_4s_3s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_4s_3s_1s_2s_4$
19 (19)	−1, 3, −5, −1	$s_4s_3s_2s_4s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4s_1s_2s_4s_3$
20 (11)	3, −4, −1, 3	$s_3s_2s_4s_3s_2s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4s_1s_2s_4s_3s_2$
21 (8)	3, −2, 1, −3	$s_4s_2s_4s_3s_2s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_2s_3s_1s_2s_4s_3s_2$
22 (18)	2, 1, −2, −4	$s_4s_3s_2s_4s_3s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_2s_4s_3s_1s_2s_4s_3$

Table 20 Weyl group D_4, level 9, elements 0–13

N^o	Weight	Element	Matrix	Inverse
0 (3)	−4, 3, −2, −2	$s_4s_3s_2s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_2s_4s_3s_1s_2s_4s_3s_2s_1$
1 (5)	−3, −1, −1, 3	$s_3s_2s_3s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$	$s_4s_2s_4s_1s_2s_4s_3s_2s_1$
2 (2)	−3, 1, 1, −3	$s_4s_2s_3s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_2s_3s_1s_2s_4s_3s_2s_1$
3 (0)	−2, −3, 2, 2	$s_2s_4s_3s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_1s_2s_4s_3s_2s_1$
4 (4)	−3, 1, −3, 1	$s_3s_2s_4s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_3s_2s_4s_1s_2s_4s_3s_2s_1$
5 (1)	−3, −1, 3, −1	$s_4s_2s_4s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_3s_2s_3s_1s_2s_4s_3s_2s_1$
6 (6)	1, −4, 1, 1	$s_2s_4s_3s_2s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_2s_4s_3s_2s_1s_2s_4s_3s_2$
7 (13)	−2, 3, −2, −4	$s_4s_3s_2s_3s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_2s_4s_3s_2s_1s_2s_4s_3$
8 (11)	−1, −1, −3, 3	$s_3s_2s_4s_3s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4s_3s_2s_1s_2s_3$
9 (15)	−1, −1, 3, −3	$s_4s_2s_4s_3s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_4s_3s_2s_1s_2s_4$
10 (12)	−2, 3, −4, −2	$s_4s_3s_2s_4s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_3s_2s_4s_3s_2s_1s_2s_4s_3$
11 (8)	3, −1, −3, −1	$s_4s_3s_2s_4s_3s_2s_1s_2s_3$	$\begin{bmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_3s_2s_4s_3s_1s_2s_4s_3s_2$
12 (10)	2, −3, −2, 2	$s_3s_2s_4s_3s_2s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4s_1s_2s_4s_3s_2$
13 (7)	2, −3, 2, −2	$s_4s_2s_4s_3s_2s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_3s_1s_2s_4s_3s_2$

Table 21 Weyl group D_4, level 9, elements 14–15

N^o	Weight	Element	Matrix	Inverse
14 (14)	1, 1, −3, −3	$s_4s_3s_2s_4s_3s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_4s_3s_1s_2s_4s_3$
15 (9)	3, −1, −1, −3	$s_4s_3s_2s_4s_3s_2s_1s_2s_4$	$\begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_2s_4s_3s_1s_2s_4s_3s_2$

Table 22 Weyl group D_4, level 10, elements 0–8

N^o	Weight	Element	Matrix	Inverse
0 (0)	−1, −3, 1, 1	$s_2s_4s_3s_2s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_2s_4s_3s_2s_1s_2s_4s_3s_2s_1$
1 (3)	−3, 2, −1, −3	$s_4s_3s_2s_3s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_2s_4s_3s_1s_2s_4s_3s_2s_1$
2 (4)	−2, −1, −2, 2	$s_3s_2s_4s_3s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$	$s_4s_3s_2s_4s_1s_2s_4s_3s_2s_1$
3 (1)	−2, −1, 2, −2	$s_4s_2s_4s_3s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_3s_1s_2s_4s_3s_2s_1$
4 (2)	−3, 2, −3, −1	$s_4s_3s_2s_4s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_3s_2s_4s_3s_1s_2s_4s_3s_2s_1$
5 (5)	1, −3, −1, 1	$s_3s_2s_4s_3s_2s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$s_3s_2s_4s_3s_2s_1s_2s_4s_3s_2$
6 (6)	1, −3, 1, −1]	$s_4s_2s_4s_3s_2s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_2s_4s_3s_2s_1s_2s_4s_3s_2$
7 (8)	−1, 2, −3, −3	$s_4s_3s_2s_4s_3s_1s_2s_4s_3s_2$	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_4s_3s_2s_1s_2s_4s_3$
8 (7)	2, −1, −2, −2	$s_4s_3s_2s_4s_3s_2s_1s_2s_4s_3$	$\begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$s_4s_3s_2s_4s_3s_1s_2s_4s_3s_2$

Table 23 Weyl group D_4, level 11, elements 0–3

N^o	Weight	Element	Matrix	Inverse
0 (0)	−1, −2, −1, 1	$s_3s_2s_4s_3s_2s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}$	$s_3s_2s_4s_3s_2s_1s_2s_4s_3s_2s_1$
1 (1)	−1, −2, 1, −1	$s_4s_2s_4s_3s_2s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}$	$s_4s_2s_4s_3s_2s_1s_2s_4s_3s_2s_1$
2 (2)	−2, 1, −2, −2	$s_4s_3s_2s_4s_3s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix}-1 & 0 & 0 & 0\\ -1 & 1 & -1 & -1\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}$	$s_4s_3s_2s_4s_3s_1s_2s_4s_3s_2s_1$
3 (3)	1, −2, −1, −1	$s_4s_3s_2s_4s_3s_2s_1s_2s_4s_3s_2$	$\begin{bmatrix}1 & -1 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}$	$s_4s_3s_2s_4s_3s_2s_1s_2s_4s_3s_2$

Table 24 Weyl group D_4, level 12, one element

N^o	Weight	Element	Matrix	Inverse
0 (0)	−1, −1, −1, −1	$s_4s_3s_2s_4s_3s_2s_1s_2s_4s_3s_2s_1$	$\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}$	itself

6 Thirteen conjugacy classes of $W(D_4)$

See Tables 25–37.

Table 25 Weyl group D_4, conjugacy class 0, order 1

N^o in CCL	Element	Level	N^o in Level
1	e	0	1

Table 26 Weyl group D_4, conjugacy class 1, order 2

N^o in CCL	Element	Level	N^o in Level
1	s_1	1	1
2	s_2	1	2
3	s_3	1	3
4	s_4	1	4
5	$s_2s_1s_2$	3	6
6	$s_3s_2s_3$	3	11
7	$s_4s_2s_4$	3	16
8	$s_3s_2s_1s_2s_3$	5	15
9	$s_4s_3s_2s_4s_3$	5	23
10	$s_4s_2s_1s_2s_4$	5	25
11	$s_4s_3s_2s_1s_2s_4s_3$	7	18
12	$s_2s_4s_3s_2s_1s_2s_4s_3s_2$	9	7

Table 27 Weyl group D_4, conjugacy class 2, order 3

N^o in CCL	Element	Level	N^o in Level
1	s_2s_1	2	0
2	s_1s_2	2	3
3	s_3s_2	2	4
4	s_4s_2	2	5
5	s_2s_3	2	6
6	s_2s_4	2	8
7	$s_3s_2s_3s_1$	4	1
8	$s_4s_2s_4s_1$	4	5
9	$s_3s_2s_1s_2$	4	6
10	$s_4s_2s_1s_2$	4	7
11	$s_2s_1s_2s_3$	4	12
12	$s_3s_1s_2s_3$	4	13
13	$s_4s_3s_2s_3$	4	15
14	$s_3s_2s_4s_3$	4	17
15	$s_4s_2s_4s_3$	4	18
16	$s_2s_1s_2s_4$	4	19
17	$s_4s_1s_2s_4$	4	21
18	$s_4s_3s_2s_4$	4	22
19	$s_4s_3s_2s_4s_3s_1$	6	3
20	$s_4s_3s_2s_1s_2s_3$	6	13
21	$s_3s_2s_1s_2s_4s_3$	6	19
22	$s_4s_2s_1s_2s_4s_3$	6	20
23	$s_4s_3s_1s_2s_4s_3$	6	22
24	$s_4s_3s_2s_1s_2s_4$	6	24
25	$s_4s_3s_2s_4s_3s_2s_1s_2$	8	5
26	$s_4s_3s_2s_1s_2s_4s_3s_2$	8	6
27	$s_3s_2s_3s_1s_2s_4s_3s_2$	8	7
28	$s_2s_4s_3s_1s_2s_4s_3s_2$	8	9
29	$s_4s_2s_4s_1s_2s_4s_3s_2$	8	11
30	$s_4s_2s_4s_3s_2s_1s_2s_3$	8	13
31	$s_2s_4s_3s_2s_1s_2s_4s_3$	8	15
32	$s_3s_2s_4s_3s_2s_1s_2s_4$	8	20

Table 28 Weyl group D_4, conjugacy class 3, order 2

N^o in CCL	Element	Level	N^o in Level
1	s_3s_1	2	1
2	$s_2s_3s_1s_2$	4	8
3	$s_3s_2s_3s_1s_2s_3$	6	14
4	$s_4s_2s_3s_1s_2s_4$	6	26
5	$s_4s_3s_2s_3s_1s_2s_4s_3$	8	16
6	$s_4s_2s_4s_3s_2s_1s_2s_4s_3s_2$	10	6

Table 29 Weyl group D_4, conjugacy class 4, order 2

N^o in CCL	Element	Level	N^o in Level
1	s_4s_1	2	2
2	$s_2s_4s_1s_2$	4	10
3	$s_4s_2s_4s_1s_2s_4$	6	29
4	$s_3s_2s_4s_1s_2s_3$	6	17
5	$s_4s_3s_2s_4s_1s_2s_4s_3$	8	19
6	$s_3s_2s_4s_3s_2s_1s_2s_4s_3s_2$	10	5

Table 30 Weyl group D_4, conjugacy class 5, order 2

N^o in CCL	Element	Level	N^o in Level
1	s_4s_3	2	7
2	$s_2s_4s_3s_2$	4	11
3	$s_1s_2s_4s_3s_2s_1$	6	0
4	$s_4s_3s_2s_4s_3s_2$	6	12
5	$s_4s_3s_1s_2s_4s_3s_2s_1$	8	3
6	$s_2s_4s_3s_2s_1s_2s_4s_3s_2s_1$	10	0

Table 31 Weyl group D_4, conjugacy class 6, order 4

N^o in CCL	Element	Level	N^o in Level
1	$s_3s_2s_1$	3	0
2	$s_2s_3s_1$	3	2
3	$s_3s_1s_2$	3	6
4	$s_1s_2s_3$	3	9
5	$s_4s_2s_4s_3s_1$	5	3
6	$s_4s_3s_2s_4s_1$	5	4
7	$s_3s_2s_3s_1s_2$	5	6
8	$s_4s_2s_3s_1s_2$	5	7
9	$s_2s_3s_1s_2s_3$	5	16
10	$s_4s_1s_2s_4s_3$	5	21
11	$s_2s_3s_1s_2s_4$	5	25
12	$s_4s_3s_1s_2s_4$	5	26
13	$s_3s_2s_4s_3s_2s_1s_2$	7	4
14	$s_3s_2s_1s_2s_4s_3s_2$	7	7
15	$s_2s_4s_1s_2s_4s_3s_2$	7	11
16	$s_2s_4s_3s_2s_1s_2s_3$	7	12
17	$s_4s_3s_2s_3s_1s_2s_3$	7	13
18	$s_3s_2s_3s_1s_2s_4s_3$	7	18
19	$s_4s_2s_3s_1s_2s_4s_3$	7	19
20	$s_4s_3s_2s_3s_1s_2s_4$	7	24
21	$s_4s_3s_2s_3s_1s_2s_4s_3s_2$	9	7
22	$s_4s_2s_4s_3s_1s_2s_4s_3s_2$	9	9
23	$s_4s_2s_4s_3s_2s_1s_2s_4s_3$	9	13
24	$s_4s_3s_2s_4s_3s_2s_1s_2s_4$	9	15

Table 32 Weyl group D_4, conjugacy class 7, order 4

N^o in CCL	Element	Level	N^o in Level
1	$s_4s_2s_1$	3	1
2	$s_2s_4s_1$	3	4
3	$s_4s_1s_2$	3	7
4	$s_1s_2s_4$	3	13
5	$s_4s_3s_2s_3s_1$	5	1
6	$s_3s_2s_4s_3s_1$	5	2
7	$s_3s_2s_4s_1s_2$	5	9
8	$s_4s_2s_4s_1s_2$	5	10
9	$s_4s_3s_1s_2s_3$	5	17
10	$s_2s_4s_1s_2s_3$	5	18
11	$s_3s_1s_2s_4s_3$	5	20
12	$s_2s_4s_1s_2s_4$	5	27
13	$s_4s_2s_4s_3s_2s_1s_2$	7	5
14	$s_4s_2s_1s_2s_4s_3s_2$	7	8
15	$s_2s_3s_1s_2s_4s_3s_2$	7	9
16	$s_4s_3s_2s_4s_1s_2s_3$	7	16
17	$s_3s_2s_4s_1s_2s_4s_3$	7	21
18	$s_4s_2s_4s_1s_2s_4s_3$	7	22
19	$s_2s_4s_3s_2s_1s_2s_4$	7	23
20	$s_4s_3s_2s_4s_1s_2s_4$	7	27
21	$s_3s_2s_4s_3s_1s_2s_4s_3s_2$	9	8
22	$s_4s_3s_2s_4s_1s_2s_4s_3s_2$	9	10
23	$s_4s_3s_2s_4s_3s_2s_1s_2s_3$	9	11
24	$s_3s_2s_4s_3s_2s_1s_2s_4s_3$	9	12

Table 33 Weyl group D_4, conjugacy class 8, order 2

N^o in CCL	Element	Level	N^o in Level
1	$s_4s_3s_1$	3	3
2	$s_2s_4s_3s_1s_2$	5	8
3	$s_2s_1s_2s_4s_3s_2s_1$	7	0
4	$s_3s_2s_4s_3s_1s_2s_3$	7	14
5	$s_4s_2s_4s_3s_1s_2s_4$	7	26
6	$s_4s_2s_3s_1s_2s_4s_3s_2s_1$	9	2
7	$s_3s_2s_4s_1s_2s_4s_3s_2s_1$	9	4
8	$s_4s_3s_2s_4s_3s_1s_2s_4s_3$	9	14
9	$s_3s_2s_4s_3s_2s_1s_2s_4s_3s_2s_1$	11	0
10	$s_4s_2s_4s_3s_2s_1s_2s_4s_3s_2s_1$	11	1
11	$s_4s_3s_2s_4s_3s_1s_2s_4s_3s_2s_1$	11	2
12	$s_4s_3s_2s_4s_3s_2s_1s_2s_4s_3s_2$	11	3

Table 34 Weyl group D_4, conjugacy class 9, order 4

N^o in CCL	Element	Level	N^o in Level
1	$s_4s_3s_2$	3	8
2	$s_4s_2s_3$	3	11
3	$s_2s_4s_3$	3	12
4	$s_3s_2s_4$	3	14
5	$s_2s_4s_3s_2s_1$	5	0
6	$s_4s_3s_2s_1s_2$	5	5
7	$s_1s_2s_4s_3s_2$	5	11
8	$s_3s_2s_4s_3s_2$	5	12
9	$s_4s_2s_4s_3s_2$	5	13
10	$s_4s_2s_1s_2s_3$	5	15
11	$s_2s_1s_2s_4s_3$	5	19
12	$s_3s_2s_1s_2s_4$	5	23
13	$s_3s_1s_2s_4s_3s_2s_1$	7	1
14	$s_4s_1s_2s_4s_3s_2s_1$	7	2
15	$s_4s_3s_2s_4s_3s_2s_1$	7	3
16	$s_4s_3s_2s_4s_3s_1s_2$	7	6
17	$s_4s_3s_1s_2s_4s_3s_2$	7	10
18	$s_4s_2s_4s_3s_1s_2s_3$	7	15
19	$s_2s_4s_3s_1s_2s_4s_3$	7	20
20	$s_3s_2s_4s_3s_1s_2s_4$	7	25
21	$s_4s_3s_2s_1s_2s_4s_3s_2s_1$	9	0
22	$s_3s_2s_3s_1s_2s_4s_3s_2s_1$	9	1
23	$s_2s_4s_3s_1s_2s_4s_3s_2s_1$	9	3
24	$s_4s_2s_4s_1s_2s_4s_3s_2s_1$	9	5

Table 35 Weyl group D_4, conjugacy class 10, order 6

N^o in CCL	Element	Level	N^o in Level
1	$s_4s_3s_2s_1$	4	0
2	$s_4s_2s_3s_1$	4	2
3	$s_2s_4s_3s_1$	4	3
4	$s_3s_2s_4s_1$	4	4
5	$s_4s_3s_1s_2$	4	9
6	$s_4s_1s_2s_3$	4	14
7	$s_1s_2s_4s_3$	4	16
8	$s_3s_1s_2s_4$	4	20
9	$s_2s_4s_3s_2s_1s_2$	6	4
10	$s_3s_2s_4s_3s_1s_2$	6	6
11	$s_4s_2s_4s_3s_1s_2$	6	7
12	$s_2s_1s_2s_4s_3s_2$	6	9
13	$s_2s_4s_3s_1s_2s_3$	6	16
14	$s_2s_4s_3s_1s_2s_4$	6	27
15	$s_3s_2s_1s_2s_4s_3s_2s_1$	8	0
16	$s_4s_2s_1s_2s_4s_3s_2s_1$	8	1
17	$s_2s_3s_1s_2s_4s_3s_2s_1$	8	2
18	$s_2s_4s_1s_2s_4s_3s_2s_1$	8	4
19	$s_4s_2s_3s_1s_2s_4s_3s_2$	8	8
20	$s_3s_2s_4s_1s_2s_4s_3s_2$	8	10
21	$s_3s_2s_4s_3s_2s_1s_2s_3$	8	12
22	$s_4s_3s_2s_4s_3s_1s_2s_3$	8	14
23	$s_3s_2s_4s_3s_1s_2s_4s_3$	8	17
24	$s_4s_2s_4s_3s_1s_2s_4s_3$	8	18
25	$s_4s_2s_4s_3s_2s_1s_2s_4$	8	21
26	$s_4s_3s_2s_4s_3s_1s_2s_4$	8	22
27	$s_4s_3s_2s_3s_1s_2s_4s_3s_2s_1$	10	1
28	$s_3s_2s_4s_3s_1s_2s_4s_3s_2s_1$	10	2
29	$s_4s_2s_4s_3s_1s_2s_4s_3s_2s_1$	10	3
30	$s_4s_3s_2s_4s_1s_2s_4s_3s_2s_1$	10	4
31	$s_4s_3s_2s_4s_3s_1s_2s_4s_3s_2$	10	7
32	$s_4s_3s_2s_4s_3s_2s_1s_2s_4s_3$	10	8

Table 36 Weyl group D_4, conjugacy class 11, order 4

N^o in CCL	Element	Level	N^o in Level
1	$s_3s_2s_4s_3s_2s_1$	6	1
2	$s_4s_2s_4s_3s_2s_1$	6	2
3	$s_4s_3s_2s_3s_1s_2$	6	5
4	$s_4s_3s_2s_4s_1s_2$	6	8
5	$s_3s_1s_2s_4s_3s_2$	6	10
6	$s_4s_1s_2s_4s_3s_2$	6	11
7	$s_4s_2s_3s_1s_2s_3$	6	15
8	$s_4s_2s_4s_1s_2s_3$	6	18
9	$s_2s_3s_1s_2s_4s_3$	6	21
10	$s_2s_4s_1s_2s_4s_3$	6	23
11	$s_3s_2s_3s_1s_2s_4$	6	25
12	$s_3s_2s_4s_1s_2s_4$	6	28

Table 37 Weyl group D_4, conjugacy class 12, order 2

N^o in CCL	Element	Level	N^o in Level
1	$s_4s_3s_2s_4s_3s_2s_1s_2s_4s_3s_2s_1$	12	0

Appendix A: Some properties of weights

This section lists some properties of finite Weyl groups, weights related to Lie algebras and Weyl groups, as well as some other concepts related to weights.

A.1 Fundamental Weyl chamber

Let $\varPhi$ be a root system, W be the Weyl group associated to $\varPhi$, Δ be the set of the simple roots, $\varPhi^+$ (resp. $\varPhi^-$) be the set of positive (resp. negative) roots, $\Delta = \{\alpha_1, \dots, \alpha_l\}$, and $\mathcal{E}$ be the linear space spanned by roots of Δ. For any root $\alpha \in \varPhi$, let H_α be the hyperplane $\{x \in \mathcal{E} \mid (\alpha, x) = 0\}$. There is the finite number of the connected components of

$$\mathcal{E} - \bigcup_{\alpha \in \varPhi} H_\alpha.$$

These components are called the open *Weyl chambers*. There is the unique chamber C such that for any $\xi \in C$, the following inequality holds:

$$(\xi, \alpha_i) > 0 \quad \text{for all } \alpha_i \in \Delta, \tag{A.1}$$

where $(\cdot, \cdot)$ is the Cartan–Killing bilinear form. The unique Weyl chamber C is called the *fundamental Weyl chamber*.[3]

Equation (A.1) is equivalent to each of the following two statements:

$$\begin{aligned} &(\xi, \alpha) > 0 \quad \text{for all } \alpha \in \varPhi^+, \\ &(\xi, \alpha) < 0 \quad \text{for all } \alpha \in \varPhi^-. \end{aligned} \tag{A.2}$$

Theorem A.1 ([2, ch. VI, §1, n°5, Th. 2])

(i) *The Weyl group acts simply-transitively on the Weyl chambers. Thus, the order of the Weyl group is equal to the number of Weyl chambers.*

(ii) *Each $\xi \in \mathcal{E}$ is conjugate to a unique point in the closure $\overline{C}$ of the fundamental Weyl chamber (i.e. $\overline{C}$ is a fundamental domain for W).*

The word "conjugate" means "in the same Weyl group orbit".

A.2 Dominant weights

For any vectors $\alpha, \beta \in \varPhi$, let us define $\langle \alpha, \beta \rangle$ as follows:

$$\langle \alpha, \beta \rangle := \frac{2(\alpha, \beta)}{(\beta, \beta)}. \tag{A.3}$$

For the simply-laced Dynkin diagrams, if β is a root, then $\langle \alpha, \beta \rangle = (\alpha, \beta)$. A *weight* (resp. *dominant weight*) is an element $\lambda \in \mathcal{E}$ such that

$$\langle \lambda, \alpha \rangle \in \mathbb{Z} \qquad \big(\text{resp. } \langle \lambda, \alpha \rangle \in \mathbb{Z} \text{ and } \langle \lambda, \alpha \rangle \geq 0\big) \quad \text{for all } \alpha \in \Delta. \tag{A.4}$$

---

[3]The fundamental domains of usual and affine Weyl groups (Weyl chambers) were first described by E. Cartan in 1927, [4], see [1, p.62].

The set of weights Λ forms a subgroup of $\mathcal{E}$ containing the root system Φ, i.e. $\Phi \subset \Lambda \subset \mathcal{E}$. The concept of a dominant weight was introduced by Cartan in [3], however his definition was differ from the definition (A.4), see [6, p. 311].

A.2.1 Partial ordering on the set of weights
Consider two weights μ and λ. We say that μ is *higher* than λ, and we write $\mu \geq \lambda$ if $\mu - \lambda$ is expressible as a linear combination of positive roots with non-negative real coefficients. This order is only partial.

Proposition A.2 ([2, ch. VI, §1, n°6, Prop. 18]) *The weight λ is dominant if and only if*

$$\lambda \geq w\lambda \quad \textit{for any } w \in W. \tag{A.5}$$

The set of dominant vectors is denoted by Λ^+. We have

$$\Lambda^+ = \Lambda \cap \overline{C}. \tag{A.6}$$

Proposition A.3 *Any weight is conjugate to unique dominant weight.*

For details, see [2, Ch. VI, §1, n°10].

A.2.2 Fundamental dominant weights
The weights $\bar{\omega}_i$ satisfying the following relations

$$\langle \bar{\omega}_i, \alpha_j \rangle = \delta_{ij}, \quad \text{where } i, j \in \{1, \dots, l\} \tag{A.7}$$

are called *fundamental dominant weights*. Any weight $\lambda \in \mathcal{E}$ can be written as an integral linear combination of the vectors $\{\bar{\omega}_1, \dots, \bar{\omega}_l\}$. The basis of the fundamental dominant weights is dual to the basis of simple roots on $\mathcal{E}$ relative to the bilinear form Cartan–Killing.

A.3 The action of the Weyl group on the the weights

A.3.3 Length $l(w)$
Each element w in the Weyl group W is the product of reflections s_i, where

$$s_i(x) = x - \langle x, \alpha_i \rangle \alpha_i. \tag{A.8}$$

The minimal number of simple reflections s_i in the decomposition

$$w = s_{i_1} \cdots s_{i_n}$$

is called the *length* of the element w and is denoted by $l(w)$.

Proposition A.4 ([7, p. 1.7, Corollary]) *The length of w is equal to the number of positive roots, which are transformed to negative roots under w.*

The reflection s_i transforms α_i to $-\alpha_i$ and permutes the other positive roots, then by Proposition A.4, we have the following

Theorem A.5 ([7, p. 1.6, Lemma])

$$l(s_i w) = \begin{cases} l(w) + 1, & \textit{if } w^{-1}(\alpha_i) \in \Phi^+, \\ l(w) - 1, & \textit{if } w^{-1}(\alpha_i) \in \Phi^-. \end{cases}$$

A.3.4 The element of the maximal length

Proposition A.6 ([2, ch. VI, §1, n°6, Corollary 3]) *There exists the unique element w_0 of the maximal length in the Weyl group W. Length $l(w_0)$ is equal to the number of positive roots.*

A.3.5 The action s_i on a weight

Let us expand an arbitrary vector $\lambda \in \mathcal{E}$ in the basis consisting of all fundamental dominant weights $\{\bar{\omega}_1, \dots, \bar{\omega}_l\}$:

$$\lambda = \sum_{i=1}^{l} m_i \bar{\omega}_i. \tag{A.9}$$

Here, $(m_1, \dots, m_l)$ are the coordinates of the weight λ in the basis $\{\bar{\omega}_1, \dots, \bar{\omega}_l\}$. By (A.7) we have $m_j = \langle \lambda, \alpha_j \rangle$ for any $j \in \{1, \dots, l\}$. If λ is one of roots, i.e., $\lambda = \alpha_j$, then

$$\alpha_j = \sum_{i=1}^{l} c_{ij} \bar{\omega}_i, \tag{A.10}$$

where $c_{ij} = \langle \alpha_i, \alpha_j \rangle$. Let $\overline{c}_i = (c_{i1}, \dots, c_{il})$ be the ith row of the Cartan matrix $(\langle \alpha_i, \alpha_j \rangle)_{i,j=1}^{l}$. The vector $\overline{c}_i$ is the root α_i in the basis of fundamental weights. Then

$$\begin{aligned} &s_i(\lambda) = \lambda - \langle \lambda, \alpha_i \rangle \alpha_i = \lambda - m_i (c_{i1}, \dots, c_{il}), \quad \text{i.e.,} \\ &s_i(\lambda) = (m_1 - m_i c_{i1}, \dots, m_l - m_i c_{il}). \end{aligned} \tag{A.11}$$

Equation (A.11) is the main formula of Snow's algorithm.

A.3.6 Dual bases and the Cartan matrix

The Cartan matrix $B = \{c_{ij}\}$ relates dual bases $\{\bar{\omega}\} = \{\bar{\omega}_i\}_{i=1,\dots,l}$ and $\{\alpha\} = \{\alpha_i\}_{i=1,\dots,l}$ as follows:

$$\bar{\omega}_i = B^{-1} \alpha_i \quad \text{for } i = 1, \dots, l, \tag{A.12}$$

see (A.7), (A.10). In other words, B is the transition matrix from the basis of the fundamental weights $\{\bar{\omega}\}$ to the basis of the simple roots $\{\alpha\}$.

Let s be a reflection in the basis $\{\alpha\}$. Since elements of the Weyl group preserve the Cartan–Killing bilinear form, for any vectors $u, v \in \mathcal{E}$, we have

$$\begin{aligned} &(su, v) = (u, sv), \quad \text{i.e.,} \\ &\langle Bsu, v \rangle = \langle Bu, sv \rangle = \langle {}^t sBu, v \rangle, \end{aligned} \tag{A.13}$$

where ${}^t s$ is the transposed matrix for the reflection matrix s. Then,

$$Bs = {}^t sB, \quad \text{or} \quad BsB^{-1} = {}^t s. \tag{A.14}$$

Here, BsB^{-1} is the reflection s in the basis of $\{\bar{\omega}\}$, we get the following

Proposition A.7 *If s is the reflection matrix in the basis $\{\alpha\}$, the transposed matrix ${}^t s$ is the reflection matrix s in the basis of the fundamental weights $\{\bar{\omega}\}$.*

Let $s_{\{\bar{\omega}\}}$ (resp. $s_{\{\bar{\alpha}\}}$) be the representation of the matrix s in the basis $\{\bar{\omega}\}$ (resp. $\{\bar{\alpha}\}$). By Proposition A.7, the action of any reflection s on the column vector v in the basis $\{\bar{\omega}\}$ is as follows:

$$s(v)_{\{\bar{\omega}\}} = s_{\{\bar{\omega}\}} v_{\{\bar{\omega}\}} = {}^t s_{\{\bar{\alpha}\}} v_{\{\bar{\omega}\}} = {}^t(\bar{v} s_{\{\bar{\alpha}\}}), \tag{A.15}$$

where $\bar{v} = {}^t v_{\{\bar{\omega}\}}$ is the row vector in the basis $\{\bar{\omega}\}$. On the left of (A.15), we have the column vector. Transpose this vector as follows:

$${}^t\big(s(v)_{\{\bar{\omega}\}}\big) = \bar{v} s_{\{\bar{\alpha}\}} = {}^t v_{\{\bar{\omega}\}} s_{\{\alpha\}}, \tag{A.16}$$

where the vector on the left and the vector $\bar{v}$ are row vectors. This means that instead of using the column vector $v_{\{\bar{\omega}\}}$ and the reflection in the basis $\{\bar{\omega}\}$, we can use the row vector ${}^t v_{\{\bar{\omega}\}}$ and the reflection in the basis $\{\alpha\}$.

Equation (A.16) is another form of Eq. (A.11), which is the main formula of Snow's algorithm.

A.4 Representation and weight space

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ (a maximal abelian subalgebra). The roots are defined as the nonzero eigenvalues of $\mathfrak{h}$ acting on $\mathfrak{g}$ via the adjoint representation:

$$\alpha : \mathfrak{h} \longrightarrow \mathbb{C}, \qquad [h,x] = \alpha(h)x \quad \text{for all } h \in \mathfrak{h}, \tag{A.17}$$

where $x \in \mathfrak{g}$ is a corresponding eigenvector. The roots are considered as linear functionals on $\mathfrak{h}$, they span a real space E in the dual space $\mathfrak{h}^*$.

Let V be a representation of $\mathfrak{g}$ over $\mathbb{C}$ (not necessarily adjoint). A weight λ of the representation V with the *weight space* of V_λ is a linear functional on $\mathfrak{h}$ given as follows:

$$V_\lambda := \big\{x \in V, h \cdot x = \lambda(h)x \text{ for all } h \in \mathfrak{h}\big\} \tag{A.18}$$

A.5 Theorem of highest weight

Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra. A weight λ of a representation V of $\mathfrak{g}$ is called a *highest weight* if $\mu \le \lambda$ for every other weight μ of V, see Sect. A.2.1.

In 1913 the theorem of highest weight for representations of simple Lie algebras was completed by E. Cartan.

Theorem A.8 (E. Cartan, [3])

(i) *If V is a finite-dimensional irreducible representation of $\mathfrak{g}$, then V has a unique highest weight, and this highest weight is dominant integral.*

(ii) *If two finite-dimensional irreducible representations have the same highest weight, they are isomorphic.*

(iii) *For each dominant integral weight λ, there exists a finite-dimensional irreducible representation with highest weight λ.*

A.6 Fundamental weights in the case D_4

The dependencies of simple roots $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and elements of the canonical basis $\{e_1, e_2, e_3, e_4\}$ are as follows:

$$\begin{aligned} &\alpha_1 = e_1 - e_2, \qquad \alpha_2 = e_2 - e_3, \qquad \alpha_3 = e_3 - e_4, \qquad \alpha_4 = e_3 + e_4, \\ &e_1 = \alpha_1 + \alpha_2 + \frac{\alpha_3 + \alpha_4}{2}, \qquad e_2 = \alpha_2 + \frac{\alpha_3 + \alpha_4}{2}, \\ &e_3 = \frac{\alpha_3 + \alpha_4}{2}, \qquad e_4 = \frac{\alpha_4 - \alpha_4}{2}. \end{aligned} \tag{A.19}$$

By [2, Table IV], the *fundamental weights* $\{\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \bar{\omega}_4\}$ can be calculated by the following formulas:

$$\begin{aligned} &\bar{\omega}_1 = e_1 = \alpha_1 + \alpha_2 + \frac{\alpha_3 + \alpha_4}{2}, \\ &\bar{\omega}_2 = e_1 + e_2 = a_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \\ &\bar{\omega}_3 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4) = \frac{1}{2}(a_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \\ &\bar{\omega}_4 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4) = \frac{1}{2}(a_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4). \end{aligned} \tag{A.20}$$

Let B denote the Cartan matrix. Then formulas (A.20) can also be obtained using the inverse of Cartan matrix B^{-1} as follows:

$$\bar{\omega}_i = B^{-1}\alpha_i, \tag{A.21}$$

see [2, Ch. VI, (14)]. For the case D_4:

$$B = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & -2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}, \qquad B^{-1} = \begin{bmatrix} 1 & 1 & 1/2 & 1/2 \\ 1 & 2 & 1 & 1 \\ 1/2 & 1 & 1 & 1/2 \\ 1/2 & 1 & 1/2 & 1 \end{bmatrix} \tag{A.22}$$

Appendix B: The Python implementation

B.7 Root system, generators and number of levels

The file below (*reflections_D4.py*) contains information related to the current root system: reflections, the Cartan matrix and number of levels. You can change to a different root system only by modifying this file. The root system is given as string "D4", "B5", "E6", etc. The generators of the Weyl group are given as the matrices of the faithful representation. The number of levels is equal to the number of positive roots plus one, see Proposition (A.4).

```
'''reflections_D4.py'''
root_system = 'D4'

s1 = [[-1, 1, 0, 0],
      [ 0, 1, 0, 0],
      [ 0, 0, 1, 0],
      [ 0, 0, 0, 1]]
s2 = [[1, 0, 0, 0],
      [1,-1, 1, 1],
      [0, 0, 1, 0],
      [0, 0, 0, 1]]
s3 = [[1, 0, 0, 0],
      [0, 1, 0, 0],
      [0, 1,-1, 0],
      [0, 0, 0, 1]]
s4 = [[1, 0, 0, 0],
      [0, 1, 0, 0],
      [0, 0, 1, 0],
      [0, 1, 0,-1]]
refl = []
refl.append(s1)
refl.append(s2)
refl.append(s3)
refl.append(s4)
''' Cartan matrix '''
Cmatr = \
[[ 2,-1, 0, 0],
 [-1, 2,-1,-1],
 [ 0,-1, 2, 0],
 [ 0,-1, 0, 2]]

''' Number of levels = number of positive roots + 1'''
Nlevels = 13
```

B.8 Data structure

The class *Element* contains all the information related to the given element (and its inverse) of the Weyl group. Consider, for example, the first element in Table (13)

$$s_1 s_2 s_4 s_3 s_2 s_1. \tag{B.1}$$

The name of the element (B.1) in the class is the following string:

$$s1.s2.s4.s3.s2.s1. \tag{B.2}$$

The information added during the extended Snow's algorithm is the name of the inverse element, its matrix, and its location in "level". This information is needed to calculate conjugacy classes.

```
''' element.py '''
import numpy as np

def isIdentity(M):
    for i in range(len(M)):
        for j in range(len(M[0])):
           if i == j and M[i][j] != 1:
               return False
           elif i!=j and M[i][j] != 0:
               return False
    return True

class Element(object):
    def __init__(self, w, name, name_inv, m, m_inv, n_in_lvl):
        self.weight      = w
        self.name        = name
        self.name_inv    = name_inv
        self.matr        = m
        self.matr_inv    = m_inv
        self.n_in_lvl    = n_in_lvl
        '''we don't know yet the location of inverse element in "level" '''
        self.n_inv_in_lvl = -1

    def keyValAndKeyInv(self):
        key     = ''.join(str(i) for row in self.matr for i in row)
        key_inv = ''.join(str(i) for row in self.matr_inv for i in row)
        val = str(self.n_in_lvl)
        return key, key_inv, val

    def ifSelfInverseMatr(self):
        prod = np.matmul(self.matr, self.matr)
        return isIdentity(prod)
```

B.9 Calculation of all levels

This section contains the main implementation file of the extended Snow's algorithm, including the search for inverse elements. To move to another root system, you need to change the inclusion *from reflections_D4* to the appropriate one, see Sect. B.7.

```python
'''algor_Snow_InvElem.py  '''
import numpy as np
from reflections_D4 import root_system, refl, Cmatr, Nlevels
from element import Element

def buildLevel_0(oneLevel):
   start_weight = np.ones(len_weight, dtype=int).tolist()
   unit_matr = np.eye(len_weight, dtype=int).tolist()
   elm = Element(w=start_weight, name=' ', name_inv = ' ', \
                 m=unit_matr, m_inv=unit_matr, n_in_lvl = 0)
   elm.n_inv_in_lvl = 0
   oneLevel.append(elm)

def newPossibleWeight(numbRefl,  weight, mi):
    new_possible_weight = []
    for jW in range(len_weight):
        ''' The main formula of Snow's algoritm '''
        new_coord =  weight[jW] - mi*Cmatr[jW][numbRefl]
        new_possible_weight.append(new_coord)
    return new_possible_weight

def newElem(iRefl, new_weight, name, name_inv, matr, matr_inv, new_n_in_lvl):
    ''' new_n_in_lvl = the following place in the oneLevel, i.e. = len(one_level) '''
    iW = iRefl - 1
    if (name == ' '):
       new_name_inv = new_name        = str('s') + str(iRefl)
       new_matr_inv = new_matr        = refl[iW]
    else:
       new_name     = str('s') + str(iRefl) + str('.') + name
       new_name_inv = name_inv + str('.s') + str(iRefl)
       new_matr     = np.matmul(refl[iW], matr)
       new_matr_inv = np.matmul(matr_inv, refl[iW])

    new_elem = \
      Element(new_weight, new_name, new_name_inv, new_matr, new_matr_inv, new_n_in_lvl)
    return new_elem

def findAllLevels_to_LvlK(root_system, list_of_all_levels, lvlK):

  len_by_all_levels = 0
  for ik in range(lvlK):
     ''' Get Lvl(k) and create Lvl(k+1) '''
     oneLevel = list_of_all_levels[ik]
     new_level = []
     dictElemsOfLevel =  {}
```

```
        len_by_all_levels =  len_by_all_levels + len(oneLevel)

        for iElem in range(len(oneLevel)):
            elem = oneLevel[iElem]
            ''' get elements of lvl = ik to construct the lvl = (ik+1) '''
            weight = elem.weight

            ''' iRefl is  the numb of reflection '''
            for iW in range(len_weight):
              iRefl = iW + 1
              if  weight[iW] > 0:
                  mi = weight[iW]
                  new_candidate_weight = newPossibleWeight(iW, weight, mi)
                  ''' should be unique weight '''
                  uniqueFlag = True
                  if (iW == len_weight - 1):
                      uniqueFlag = True
                  else:
                      for iUniq in range(iW+1, len_weight):
                         if new_candidate_weight[iUniq] < 0:
                             uniqueFlag = False
                             break

                  if uniqueFlag is True:
                      ''' This the element of order 2 '''
                      new_n_in_lvl = len(new_level)
                      new_elm = newElem(iRefl, new_candidate_weight, \
                          elem.name, elem.name_inv, \
                          elem.matr, elem.matr_inv, new_n_in_lvl)

                      if new_elm.ifSelfInverseMatr():
                          new_elm.n_inv_in_lvl = new_elm.n_in_lvl
                          ''' no need to save this elem in dictionary'''
                          new_level.append(new_elm)
                      else:
                          key, key_inv, val = new_elm.keyValAndKeyInv()

                          if key in dictElemsOfLevel.keys():
                             ''' the partner (inverse) already waits for this key'''
                             ''' relate 'new_elem_inv' and 'new_elm'  '''
                             val = dictElemsOfLevel[key]
                             n_in_lvl = int(val)
                             new_elem_inv = new_level[n_in_lvl]
                             new_elem_inv.n_inv_in_lvl = new_elm.n_in_lvl
                             new_elm.n_inv_in_lvl = new_elem_inv.n_in_lvl
                             new_level.append(new_elm)
```

```
                    else:
                        new_elm.n_in_lvl = len(new_level)
                        ''' inform the partner(inv) about location of new element'''
                        val = str(new_elm.n_in_lvl)
                        dictElemsOfLevel[key_inv] = val
                        new_level.append(new_elm)

     list_of_all_levels.append(new_level)

     ''' write down new_level on a disk '''
     writeOneLevel(ik+1, new_level, prefix)


''' single level recoding procedure. Parameters are as follows:
ik - number of level, oneLevel - one level from list_of_all_levels,
prefix - string root_system, like "D4", "B5", "E6", etc.'''
def writeOneLevel(ik, oneLevel, prefix):

    n_elems = len(oneLevel)

    if n_elems == 0:
        return

    file_name = prefix + '_WeightMatrByLevel_' + str(ik) +\
            '_elems=' + str(n_elems) + '.txt'
    path_to_file = prefix + '_DataFiles\\' + file_name
    print('write file: ', path_to_file)
    with  open(path_to_file, 'w') as f:

        '''in weight the last comma already there '''
        for iElem in range(len(oneLevel)):
            elem = oneLevel[iElem]
            wStr = weightToStr(elem.weight)
            lineElem = 'n='+ str(elem.n_in_lvl) +\
                        ', name=' + elem.name +\
                        ', w=' + wStr +\
                        ', n_inv=' + str(elem.n_inv_in_lvl)
            f.write(lineElem)
            for r in elem.matr:
                line = list(r)
                f.write('\n')
                f.write(str(line))
            f.write('\n')
        f.close()

if __name__ == "__main__":
```

```
    list_of_all_levels = []
    len_weight = len(Cmatr)
    oneLevel = []

    ''' Step 0'''
    buildLevel_0(oneLevel)
    writeOneLevel(0, oneLevel, root_system)
    list_of_all_levels.append(oneLevel)

    ''' Here, function writeOneLevel is called for each level  '''
    findAllLevels_to_LvlK(root_system, list_of_all_levels, lvlK=Nlevels)
```

B.10 Sample output: file containing one level

For the root system D_4, we get 13 files corresponding to 13 levels. Here is the file containing level 2 consisting of 9 elements.

```
n=0, name=s2.s1, w=1,-2,3,3, n_inv=3
[-1, 1, 0, 0]
[-1, 0, 1, 1]
[0, 0, 1, 0]
[0, 0, 0, 1]
n=1, name=s3.s1, w=-1,3,-1,1, n_inv=1
[-1, 1, 0, 0]
[0, 1, 0, 0]
[0, 1, -1, 0]
[0, 0, 0, 1]
n=2, name=s4.s1, w=-1,3,1,-1, n_inv=2
[-1, 1, 0, 0]
[0, 1, 0, 0]
[0, 0, 1, 0]
[0, 1, 0, -1]
n=3, name=s1.s2, w=-2,1,2,2, n_inv=0
[0, -1, 1, 1]
[1, -1, 1, 1]
[0, 0, 1, 0]
[0, 0, 0, 1]
n=4, name=s3.s2, w=2,1,-2,2, n_inv=6
[1, 0, 0, 0]
[1, -1, 1, 1]
[1, -1, 0, 1]
[0, 0, 0, 1]
n=5, name=s4.s2, w=2,1,2,-2, n_inv=8
[1, 0, 0, 0]
[1, -1, 1, 1]
```

```
[0, 0, 1, 0]
[1, −1, 1, 0]
n=6, name=s2.s3, w=3,−2,1,3, n_inv=4
[1, 0, 0, 0]
[1, 0, −1, 1]
[0, 1, −1, 0]
[0, 0, 0, 1]
n=7, name=s4.s3, w=1,3,−1,−1, n_inv=7
[1, 0, 0, 0]
[0, 1, 0, 0]
[0, 1, −1, 0]
[0, 1, 0, −1]
n=8, name=s2.s4, w=3,−2,3,1, n_inv=5
[1, 0, 0, 0]
[1, 0, 1, −1]
[0, 0, 1, 0]
[0, 1, 0, −1]
```

Appendix C: Procedure for obtaining CCLs

This section describes an example algorithm (using pseudocode) for obtaining CCLs. This algorithm uses information about the inverse elements found by ESA.

First, read all elements of all levels into 2-dimensional list 'list_of_levels'. Add to each element information about its inverse element.

Further, create a dictionary 'dictAllElems' containing information on each element. The Python dictionary used here is similar to the dictionary in ESA, see Sect. 3.2.

```
''' create 'dictAllElems' with 'key' and 'value' for each element '''
def consructKey(level, matrl):
    key = ''.join(str(i) for row in matr for i in row)
    return key
def consructValue(level, elem):
    value = str(level) + ',' + str(elem.n_in_lvl)
    return value
```

If some element is not yet included in any CCL ("elem.ccl" is −1), then a new conjugacy class "oneCCL" is created. Each candidate to be included in "oneCCL" is checked to see if it has been included before. The function "createCCL" is executed in a loop for all elements not yet covered.

```
def createCCL(list_of_levels, dictAllElems, elem, oneCCL, ccl_number):
    setOneCCL = set()
    oneCCL.append(elem)
    setOneCCL.add(consructKey(elem.matr))
    for ik in range(1, Nlevels):
```

```
oneLevel = list_of_levels[ik]
for iElem in range(len(oneLevel)):
   an_elem = oneLevel[iElem]
   matr_inv_an_elem = oneLevel[an_elem.n_inv_in_lvl].matr
   matr_prom = np.matmul(an_elem.matr, elem.matr)
   conj_matr = np.matmul(matr_prom, matr_inv_an_elem)
   key = consructKey(conj_matr)
   if key not in setOneCCL:
      level, n_in_lvl =  getFromValue(dictAllElems[key])
      conj_elem = list_of_levels[level][n_in_lvl]
      conj_elem.ccl = ccl_number
      oneCCL.append(conj_elem)
      setOneCCL.add(key)
```

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