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Equivalence of some results and fixed-point theorems in S-multiplicative metric spaces

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Abstract

In this paper, some fixed-point theorems are stated and proved in S-multiplicative metric spaces. We also show in this paper that some fixed-point results for various S-multiplicative metric spaces are equivalent to those of corresponding fixed-point results in S-metric spaces. Some examples are presented to validate the originality and applicability of our main results.

Keywords: Multiplicative metric spaces; S-metric spaces; Usual metric spaces; Fixed point; Contractive maps

1 Introduction

In 1922, the Banach contraction principle (BCP) was introduced by Banach [8]. This was preceded by the introduction of metric spaces by Frechet in 1906 [17]. The BCP was used as an alternative method to prove the existence and uniqueness of solutions of ODE [8, 16]. Since then, metric space had been a crucial device in functional analysis, nonlinear analysis, and topology. The topological structure of this space with its usefulness in fixed-point theory has attracted the attention of many mathematicians (see [1–32]). In recent years, diverse applications of fixed-point theorems have challenged researchers to introduce different generalizations of metric spaces. These generalized spaces include 2-metric spaces, D-metric spaces, D^*-metric spaces, G-metric spaces, b-metric spaces, quasimetric spaces, G_b-metric spaces, complex-valued G_b-metric spaces, quaternion-valued G-metric spaces, S-metric spaces, S_b-metric spaces, complex-valued S_b-metric spaces, A-metric spaces, γ-generalized quasimetric spaces, S_p-metric spaces and A_p-metric spaces, multiplicative metric spaces, G_b-multiplicative metric spaces, and most recently, rectangular S-metric spaces (see [1–3, 5, 7, 16, 21, 26, 28, 31]).

In this paper, some fixed-point theorems are stated and proved in multiplicative S-metric spaces. We also show that some fixed-point theorems are equivalent in both multiplicative S-metric spaces and S-metric spaces.

In [17], Frechet defined metric spaces as follows:

Definition 1.1 ([17]) For a nonempty set X and a function $d : X^2 \to [0,\infty)$ satisfying the following properties:

(i) $d(x,y) \geq 0$ for all $x, y \in X$;
(ii) $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(iii) $d(x,y) = d(y,x), \forall x, y \in X$;
(iv) $d(x,y) \leq d(x,z) + d(z,y) \forall x, y, z \in X$.

d is called a metric on X and (X, d) is called a rectangular metric space.

In [28], Sedghi *et al.* introduced the notion of an S-metric space as follows.

Definition 1.2 ([28]) Let X be a nonempty set and $\overline{S} : X^3 \to \mathbb{R}^+$, a function satisfying the following properties:
(i) $\overline{S}(x,y,z) = 0$ if and only if $x = y = z$;
(ii) $\overline{S}(x,y,z) \leq \overline{S}(x,x,a) + \overline{S}(y,y,a) + \overline{S}(z,z,a) \forall a, x, y, z \in X$ (rectangle inequality).

Then, $(X, \overline{S})$ is called a S-metric-metric space.

The following is the definition of S_p-metric spaces, a generalization of both S-metric spaces and S_b-metric spaces.

Definition 1.3 ([24]) Let X be a nonempty set and $\overline{S} : X^3 \to \mathbb{R}^+$, a function with a strictly increasing continuous function, $\Omega : [0,\infty) \to [0,\infty)$ such that $\Omega(t) \geq t$ for all $t > 0$ and $\Omega(0) = 0$, satisfying the following properties:
(i) $\overline{S}(x,y,z) = 0$ if and only if $x = y = z$;
(ii) $\overline{S}(x,y,z) \leq \Omega(\overline{S}(x,x,a) + \overline{S}(y,y,a) + \overline{S}(z,z,a)) \forall a, x, y, z \in X$ (rectangle inequality).

Then, $(X, \overline{S})$ is called an S_p-metric-metric space.

Remark 1.4
(i) If $\Omega(z) = z$, S_p-metric space reduces to S-metric space.
(ii) If $\Omega(z) = bz$, S_p-metric space reduces to S_b-metric space.

In 2008, Bashirov *et al.* [9], introduced multiplicative metric spaces in the following way.

Definition 1.5 ([9]) For a nonempty set X and a function $d : X^2 \to [0,\infty)$ satisfying the following properties:
(i) $d(x,y) \geq 1$ for all $x, y \in X$;
(ii) $d(x,y) = 1$ if and only if $x = y$ for all $x, y \in X$;
(iii) $d(x,y) = d(y,x), \forall x, y \in X$;
(iv) $d(x,y) \leq d(x,z).d(z,y) \forall x, y, z \in X$.

d is called a multiplicative metric on X and (X, d) is called a multiplicative metric space. By taking logarithms of (iv), the multiplicative metric space is equivalent to the standard metric space.

2 Main results

We introduce the following definitions:

Definition 2.1 Let X be a nonempty set and $\overline{S} : X^3 \to \mathbb{R}^+$, a function satisfying the following properties:
(i) $\overline{S}(x,y,z) = 1$ if and only if $x = y = z$;
(ii) $\overline{S}(x,y,z) \leq \overline{S}(x,x,a) \times \overline{S}(y,y,a) \times \overline{S}(z,z,a)$.

Then, $(X, \overline{S})$ is called a S-multiplicative metric space.

Example 2.2 Let $X = Z$ and define $S : X^3 \to R^+ \cup \{0\}$ by

$$\overline{S}(x,y,z) = \begin{cases} 1, & x = y = z; \\ e^{x+y+z}, & \text{otherwise.} \end{cases}$$

Then, (X,S) is a S-multiplicative metric space but neither a S-metric space nor metric space as

$$S(0,1,2) < S(0,0,-1).$$

Example 2.3 Let $X = Z$ and define $S : X^3 \to R^+ \cup \{0\}$ by

$$\overline{S}(x,y,z) = \begin{cases} 1, & x = y = z; \\ e^{x}, & \text{otherwise.} \end{cases}$$

Then, (X,S) is a S-multiplicative metric space but neither a S-metric space nor metric space as

$$S(-6,-4,-2) < S(-6,-6,1).$$

Example 2.4 Let $X = N \cup \{0\}$ and define $S : X \times X \times X \to R^+ \cup \{0\}$ by

$$\overline{S}(x,y,z) = \begin{cases} 1, & x = y = z; \\ xyz, & \text{otherwise.} \end{cases}$$

Then, (X,S) is a S-multiplicative metric space but neither a S-metric space nor metric space as

$$S(x,y,z) = S(x,x,y).$$

Definition 2.5 Let (X,S) be a S-multiplicative metric space. For $y \in X$, $r > 0$, the S-sphere with center y and radius r is

$$\underline{S}_(y,r) = \{z \in X : \overline{S}(y,z,z) < r\}.$$

Definition 2.6 Let $(X.S)$ be a S-multiplicative metric space. A sequence $\{x_n\} \subset X$ is S-convergent to z if it converges to z in the S-multiplicative metric topology.

Definition 2.7 Let (X,S) and $(\overline{X},\overline{S})$ be two S-multiplicative metric spaces, a function $T : X \to \overline{X}$ is S-continuous at a point $x \in X$ if $T^{-1}(S_{\overline{S}}(T(x),r)) \in \tau(S)$, for all $r > 1$. T is S-continuous if it is S-continuous at all points of X.

Lemma 2.8 *Let (X,S) be a S-multiplicative metric space and $\{x_n\}$ a sequence in X. Then, $\{x_n\}$ converges to x if and only if $\overline{S}(x_n,x,x) \to 1$ as $n \to \infty$.*

Lemma 2.9 *Let (X,S) be a S-multiplicative metric space and $\{x_n\}$ a sequence in X. Then, $\{x_n\}$ is said to be a Cauchy sequence if and only if $\overline{S}(x_n,x_m,x_l) \to 1$ as $n,m,l \to \infty$.*

Theorem 2.10 *Let X be a complete S-multiplicative metric space and $T : X \to X$ a map for which there exist the real number, k satisfying $0 \leq k < 1$ such that for each pair $x, y, z \in X$.*

$$\overline{S}(Tx, Ty, Tz) \leq \left(\overline{S}(x, y, z)\right)^k. \tag{1}$$

Then, T has a unique fixed point.

Proof Considering (1),

$$\overline{S}(Tx, Ty, Ty) \leq \left(\overline{S}(x, y, y)\right)^k. \tag{2}$$

Suppose T satisfies condition (2) and $x_0 \in X$ is an arbitrary point and define a sequence x_n by $x_n = T^n x_0$, then

$$\overline{S}(x_n, x_n, x_{n+1}) = \overline{S}(Tx_{n-1}, Tx_{n-1}, Tx_n) \tag{3}$$

$$\leq \left(\overline{S}(x_{n-1}, x_{n-1}, x_n)\right)^k \tag{4}$$

$$\leq \left(\overline{S}(x_{n-2}, x_{n-2}, x_{n-1})\right)^{k^2}. \tag{5}$$

Using this repeatedly, we obtain

$$\overline{S}(x_n, x_n, x_{n+1}) \leq \left(\overline{S}(x_0, x_0, x_1)\right)^{k^n}. \tag{6}$$

By using (ii) in Definition 2.1, we have

$$\overline{S}(x_n, x_m, x_m) \leq \overline{S}(x_n, x_n, x_{n+1}) \times \left(\overline{S}(x_m, x_m, x_{n+1})\right)^2 \tag{7}$$

$$= \overline{S}(x_n, x_n, x_{n+1})\left(\overline{S}(x_m, x_m, x_{n+1})\right)^2. \tag{8}$$

Using this repeatedly with $m > n$, we obtain

$$\overline{S}(x_n, x_m, x_m) \leq \overline{S}(x_n, x_n, x_{n+1}) \times \overline{S}\left(\overline{S}(x_{n+1}, x_{n+1}, x_{n+2})\right)^2 \tag{9}$$

$$\times \left(\overline{S}(x_{n+2}, x_{n+2}, x_{n+3})\right)^4 \times \therefore \tag{10}$$

$$\times \left(\overline{S}(x_{m-1}, x_{m-1}, x_m)\right)^{2n}. \tag{11}$$

From (6) and (11), we have

$$\overline{S}(x_n, x_m, x_m) \leq \left(\overline{S}(x_0, x_0, x_1)\right)^{k^n} \times \overline{S}\left(\overline{S}(x_0, x_0, x_1)\right)^{k^{2n}} \tag{12}$$

$$\times \left(\overline{S}(x_0, x_0, x_1)\right)^{k^{4n}} \times \therefore \tag{13}$$

$$\times \left(\overline{S}(x_0, x_0, x_1)\right)^{k^{2n^2}}. \tag{14}$$

Taking the limit of $\overline{S}(x_n, x_m, x_m)$ as $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} \overline{S}(x_n, x_m, x_m) = 1. \tag{15}$$

For $n, m, l \in N$ with $n > m > l$,

$$\overline{S}(x_n, x_m, x_l) \leq \overline{S}(x_n, x_n, x_{n-1}) + \overline{S}(x_m, x_m, x_{n-1}) \tag{16}$$

$$+ \overline{S}(x_l, x_l, x_{n-1}). \tag{17}$$

Taking the limit of $\overline{S}(x_n, x_m, x_l)$ as $n, m, l \to \infty$, we have

$$\lim_{n,m,l\to\infty} \overline{S}(x_n, x_m, x_l) = 1. \tag{18}$$

Hence, $\{x_n\}$ is a S-Cauchy sequence.

By the completeness of (X, S), there exist $u \in X$ such that x_n is S-convergent to u.

Suppose $Tu \neq u$

$$\overline{S}(x_n, Tu, Tu) \leq \left(\overline{S}(x_{n-1}, u, u)\right)^k. \tag{19}$$

Taking the limit as $n \to \infty$ and using the fact that function is S-continuous in its variables, we obtain

$$\overline{S}(u, Tu, Tu) \leq \left(\overline{S}(u, u, u)\right)^k. \tag{20}$$

Hence,

$$\overline{S}(u, Tu, Tu) \leq 1. \tag{21}$$

This is a contradiction. Hence, $Tu = u$.

To show the uniqueness, suppose $v \neq u$ is such that $Tv = v$, then

$$\overline{S}(Tu, Tv, Tv) \leq \left(\overline{S}(u, v, v)\right)^k. \tag{22}$$

Since $Tu = u$ and $Tv = v$, we have

$$\overline{S}(u, v, v) \leq 1, \tag{23}$$

which implies that $v = u$. □

Remark 2.11 Let (X, S) be a S-multiplicative metric space and $d : X \times X \to [0, \infty)$ a function defined by $d(x, y) = \overline{S}(x, y, y)$, then Theorem 2.10 reduces to the Banach contraction principle in multiplicative metric space (an analog of the Banach contraction principle in multiplicative metric space).

Lemma 2.12 *Let (X, S) be a S-multiplicative metric space and $\{x_n\}$ a sequence in X. Then, $\overline{S}(x_n, x_{n+1}, x_{n+1}) \leq \overline{S}(x_n, x_n, x_{n+1})$.*

Proof By (ii) of Definition 2.1,

$$\overline{S}(x_n, x_{n+1}, x_{n+1}) \leq \overline{S}(x_n, x_n, x_{n+1}) \times \left(\overline{S}(x_{n+1}, x_{n+1}, x_{n+1})\right)^2 \tag{24}$$

$$\leq \overline{S}(x_n, x_n, x_{n+1}) \times (1)^2 \tag{25}$$

$$\leq \overline{S}(x_n, x_n, x_{n+1}). \tag{26}$$

□

Theorem 2.13 *Let X be a complete S-multiplicative metric space and $T : X \to X$ a map for which there exist the real number, b satisfying $b < -1$ such that for each pair $x, y, z \in X$.*

$$\overline{S}(Tx, Ty, Tz) \leq \left[\overline{S}(x, Tx, Tx) \times \overline{S}(y, Ty, Ty) \times \overline{S}(z, Tz, Tz)\right]^b. \tag{27}$$

Then, T has a unique fixed point.

Proof Suppose T satisfies condition (2) and $x_0 \in X$ is an arbitrary point and define a sequence x_n by $x_n = T^n x_0$, then

$$\overline{S}(x_n, x_n, x_{n+1}) = \overline{S}(Tx_{n-1}, Tx_{n-1}, Tx_n) \tag{28}$$

$$\leq \left[\left(\overline{S}(x_{n-1}, x_n, x_n)\right)^2 \times \overline{S}(x_n, x_{n+1}, x_{n+1})\right]^b \tag{29}$$

$$\leq \left[\left(\overline{S}(x_{n-1}, x_n, x_n)\right)^2 \times \overline{S}(x_n, x_n, x_{n+1})\right]^b \tag{30}$$

$$\leq \left(\overline{S}(x_{n-1}, x_n, x_n)\right)^{\frac{2}{1-b}}. \tag{31}$$

Using $k = \frac{2}{1-b}$ and (31) repeatedly, we obtain

$$\overline{S}(x_n, x_n, x_{n+1}) \leq \left(\overline{S}(x_0, x_0, x_1)\right)^{k^n}. \tag{32}$$

By using (ii) in Definition 2.1, we have

$$\overline{S}(x_n, x_m, x_m) \leq \overline{S}(x_n, x_n, x_{n+1}) \times \left(\overline{S}(x_m, x_m, x_{n+1})\right)^2 \tag{33}$$

$$= \overline{S}(x_n, x_n, x_{n+1})\left(\overline{S}(x_m, x_m, x_{n+1})\right)^2. \tag{34}$$

Using this repeatedly with $m > n$, we obtain

$$\overline{S}(x_n, x_m, x_m) \leq \overline{S}(x_n, x_n, x_{n+1}) \times \overline{S}\left(\overline{S}(x_{n+1}, x_{n+1}, x_{n+2})\right)^2 \tag{35}$$

$$\times \left(\overline{S}(x_{n+2}, x_{n+2}, x_{n+3})\right)^4 \times \therefore \tag{36}$$

$$\times \left(\overline{S}(x_{m-1}, x_{m-1}, x_m)\right)^{2n}. \tag{37}$$

From (6) and (11), we have

$$\overline{S}(x_n, x_m, x_m) \leq \left(\overline{S}(x_0, x_0, x_1)\right)^{k^n} \times \overline{S}\left(\overline{S}(x_0, x_0, x_1)\right)^{k^{2n}} \tag{38}$$

$$\times \left(\overline{S}(x_0, x_0, x_1)\right)^{k^{4n}} \times \therefore \tag{39}$$

$$\times \left(\overline{S}(x_0, x_0, x_1)\right)^{k^{2n^2}}. \tag{40}$$

Taking the limit of $\overline{S}(x_n, x_m, x_m)$ as $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} \overline{S}(x_n, x_m, x_m) = 1. \tag{41}$$

For $n, m, l \in N$ with $n > m > l$,

$$\overline{S}(x_n, x_m, x_l) \leq \overline{S}(x_n, x_n, x_{n-1}) + \overline{S}(x_m, x_m, x_{n-1}) \tag{42}$$

$$+ \overline{S}(x_l, x_l, x_{n-1}). \tag{43}$$

Taking the limit of $\overline{S}(x_n, x_m, x_l)$ as $n, m, l \to \infty$, we have

$$\lim_{n,m,l\to\infty} \overline{S}(x_n, x_m, x_l) = 1. \tag{44}$$

Hence, $\{x_n\}$ is a S-Cauchy sequence.

By the completeness of (X, S), there exist $u \in X$ such that x_n is S-convergent to u.

Suppose $Tu \neq u$

$$\overline{S}(x_n, Tu, Tu) \leq \left[\overline{S}(x_{n-1}, x_n, x_n) \times \left(\overline{S}(u, Tu, Tu)\right)^2\right]^b. \tag{45}$$

Taking the limit as $n \to \infty$ and using the fact that function is S-continuous in its variables, we obtain

$$\overline{S}(u, Tu, Tu) \leq \left[\overline{S}(u, u, u) \times \left(\overline{S}(u, Tu, Tu)\right)^2\right]^b. \tag{46}$$

Hence,

$$\overline{S}(u, Tu, Tu) \leq \frac{1}{1 - 2b}. \tag{47}$$

This is a contradiction. Hence, $Tu = u$.

To show the uniqueness, suppose $v \neq u$ is such that $Tv = v$, then

$$\overline{S}(Tu, Tv, Tv) \leq \left[\overline{S}(u, Tu, Tu)) \times \left(\overline{S}(v, Tv, Tv)\right)^2\right]^b. \tag{48}$$

Since $Tu = u$ and $Tv = v$, we have

$$\overline{S}(u, v, v) \leq 1, \tag{49}$$

which implies that $v = u$. □

Remark 2.14 Let (X, S) be a S-multiplicative metric space and $d : X \times X \to [0, \infty)$ a function defined by $d(x, y) = \overline{S}(x, y, y)$, then Theorem 2.10 reduces to the Banach contraction principle in multiplicative metric space (an analog of the Banach contraction principle in multiplicative metric space).

The following theorem is the Banach contraction principle in S-metric space.

Theorem 2.15 *Let (X, S) be a sequentially compact S-metric space and let $f : X \longrightarrow X$ satisfy the following condition*:

$$S(fx, fy, fz) < kS(x, y, z),$$

whenever $x, y, z \in X$. and $x \neq y \neq z$ with $k \in [0, 1)$. Then, f has a unique fixed point.

Theorem 2.15 in S-multiplicative metric space is as follows:

Theorem 2.16 *Let* $(X,\overline{S})$ *be a sequentially compact* S*-multiplicative metric space and let* $f : X \longrightarrow X$ *satisfy the following condition*:

$$\overline{S}(fx,fy,fz) \leq \left(\overline{S}(x,y,z)\right)^k,$$

whenever $x,y,z \in X$ *and* $x \neq y \neq z$ *with* $k \in [0,1)$. *Then,* f *has a unique fixed point.*

Remark 2.17 The Banach contraction in S-metric space and S-multiplicative metric space are equivalent.

Verification:
Let $\overline{S}(x,y,z) = e^{S(x,y,z)}$. Then,

$$e^{S(fx,fy,fz)} = \overline{S}(fx,fy,fz) \tag{50}$$
$$\leq \left(\overline{S}(x,y,z)\right)^k \tag{51}$$
$$= \left(e^{S(x,y,z)}\right)^k \tag{52}$$
$$= \left(e^{kS(x,y,z)}\right), \tag{53}$$

which implies $S(fx,fy,fz) \leq kS(x,y,z)$.
The following theorem is the Edelstein–Nemytskii theorem in S-metric space.

Theorem 2.18 *Let* (X,S) *be a sequentially compact* S*-metric space and let* $f : X \longrightarrow X$ *satisfy the following condition*:

$$S(fx,fy,fz) < S(x,y,z),$$

whenever $x,y,z \in X$ *and* $x \neq y \neq z$. *Then,* f *has a unique fixed point.*

Theorem 2.18 in S-multiplicative metric space is as follows:

Theorem 2.19 *Let* $(X,\overline{S})$ *be a sequentially compact* S*-multiplicative metric space and let* $f : X \longrightarrow X$ *satisfy the following condition*:

$$\overline{S}(fx,fy,fz) < \overline{S}(x,y,z),$$

whenever $x,y,z \in X$ *and* $x \neq y \neq z$. *Then,* f *has a unique fixed point.*

Remark 2.20 The Edelstein–Nemytskii theorem in S-metric space and S-multiplicative metric space are equivalent.

Verification:
Let $\overline{S}(x,y,z) = e^{S(x,y,z)}$. Then,

$$e^{S(fx,fy,fz)} = \overline{S}(fx,fy,fz) \tag{54}$$

$$\leq \overline{S}(x,y,z) \tag{55}$$

$$= e^{S(x,y,z)}, \tag{56}$$

which implies $S(fx,fy,fz) \leq S(x,y,z)$.

The following theorem is the Kannan theorem in S-metric space.

Theorem 2.21 *Let (X,S) be a sequentially compact S-metric space and let $f : X \longrightarrow X$ satisfy the following condition*:

$$S(fx,fy,fz) \leq a\big[S(x,fx,fx) + S(y,fy,fy) + S(z,fz,fz)\big],$$

whenever $x,y,z \in X$ and $a \in [0,\frac{1}{2})$. Then, f has a unique fixed point.

Theorem 2.21 in S-multiplicative metric space is as follows:

Theorem 2.22 *Let $(X,\overline{S})$ be a sequentially compact S-multiplicative metric space and let $f : X \longrightarrow X$ satisfy the following condition*:

$$\overline{S}(fx,fy,fz) \leq \big[\overline{S}(x,fx,fx)\overline{S}(y,fy,fy)\overline{S}(z,fz,fz)\big]^a,$$

whenever $x,y,z \in X$ and $x \neq y \neq z$. Then, f has a unique fixed point.

Remark 2.23 The Kannan theorem in S-metric space and S-multiplicative metric space are equivalent.

Verification:

Let $\overline{S}(x,y,z) = e^{S(x,y,z)}$. Then,

$$e^{S(fx,fy,fz)} = \overline{S}(fx,fy,fz) \tag{57}$$

$$\leq \big[\overline{S}(x,fx,fx)\overline{S}(y,fy,fy)\overline{S}(z,fz,fz)\big]^a \tag{58}$$

$$= \big[e^{S(x,fx,fx)} \times e^{S(y,fy,fy)} \times e^{S(z,fz,fz)}\big]^a \tag{59}$$

$$= \big[e^{S(x,fx,fx)+S(y,fy,fy)+S(z,fz,fz)}\big]^a \tag{60}$$

$$= e^{a[S(x,fx,fx)+S(y,fy,fy)+S(z,fz,fz)]}, \tag{61}$$

which implies $S(fx,fy,fz) \leq a[S(x,fx,fx) + S(y,fy,fy) + S(z,fz,fz)]$.

The following theorem is a Chatterjea-type theorem in S-metric space.

Theorem 2.24 *Let (X,S) be a sequentially compact S-metric space and let $f : X \longrightarrow X$ satisfy the following condition*:

$$S(fx,fy,fz) \leq a\big[S(x,fy,fy) + S(y,fz,fz) + S(z,fx,fx) \tag{62}$$

$$+ S(x,fz,fz) + S(y,fx,fx) + S(z,fy,fy)\big], \tag{63}$$

whenever $x,y,z \in X$ and $a \in [0,\frac{1}{2})$. Then, f has a unique fixed point.

Theorem 2.24 in S-multiplicative metric space is as follows:

Theorem 2.25 *Let $(X,\overline{S})$ be a sequentially compact S-multiplicative metric space and let $f: X \longrightarrow X$ satisfy the following condition*:

$$\overline{S}(fx,fy,fz) \leq \left[\overline{S}(x,fy,fy)\overline{S}(y,fz,fz)\overline{S}(z,fx,fx)\overline{S}(x,fz,fz)\overline{S}(y,fx,fx)\overline{S}(z,fy,fy)\right]^a,$$

whenever $x, y, z \in X$ and $x \neq y \neq z$. Then, f has a unique fixed point.

Remark 2.26 The Chatterjea-type theorem in S-metric space and S-multiplicative metric space are equivalent.

Verification:
Let $\overline{S}(x,y,z) = e^{S(x,y,z)}$. Then,

$$\begin{aligned}
e^{S(fx,fy,fz)} &= \overline{S}(fx,fy,fz) && (64)\\
&\leq \big[\overline{S}(x,fy,fy)\overline{S}(y,fz,fz)\overline{S}(z,fx,fx)\overline{S}(x,fz,fz) && (65)\\
&\quad \times \overline{S}(y,fx,fx)\overline{S}(z,fy,fy)\big]^a && (66)\\
&= \big[e^{S(x,fy,fy)} \times e^{S(y,fz,fz)} \times e^{S(z,fx,fx)} \times e^{S(x,fz,fz)} && (67)\\
&\quad \times e^{S(y,fx,fx)} \times e^{S(z,fy,fy)}\big]^a && (68)\\
&= \left[e^{S(x,fy,fy)+S(y,fz,fz)+S(z,fx,fx)+S(x,fz,fz)+S(y,fx,fx)}\right]^a && (69)\\
&\quad \times e^{aS(z,fy,fy)} && (70)\\
&= e^{a[S(x,fy,fy)+S(y,fz,fz)+S(z,fx,fx)+S(x,fz,fz)+S(y,fx,fx)]} && (71)\\
&\quad \times e^{aS(z,fy,fy)}, && (72)
\end{aligned}$$

which implies $S(fx,fy,fz) \leq a[S(x,fy,fy) + S(y,fz,fz) + S(z,fx,fx) + S(x,fz,fz) + S(y,fx,fx) + S(z,fy,fy)]$.

The following theorem is a Boyd and Wong-type contraction in S-metric space.

Theorem 2.27 *Let (X,S) be a sequentially compact S-metric space and let $f: X \longrightarrow X$ satisfy the following condition*:

$$S(fx,fy,fz) \leq \phi\big(S(x,y,z)\big),$$

where $\phi: [0,\infty] \to [0,\infty]$ is upper semicontinuous from the right, satisfying $\phi(t) < t$ for $t > 0$. Then, f has a unique fixed point.

Theorem 2.27 in S-multiplicative metric space is as follows:

Theorem 2.28 *Let $(X,\overline{S})$ be a sequentially compact S-multiplicative metric space and let $f: X \longrightarrow X$ satisfy the following condition*:

$$\overline{S}(fx,fy,fz) \leq \Phi\big(\overline{S}(x,y,z)\big),$$

where $\Phi : [0,\infty] \to [0,\infty]$ *is upper semicontinuous from the right, satisfying* $\Phi(t) \geq t$ *for* $t > 0$. *Then,* f *has a unique fixed point.*

Remark 2.29 The Boyd and Wong-type contraction in S-metric space and S-multiplicative metric space are equivalent.

Verification:
Let $\overline{S}(x,y,z) = e^{S(x,y,z)}$. Then,

$$\begin{aligned} e^{S(fx,fy,fz)} &= \overline{S}(fx,fy,fz) && (73)\\ &\leq \phi\big(\overline{S}(x,y,z)\big) && (74)\\ &= \phi\big(e^{S(x,y,z)}\big) && (75)\\ &\leq e^{S(x,y,z)} && (76)\\ &\leq e^{\Phi(S(x,y,z))}, && (77) \end{aligned}$$

which implies $S(fx,fy,fz) \leq \Phi(S(x,y,z))$

The following theorem is a generalization of both Kannan- and Chatterjea-type theorems in S-metric spaces.

Theorem 2.30 *Let* (X,S) *be a sequentially compact* S*-metric space and let* $f : X \longrightarrow X$ *satisfy the following condition*:

$$\begin{aligned} S(fx,fy,fz) \leq a\max\big\{&S(x,y,z), S(z,fz,fz), S(y,fy,fy), && (78)\\ &S(x,fx,fx), S(x,fy,fy), S(y,fz,fz), S(z,fx,fx), && (79)\\ &S(x,fz,fz), S(y,fx,fx), S(z,fy,fy)\big\} && (80) \end{aligned}$$

whenever $x,y,z \in X$ *and* $a \in [0,\frac{1}{2})$. *Then,* f *has a unique fixed point.*

Theorem 2.30 in S-multiplicative metric space is as follows:

Theorem 2.31 *Let* $(X,\overline{S})$ *be a sequentially compact* S*-multiplicative metric space and let* $f : X \longrightarrow X$ *satisfy the following condition*:

$$\begin{aligned} \overline{S}(fx,fy,fz) \leq \max\big\{&\overline{S}(x,y,z), \overline{S}(z,fz,fz), \overline{S}(y,fy,fy), && (81)\\ &\overline{S}(x,fx,fx), \overline{S}(x,fy,fy), \overline{S}(y,fz,fz), \overline{S}(z,fx,fx), && (82)\\ &\overline{S}(x,fz,fz), \overline{S}(y,fx,fx), \overline{S}(z,fy,fy)\big\}^{a}, && (83) \end{aligned}$$

whenever $x,y,z \in X$. *and* $x \neq y \neq z$. *Then,* f *has a unique fixed point.*

Remark 2.32 The generalizations of both Kannan- and Chatterjea-type theorems in S-metric space and S-multiplicative metric space are equivalent.

Verification:
Let $\overline{S}(x,y,z) = e^{S(x,y,z)}$. Then,

$$e^{S(fx,fy,fz)} = \overline{S}(fx,fy,fz) \tag{84}$$
$$\leq \max\Big\{\overline{S}(x,y,z), \overline{S}(z,fz,fz), \overline{S}(y,fy,fy), \tag{85}$$
$$\overline{S}(x,fx,fx), \overline{S}(x,fy,fy), \overline{S}(y,fz,fz), \overline{S}(z,fx,fx), \tag{86}$$
$$\overline{S}(x,fz,fz), \overline{S}(y,fx,fx), \overline{S}(z,fy,fy)\Big\}^a \tag{87}$$
$$= \max\Big\{e^{S(x,y,z)}, e^{S(z,fz,fz)}, e^{S(y,fy,fy)}, e^{S(x,fx,fx)}, e^{S(x,fy,fy)}, \tag{88}$$
$$e^{S(y,fz,fz)}, e^{S(z,fx,fx)}, e^{S(x,fz,fz)}, e^{S(y,fx,fx)}, e^{S(z,fy,fy)}\Big\}^a \tag{89}$$
$$= \max\Big\{e^{aS(x,y,z)}, e^{aS(z,fz,fz)}, e^{aS(y,fy,fy)}, e^{aS(x,fx,fx)}, \tag{90}$$
$$e^{aS(x,fy,fy)}, e^{aS(y,fz,fz)}, e^{aS(z,fx,fx)}, e^{aS(x,fz,fz)}, \tag{91}$$
$$e^{aS(y,fx,fx)}, e^{aS(z,fy,fy)}\Big\}, \tag{92}$$

which implies $S(fx,fy,fz) \leq a\max\{S(x,y,z), S(z,fz,fz), S(y,fy,fy), S(x,fx,fx), S(x,fy,fy), S(y,fz,fz), S(z,fx,fx), S(x,fz,fz), S(y,fx,fx), S(z,fy,fy)\}$.

3 Conclusion

Some fixed-point theorems are stated and proved in S-multiplicative metric spaces. We also show that some fixed-point results for various S-multiplicative metric spaces are equivalent to those of corresponding fixed-point results in S-metric spaces. Some examples are presented to validate the originality and applicability of our main results.

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