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Fixed point theorems for generalized (α,ϕ)-Meir–Keeler type hybrid contractive mappings via simulation function in b-metric spaces

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Abstract

In this paper, we introduce the notion of generalized (α,ϕ)-Meir–Keeler hybrid contractive mappings of type I and II via simulation function and establish fixed point theorems for such mappings in the setting of complete b-metric spaces. Our results extend and generalize many related fixed point results in the existing literature. Finally, we provide an example in support of our main finding.

Keywords: Fixed points; b-metric spaces; Generalized (α,ϕ)-Meir–Keeler type hybrid contractive mappings

1 Introduction

Fixed point theory is one of the most important topics in development of nonlinear and mathematical analysis in general. Also, fixed point theory has been effectively used in many other branches of science such as chemistry, physics, biology, economics, computer science, all engineering fields, and so on. In 1922, Banach [1] introduced a well-known fixed point result, now called Banach contraction principle, which is one of the pivotal results in nonlinear analysis. Due to its importance and fruitful applications, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle in several direction (see, e.g., [2, 3]). These generalizations are achieved either by using contractive conditions or by imposing some additional conditions on the ambient spaces. For example, one of the important and peculiar generalizations is due to Meir and Keeler [4]. The class of Meir–Keeler contractions consists of the class of Banach contractions and many other classes of nonlinear contractions (see, for example, [5]). Meir and Keeler's theorem was the originator of further exploration in metric fixed point theory. Later on, Meir–Keeler contraction mapping has been generalized by several authors in several ways. For more works in this line of research, we refer to [6–8], as well as [9–14]. On the other hand, the notion of a b-metric space was introduced by Bakhtin [15] and Czerwik [16] as a generalization of metric spaces. Since then, several papers have been published on the fixed point theory in such spaces which are interesting extensions and generalizations of the Banach contraction principle. For further works in the setting of b-metric spaces and their generalization, we refer the readers to [17–42]. In 2020, Karapinar et al. [43] studied fixed point results for the Meir–Keeler contraction via simulation function in the setting of metric spaces. Inspired and motivated by the work of Karapinar et al. [43], the main objectives of this research is to introduce the notion of generalized (α,ϕ)-Meir–Keeler hybrid contractive mappings of type I and II via simulation function and establish fixed point theorems for the introduced mappings in the setting of b-metric spaces. The present results extend and generalize the results of Karapinar et al. [43] and many other related results in the existing literature.

2 Preliminaries

In what follows we recall basic definitions and results on the topics which we use in the sequel.

Notations 1 Throughout this paper, we denote $\mathbb{R}^+$, $\mathbb{R}$ and $\mathbb{N}$ respectively by

- $\mathbb{R}^+ = [0,\infty)$ – the set of all non-negative real numbers;
- $\mathbb{R}$ – the set of all real numbers;
- $\mathbb{N}$ – the set of all natural numbers.

Khojasteh et al. [44] introduced the notion of a simulation function as follows.

Definition 1 ([44]) A weak simulation function is a mapping $\zeta:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(0,0)=0$;

(ζ_2) $\zeta(t,s)<s-t$ for all $t,s>0$.

Note Throughout this paper we denote by Z_w the family of all simulation functions $\zeta:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$. Due to the axiom (ζ_2), we have $\zeta(t,t)<0$ for all $t>0$.

Recently, Suzuki [45] introduced the following class of mappings and proved the following interesting fixed point result to extend the coverage of Meir–Keeler theorem in the setting of metric spaces. Let (X,d) be a metric space and $T:X\to X$ be a self-mapping. Define a mapping $M:X\times X\to\mathbb{R}^+$ as follows:

$$M(x,y)=\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

And let $p:X\times X\to\mathbb{R}^+$ be a mapping satisfies the following conditions:

$(P_p^1:M)$ $x\neq y$ and $d(x,Tx)\leq d(x,y)$ imply $p(x,y)\leq M(x,y)$;

$(P_p^2:c)$ $x_n\neq y$, $\lim_{n\to\infty}d(x_n,y)=0$, and $\lim_{n\to\infty}d(x_n,Tx_n)=0$ imply

$$\limsup_{n\to\infty}d(x_n,y)\leq cd(y,Ty),\quad \text{where } c\in[0,1).$$

Theorem 1 ([45]) *Let T be a self-mapping on a complete metric space (X,d). Let $p:X\times X\to\mathbb{R}^+$ be mapping that satisfies the conditions $(P_p^1:M)$ and $(P_p^2:c)$ defined above. Suppose also that the following are satisfied*:

(i) *For any $\epsilon>0$, there exists $\delta(\epsilon)>0$ such that $x\neq y$ and $p(x,y)<\epsilon+\delta(\epsilon)$ imply $d(Tx,Ty)\leq\epsilon$*;

(ii) *$x \neq y$ and $p(x,y) > 0$ imply $d(Tx,Ty) < p(x,y)$.*
Then T has a unique fixed point z. Moreover, the sequence $\{T^n x\}$ converges to z for all $x \in X$.

Bakhtin [15] and Czerwik [16] defined a b-metric space as follows.

Definition 2 ([15, 16]) Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if and only if for all $x,y,z \in X$, the following conditions are satisfied:
(a) $d(x,y) = 0$ if and only if $x = y$;
(b) $d(x,y) = d(y,x)$;
(c) $d(x,z) \leq s[d(x,y) + d(y,z)]$.
The pair (X,d) is called a b-metric space.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces. A metric space is a b-metric with $s = 1$. But, in general, the converse is not true.

Example 1 ([32]) Let $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}^+$ be given by $d(x,y) = |x-y|^2$ for $x,y \in X$, then d is a b-metric on X with $s = 2$ but it is not a metric on X since for $x = 2$, $y = 4$, and $z = 6$, we have

$$d(2,6) > d(2,4) + d(4,6).$$

Hence, the triangle inequality for a metric does not hold.

Definition 3 ([46]) Let X be a b-metric space and $\{x_n\}$ a sequence in X. We say that
1. $\{x_n\}$ is b-convergent to $x \in X$ if $d(x_n,x) \to 0$ as $n \to \infty$.
2. $\{x_n\}$ is a b-Cauchy sequence if $d(x_n,x_m) \to 0$ as $n,m \to \infty$.
3. (X,d) is b-complete if every b-Cauchy sequence in X is b-convergent.

Definition 4 ([47]) Let (X,d) be a b-metric space with the coefficient $s \geq 1$ and let $T: X \to X$ be a given mapping. We say that T is b-continuous at $x_0 \in X$ if and only if for every sequence $x_n \in X$ such that $x_n \to x_0$ as $n \to \infty$, we have $Tx_n \to Tx_0$ as $n \to \infty$. If T is b-continuous at each point $x \in X$, then we say that T is b-continuous on X.

In general, a b-metric is not necessarily continuous.

Example 2 ([48]) Let $X = \mathbb{N} \cup \{\infty\}$.
Define a mapping $d : X \times X \to \mathbb{R}^+$ as follows:

$$d(m,n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m \text{ and } n \text{ is even and the other even or } \infty, \\ 5 & \text{if one of } m \text{ and } n \text{ is odd and the other odd or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Observe that $d(m,p) \leq \frac{3}{2}[d(m,n) + d(n,p)]$ for all $m,n,p \in X$.
Then (X,d) is a b-metric space with $s = \frac{3}{2}$.

If we choose $x_n = 2n$ for each $n \in \mathbb{N}$, then

$$d(x_n, \infty) = d(2n, \infty) = \frac{1}{2n} \to 0 \quad \text{as } n \to \infty,$$

that is, $x_n \to \infty$ as $n \to \infty$.

But $\lim_{n\to\infty} d(x_n, 1) = 2 \neq 5 = d(\infty, 1)$. Hence, d is not continuous.

The following are definitions of α-orbital admissible and triangular α-orbital admissible mappings.

Definition 5 ([49]) Let X be a nonempty set and $\alpha : X \times X \to \mathbb{R}^+$ a function. A mapping $T : X \to X$ is said to be α-orbital admissible if, for all $x \in X$, $\alpha(x, Tx) \geq 1$ implies $\alpha(Tx, T^2x) \geq 1$.

Definition 6 ([49]) Let X be a nonempty set, $T : X \to X$, and $\alpha : X \times X \to \mathbb{R}^+$. We say that T is triangular α-orbital admissible if:

(i) T is α-orbital admissible;

(ii) for all $x, y \in X$, $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$ imply that $\alpha(x, Ty) \geq 1$.

In 2020, Karapinar et al. [43] introduced the class of hybrid contraction mappings of type I and II and studied fixed point results for such mappings.

Definition 7 ([43]) Let T be a self-mapping on a metric space (X, d) and $\zeta \in Z_w$. Suppose that $p : X \times X \to \mathbb{R}^+$ is a function that satisfies only $(P^1p : M)$. Then T is called a hybrid contraction of type I if the following conditions are fulfilled:

(a) For any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $x \neq y$ and $p(x, y) < \epsilon + \delta(\epsilon)$ imply $d(Tx, Ty) \leq \epsilon$;

(b) $x \neq y$ and $p(x, y) > 0$ imply $\zeta(\alpha(x, y)d(Tx, Ty), p(x, y)) \geq 0$.

Let a mapping $N : X \times X \to \mathbb{R}^+$ be defined as follows:

$$N(x, y) = \max\left\{d(y, Ty)\frac{1 + d(x, Tx)}{1 + d(x, y)}, d(x, y)\right\},$$

where T is a self-mapping defined on a metric space (X, d). We notice that, for any $x, y \in X$ with $x = y$, we have $0 = d(Tx, Ty) \leq N(x, y)$. Moreover, if $x \neq y$, then $N(x, y) > 0$.

Definition 8 ([43]) Let T be a self-mapping on a metric space (X, d) and $\zeta \in Z_w$. Suppose that $p : X \times X \to \mathbb{R}^+$ is a function that satisfies $(P^1p : N)$ and $(P^2p : c)$, for all $c \in [0, 1)$. Then T is called a hybrid contraction of type II if the following conditions are satisfied:

(a) For any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $x \neq y$ and $p(x, y) < \epsilon + \delta(\epsilon)$ imply $d(Tx, Ty) \leq \epsilon$;

(b) $x \neq y$ and $p(x, y) > 0$ imply $\zeta(\alpha(x, y)d(Tx, Ty), p(x, y)) \geq 0$.

Theorem 2 ([43]) *Let (X, d) be a complete metric space and $T : X \to X$ be a hybrid contraction of type I. Assume that the following conditions are satisfied*:

(i) *T is triangular α-orbital admissible*;

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;

(iii) *T is continuous.*

Then T has a fixed point u. Moreover, $\{T^n x\}$ converges to u for all $x \in X$.

Theorem 3 ([43]) *Let (X, d) be a complete metric space and $T : X \times X$ be a hybrid contraction of type II. Assume that the following conditions are fulfilled*:

(i) *T is triangular α-orbital admissible*;

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;

(iii) *either T is continuous*;

(iv) *or T^2 is continuous and $\alpha(u, Tu) \geq 1$*;

(v) *or (X, d) is regular.*

Then T has a fixed point u. Moreover, $\{T^n x\}$ converges to u for all $x \in X$.

3 Results

In this section, first we introduce generalized (α, ϕ)-Meir–Keeler hybrid contractive mapping of type I in the setting of b-metric spaces and prove fixed point results for such mappings.

Note In this section, we denote the class of mappings Ψ by

$$\Psi = \left\{\phi : \mathbb{R}^+ \to \mathbb{R}^+ : \phi \text{ is continuous, monotone nondecreasing}, \phi(t) = 0 \text{ iff } t = 0\right\}.$$

Let (X, d) be a b-metric space with $s \geq 1$ and $T : X \to X$ be a self-mapping. We define a mapping $M_s : X \times X \to \mathbb{R}^+$ by

$$M_s(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$

Let also $p : X \times X \to \mathbb{R}^+$ be a mapping. The following conditions are used in this section:

($P_p^1 : M_s$) $x \neq y$ and $d(x, Tx) \leq d(x, y)$ imply $p(x, y) \leq M_s(x, y)$;

($P_p^2 : sc$) $x_n \neq y$, $\lim_{n\to\infty} d(x_n, y) = 0$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ imply $\limsup_{n\to\infty}(sd(x_n, y)) \leq cd(y, Ty)$, where $c \in [0, 1)$.

Definition 9 Let (X, d) be a b-metric space with $s \geq 1$, $T : X \to X$, $\alpha : X \times X \to \mathbb{R}^+$, $p : X \times X \to \mathbb{R}^+$ satisfy ($P_p^1 : M_s$), and $\phi \in \Psi$. Then the mapping T is said to be a generalized (α, ϕ)-Meir–Keeler hybrid contractive mapping of type I if it satisfies, for all $x, y \in X$, the following conditions:

(i) For any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $x \neq y$ and $p(x, y) < \epsilon + \delta(\epsilon)$ imply $d(Tx, Ty) \leq \frac{\epsilon}{s}$;

(ii) $x \neq y$ and $p(x, y) > 0$ imply $\zeta(\alpha(x, y)\phi(d(Tx, Ty)), \phi(p(x, y))) \geq 0$.

Remark 1 If T is a generalized (α, ϕ)-Meir–Keeler hybrid contractive mapping of type I, then

$$\alpha(x, y)\phi\big(d(Tx, Ty)\big) < \phi\big(p(x, y)\big) \leq \phi\big(M_s(x, y)\big). \tag{1}$$

Indeed, we have $d(x, y) > 0$ since $x \neq y$. If $p(x, y) = 0$, from (ii), we have $\phi(d(Tx, Ty)) < \epsilon$ for any $\epsilon > 0$. But $\epsilon > 0$ is arbitrary, thus we obtain $Tx = Ty$. In this case, $\alpha(x, y)\phi(d(Tx, Ty)) =$

$0 \leq \phi(p(x,y))$. Otherwise, $p(x,y) > 0$, and if $Tx \neq Ty$, then $d(Tx,Ty) > 0$. If $\alpha(x,y) = 0$, then (1) is satisfied. On the other hand, from (ζ_2) and Definition 9(ii), we get

$$0 \leq \zeta\big(\alpha(x,y)\phi\big(d(Tx,Ty)\big), \phi\big(p(x,y)\big)\big) < \phi\big(p(x,y)\big) - \alpha(x,y)\phi\big(d(Tx,Ty)\big),$$

so (1) holds.

Now, we give our first main result as follows:

Theorem 4 *Let (X,d) be a complete b-metric space with $s \geq 1$, $T : X \to X$, $\alpha : X \times X \to \mathbb{R}^+$ be mappings, and $\phi \in \Psi$. Suppose the following conditions hold:*
(i) *T is generalized (α,ϕ)-Meir–Keeler hybrid contractive mapping of type I;*
(ii) *T is a triangular α-orbital admissible mapping;*
(iii) *There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;*
(iv) *T is b-continuous.*
Then T has a fixed point z. Moreover, $\{T^n x\}$ converges to z for all $x \in X$.

Proof By (iii) above, there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$. We construct an iterative sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for $n \in \mathbb{N}$. Suppose first that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$. Since $Tx_{n_0} = x_{n_0+1} = x_{n_0}$, the point x_{n_0} is a fixed point of T and this completes the proof. So from now on, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is triangular α-orbital admissible, $\alpha(x_0,Tx_0) = \alpha(x_0,x_1) \geq 1 \Rightarrow \alpha(Tx_0,Tx_1) = \alpha(x_1,x_2) \geq 1 \Rightarrow \alpha(Tx_1,Tx_2) = \alpha(x_2,x_3) \geq 1$. Continuing in this manner, we get

$$\alpha(x_n,x_{n+1}) \geq 1 \quad \text{for all } n \geq 0. \tag{2}$$

Again, by using the assumption that T is triangular α-orbital admissible, for all $n \in \mathbb{N} \cup \{0\}$, (2) yields that $\alpha(x_n,x_{n+1}) \geq 1$ and $\alpha(x_{n+1},x_{n+2}) \geq 1 \Rightarrow \alpha(x_n,x_{n+1}) \geq 1$. Recursively, we conclude that $\alpha(x_n,x_{n+j}) \geq 1$ for all $n,j \in \mathbb{N}$. In what follows we prove that the sequence $\{d(x_n,x_{n+1})\}$ is monotone decreasing. Taking $x = x_n$ and $y = x_{n+1}$ in $(P_p^1 : M_s)$, we get

$$0 < d(x_n,x_{n+1}) = d(x_n,Tx_n) \leq d(x_n,x_{n+1}),$$

which implies

$$p(x_n,x_{n+1}) \leq M_s(x_n,x_{n+1}),$$

where

$$\begin{aligned} M_s(x_n,x_{n+1}) &= \max\Big\{d(x_n,x_{n+1}), d(x_n,Tx_n), d(x_{n+1},Tx_{n+1}), \\ &\qquad \frac{d(x_n,Tx_{n+1}) + d(x_{n+1},Tx_n)}{2s}\Big\} \\ &= \max\Big\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}), \frac{d(x_n,x_{n+2}) + d(x_{n+1},x_{n+1})}{2s}\Big\} \\ &= \max\Big\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}), \frac{d(x_n,x_{n+2})}{2s}\Big\}, \end{aligned}$$

and, taking the b-triangle inequality into account, we observe that

$$\begin{aligned}\frac{d(x_n,x_{n+2})}{2s} &\le \frac{sd(x_n,x_{n+1})+sd(x_{n+1},x_{n+2})}{2s}\\ &= \frac{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{2}\\ &\le \max\bigl\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\bigr\},\end{aligned}$$

which gives

$$M_s(x_n,x_{n+1}) = \max\bigl\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\bigr\}.$$

By Definition 9(ii), we get that

$$\begin{aligned}&0 \le \zeta\bigl(\alpha(x_n,x_{n+1})\phi\bigl(d(Tx_n,Tx_{n+1})\bigr),\phi\bigl(P(x_n,x_{n+1})\bigr)\bigr)\\ &\quad<\phi\bigl(P(x_n,x_{n+1})\bigr)-\alpha(x_n,x_{n+1})\phi\bigl(d(Tx_n,Tx_{n+1})\bigr),\end{aligned}$$

which is equivalent to

$$\begin{aligned}\phi\bigl(d(x_{n+1},x_{n+2})\bigr) &= \phi\bigl(d(Tx_n,Tx_{n+1})\bigr)\\ &\le \alpha(x_n,x_{n+1})\phi\bigl(d(Tx_n,Tx_{n+1})\bigr)\\ &<\phi\bigl(p(x_n,x_{n+1})\bigr)\\ &\le \phi\bigl(M_s(x_n,x_{n+1})\bigr).\end{aligned}\tag{3}$$

If $M_s(x_n,x_{n+1}) = d(x_{n+1},x_{n+2})$, then (3) yields a contradiction. Thus, we have

$$M_s(x_n,x_{n+1}) = d(x_n,x_{n+1}).\tag{4}$$

Moreover, from (3), we get

$$\phi\bigl(d(x_{n+1},x_{n+2})\bigr)<\phi\bigl(d(x_n,x_{n+1})\bigr),$$

which implies, using the monotonicity of ϕ,

$$d(x_{n+1},x_{n+2})<d(x_n,x_{n+1}) \quad \text{for all } n\in\mathbb{N}\cup\{0\},$$

that is, $\{d(x_n,x_{n+1})\}$ is a monotone decreasing sequence of nonnegative real numbers. Thus, there is some $l\ge 0$ such that $\lim_{n\to\infty} d(x_n,x_{n+1}) = l$. We need to show $l = 0$. Suppose, on the contrary, that $l>0$ and set $0<\epsilon = l$. We also note that

$$\epsilon = l < d(x_n,x_{n+1}) \quad \text{for all } n\in\mathbb{N}\cup\{0\}.\tag{5}$$

On the other hand, from (3) and (4), we have

$$p(x_n,x_{n+1}) \le d(x_n,x_{n+1}) < \epsilon+\delta(\epsilon)$$

for n sufficiently large. So, applying Definition 9(i), we have

$$d(Tx_n, Tx_{n+1}) \leq \frac{\epsilon}{s}. \tag{6}$$

Combining (5) together with (6), we obtain

$$\epsilon < d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \leq \frac{\epsilon}{s},$$

which is a contradiction. We conclude that $\epsilon = 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{7}$$

Now, we show that $\{x_n\}$ is a b-Cauchy sequence. Let $\epsilon_1 > 0$ be fixed. From (7), we can choose $k \in \mathbb{N}$ large enough such that

$$d(x_k, x_{k+1}) < \frac{\delta_1}{2s}, \tag{8}$$

for some $\delta_1 > 0$. Without loss of generality, we assume that $\delta_1 = \delta_1(\epsilon_1) < \epsilon_1$. By induction, we prove that

$$d(x_k, x_{k+m}) < \epsilon_1 + \frac{\delta_1}{2} \quad \text{for all } k, m \in \mathbb{N} \cup \{0\}. \tag{9}$$

We already have (9) from (8), for $m = 1$. Suppose that (9) is satisfied for some $m = j$. Now, we show that (9) holds for $m = j + 1$. On account of (8) and (9), we first observe that

$$\begin{aligned}
&\frac{d(x_k, x_{k+j+1}) + d(x_{k+j}, x_{k+1})}{2s} \\
&\quad\leq \frac{sd(x_k, x_{k+j}) + sd(x_{k+j}, x_{k+j+1}) + sd(x_{k+j}, x_k) + sd(x_k, x_{k+1})}{2s} \\
&\quad= \frac{d(x_k, x_{k+j}) + d(x_{k+j}, x_{k+j+1}) + d(x_{k+j}, x_k) + d(x_k, x_{k+1})}{2} \\
&\quad< \frac{1}{2}\left[2\epsilon_1 + \delta_1 + \frac{\delta_1}{s}\right] \\
&\quad\leq \frac{1}{2}[2\epsilon_1 + 2\delta_1] = \epsilon_1 + \delta_1.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&M_s(x_k, x_{k+j}) \\
&\quad= \max\left\{d(x_k, x_{k+j}), d(x_k, Tx_k), d(x_{k+j}, Tx_{k+j}), \frac{d(x_k, Tx_{k+j}) + d(x_{k+j}, Tx_k)}{2s}\right\} \\
&\quad= \max\left\{d(x_k, x_{k+j}), d(x_k, x_{k+1}), d(x_{k+j}, x_{k+j+1}), \frac{d(x_k, x_{k+j+1}) + d(x_{k+j}, x_{k+1})}{2s}\right\} \\
&\quad< \max\left\{\epsilon_1 + \frac{\delta_1}{2}, \frac{\delta_1}{2s}, \epsilon_1 + \delta_1\right\} \\
&\quad= \epsilon_1 + \delta_1.
\end{aligned}$$

From the above inequality, we have

$$p(x_k, x_{k+j}) \le M_s(x_k, x_{k+j}) = d(x_k, x_{k+j}) < \epsilon_1 + \delta_1,$$

and, by Definition 9(i), we get

$$d(x_{k+1}, x_{k+j+1}) = d(Tx_k, Tx_{k+j}) \le \frac{\epsilon_1}{s}. \tag{10}$$

Now, using the b-triangle inequality, as well as (8) and (10), we have

$$\begin{aligned} d(x_k, x_{k+j+1}) &\le sd(x_k, x_{k+1}) + sd(x_{k+1}, x_{k+j+1}) \\ &= sd(x_k, x_{k+1}) + sd(Tx_k, Tx_{k+j}) \\ &< \frac{\delta_1}{2} + \epsilon_1. \end{aligned}$$

So, (9) holds for $m = j + 1$. Therefore,

$$d(x_k, x_{k+m}) < \epsilon_1 \quad \text{for all } k, m \in \mathbb{N} \cup \{0\}.$$

In other words, for $m > n$, we have $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ and hence the sequence $\{x_n\}$ is a b-Cauchy sequence. Since, (X, d) is a complete b-metric space, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. By b-continuity of T, we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tu,$$

that is, u is a fixed point of T. □

Now, replacing continuity of T by continuity of T^2 in Theorem 4, we prove the following fixed point result.

Theorem 5 *Let (X, d) be a complete b-metric space with $s \ge 1$ and let $T : X \to X$ be a generalized (α, ϕ)-Meir–Keeler hybrid contractive mapping of type I satisfying the following conditions*:

(i) *T is a triangular α-orbital admissible mapping*;

(ii) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$*;

(iii) *T^2 is continuous.*

Then $\{T^n x\}$ is converges to z for all $x \in X$. Moreover, for $\alpha(z, Tz) \ge 1$, z is a fixed point of T, and T is discontinuous at z if and only if $\lim_{x\to z} M_s(x, z) \ne 0$.

Proof Following the proof of Theorem 4, we see that the sequence $\{x_n\}$ in X defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ is convergent to $z \in X$ and $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$. Regarding the fact that any subsequence of $\{x_n\}$ converges to z, we get $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = z$ and $\lim_{n\to\infty} x_{n+2} = \lim_{n\to\infty} T^2 x_n = z$. On the other hand, due to the continuity of T^2,

$$T^2 z = \lim_{n\to\infty} T^2 x_n = z.$$

We claim that $Tz = z$. Suppose, on the contrary, that $Tz \neq z$ and $p(z, Tz) > 0$. Then we have

$$\begin{aligned} p(z,Tz) &\leq M_s(z,Tz) \\ &= \max\left\{d(z,Tz), d(z,Tz), d\bigl(Tz,T^2z\bigr), \frac{d(z,T^2z)+d(Tz,Tz)}{2s}\right\} \\ &= d(z,Tz). \end{aligned}$$

Thus, using (1) together with the hypothesis $\alpha(z, Tz) \geq 1$, we obtain

$$0 \leq \zeta\bigl(\alpha(z,Tz)\phi\bigl(d\bigl(Tz,T^2z\bigr)\bigr), \phi\bigl(P(z,Tz)\bigr)\bigr),$$

and also

$$\begin{aligned} 0 &< \phi\bigl(d(Tz,z)\bigr) = \phi\bigl(d\bigl(Tz,T^2z\bigr)\bigr) \\ &\leq \alpha(z,Tz)\phi\bigl(d\bigl(Tz,T^2z\bigr)\bigr) \\ &< \phi\bigl(P(z,Tz)\bigr) \\ &\leq \phi\bigl(M_s(z,Tz)\bigr) \\ &= \phi\bigl(d(z,Tz)\bigr), \end{aligned}$$

which is a contradiction. So, $z = Tz$, that is, z is a fixed point of T. □

Definition 10 A b-metric space (X, d) is called regular if for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} d(x_n, z) = 0$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, one has $\alpha(x_n, z) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

In the following, we prove the following fixed point theorem, without continuity assumption of T and T^2.

Theorem 6 *Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a generalized (α, ϕ)-Meir–Keeler hybrid contractive mapping of type I. Suppose that $(P_p^2 : sc)$ and the following conditions hold*:

(i) *T is a triangular α-orbital admissible mapping*;
(ii) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *(X, d) is regular.*

Then $\{T^n x\}$ is converges to z for all $x \in X$. Moreover, z is a fixed point of T.

Proof Following the proof of Theorem 4, we see that the sequence $\{x_n\}$ in X defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ is convergent to $z \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. We notice also that all adjacent terms in $\{x_n\}$ are distinct. Moreover, we note $T^n x \neq z$ for all $n \in \mathbb{N} \cup \{0\}$. Regarding the limits $\lim_{n\to\infty} d(x_n, z) = 0$ and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, we drive from $(P_p^2 : sc)$ that

$$s \limsup_{n\to\infty} p(x_n, z) \leq c d(z, Tz) \quad \text{for any } c \in [0, 1). \tag{11}$$

So, by assumption (iii), we get $\alpha(x_n,z) \geq 1$. Now, we prove that z is a fixed point of T. Suppose, on the contrary, that $Tz \neq z$. Taking $x = x_n$ and $y = z$ in Definition 9(ii), we obtain that

$$\begin{aligned} 0 &\leq \zeta\big(\alpha(x_n,z)\phi\big(d(Tx_n,Tz)\big),\phi\big(p(x_n,z)\big)\big) \\ &< \phi\big(p(x_n,z)\big) - \alpha(x_n,z)\phi\big(d(Tx_n,Tz)\big), \end{aligned}$$

which is equivalent to

$$\begin{aligned} \phi\big(d(x_{n+1},Tz)\big) &= \phi\big(d(Tx_n,Tz)\big) \\ &\leq \alpha(x_n,z)\phi\big(d(Tx_n,Tz)\big) \\ &< \phi\big(p(x_n,z)\big). \end{aligned} \tag{12}$$

Since ϕ is monotone, (12) yields

$$d(x_{n+1},Tz) < p(x_n,z). \tag{13}$$

Applying the b-triangle inequality and using (13), we have

$$\begin{aligned} d(z,Tz) &\leq sd(z,x_{n+1}) + sd(x_{n+1},Tz) \\ &< sd(z,x_{n+1}) + sp(x_n,z). \end{aligned} \tag{14}$$

Taking the limit as $n \to \infty$ in (14) and using $(P_p^2 : sc)$, we obtain that

$$d(z,Tz) < s\limsup_{n\to\infty} p(x_n,z) \leq cd(z,Tz) \quad \text{for any } c \in [0,1),$$

which is a contradiction. Therefore, z is a fixed point of T. □

For the uniqueness of fixed point, we need the following additional condition.

Condition (U) For all $x,y \in \mathrm{Fix}(T)$, we have $\alpha(x,y) \geq 1$, where $\mathrm{Fix}(T)$ denotes the set of all fixed points of T.

Theorem 7 *Adding Condition (U) to the hypotheses of Theorem 4 (resp. Theorems 5 and 6), we prove the uniqueness of fixed point of T.*

Proof We argue by contradiction, that is, suppose there exist $z,w \in X$ such that $z = Tz$ and $w = Tw$ with $z \neq w$. By Condition (U), we have $\alpha(z,w) \geq 1$. We notice first that the case $p(z,w) = 0$ is impossible since we have $Tz = Tw$ and

$$0 < d(z,w) = d(Tz,Tw) = 0,$$

which is a contradiction. Thus, we get that $p(z,w) > 0$. Since

$$0 = d(z,Tz) \leq d(z,w),$$

by $(P_p^1 : M_s)$, we have

$$p(z,w) \leq M_s(z,w),$$

where

$$M_s(z,w) = \max\left\{d(z,w), d(z,Tz), d(w,Tw), \frac{d(z,Tw)+d(Tz,w)}{2s}\right\} = d(z,w).$$

Using Definition 9(ii), we get

$$\begin{aligned} 0 &\leq \zeta\big(\alpha(z,w)\phi\big(d(Tz,Tw)\big), \phi\big(p(z,w)\big)\big) \\ &< \phi\big(p(z,w)\big) - \alpha(z,w)\phi\big(d(Tz,Tw)\big), \end{aligned}$$

which imply

$$\begin{aligned} 0 &< \phi\big(d(z,w)\big) = \phi\big(d(Tz,Tw)\big) \\ &\leq \alpha(z,w)\phi\big(d(Tz,Tw)\big) \\ &< \phi\big(p(z,w)\big) \\ &\leq \phi\big(d(z,w)\big), \end{aligned}$$

which is a contradiction. Hence, $d(z,w) = 0$, that is, the fixed point of T is unique. □

In the following, we introduce generalized (α,ϕ)-Meir–Keeler hybrid contractive mapping of type II and study fixed point results for such mappings.

Definition 11 Let (X,d) be a b-metric space with $s \geq 1$, $T : X \to X$, $\alpha : X \times X \to \mathbb{R}^+$, $\zeta \in Z_w$, $\phi \in \Psi$, and suppose $p : X \times X \to \mathbb{R}^+$ is a function that satisfies $(P_p^1 : N_s)$ and $(P_p^2 : sc)$. The mapping T is said to be a generalized (α,ϕ)-Meir–Keeler hybrid contractive mapping of type II if it satisfies for all $x, y \in X$ the following conditions:

(a) For any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $x \neq y$ and $p(x,y) < \epsilon + \delta(\epsilon)$ imply $d(Tx,Ty) \leq \frac{\epsilon}{s}$;

(b) $x \neq y$ and $p(x,y) > 0$ imply

$$\zeta\big(\alpha(x,y)\phi\big(d(Tx,Ty)\big), \phi\big(p(x,y)\big)\big) \geq 0. \tag{15}$$

We define a mapping $N_s : X \times X \to \mathbb{R}^+$ by

$$\begin{aligned} N_s(x,y) = \max\bigg\{ & d(x,y), d(x,Tx), d(y,Ty), \\ & \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(x,Tx)[1+d(y,Ty)]}{1+d(Tx,Ty)}\bigg\}. \end{aligned}$$

We note that, for any $x, y \in X$ with $x = y$, we have $0 = d(Tx,Ty) \leq N_s(x,y)$. Moreover, if $x \neq y$, then $N_s(x,y) > 0$.

Now, we state and prove the following fixed point theorem.

Theorem 8 *Let (X,d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a generalized (α,ϕ)-Meir–Keeler hybrid contractive mapping of type II satisfying the following conditions*:

(i) *T is a triangular α-orbital admissible mapping*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *either T is continuous*;
(iv) *or T^2 is continuous and $\alpha(z, Tz) \geq 1$*;
(v) *or (X,d) is regular.*

Then T has a fixed point z. Moreover, $\{T^n x\}$ is convergent to z for all $x \in X$.

Proof As in the proof of Theorem 4, we construct a recursive sequence $\{x_n\}$ as follows:

$$x_n = Tx_{n-1} \quad \text{for all } n \in \mathbb{N}.$$

One can conclude that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, due to conditions (i) and (ii). Throughout the proof, we assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Indeed, as it was discussed in the proof of Theorem 4, the other case is trivial and is excluded. Now, by letting $x = x_n$ and $y = x_{n+1}$ in $(P_p^1 : N_s)$, we have

$$d(x_n, Tx_n) \leq d(x_n, x_{n+1}),$$

which implies

$$p(x_n, x_{n+1}) \leq N_s(x_n, x_{n+1}),$$

where

$$\begin{aligned} N_s(x_n, x_{n+1}) = \max\Bigg\{ & d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\ & \frac{d(x_{n+1}, Tx_{n+1})[1 + d(x_n, Tx_n)]}{1 + d(x_n, x_{n+1})}, \frac{d(x_n, Tx_n)[1 + d(x_{n+1}, Tx_{n+1})]}{1 + d(Tx_n, Tx_{n+1})} \Bigg\} \\ = \max\Bigg\{ & d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\ & \frac{d(x_{n+1}, x_{n+2})[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_{n+1})}, \frac{d(x_n, x_{n+1})[1 + d(x_{n+1}, x_{n+2})]}{1 + d(x_{n+1}, x_{n+2})} \Bigg\} \\ = \max\big\{ & d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}) \big\}. \end{aligned}$$

By Definition 11(b), we have

$$0 \leq \zeta\big(\alpha(x_n, x_{n+1})\phi\big(d(Tx_n, Tx_{n+1})\big), \phi\big(p(x_n, x_{n+1})\big)\big).$$

Consequently, the above inequality yields

$$\begin{aligned}\phi(d(x_{n+1},x_{n+2}) &= \phi\big(d(Tx_n, Tx_{n+1})\big)\\ &\leq \alpha(x_n,x_{n+1})\phi\big(d(Tx_n, Tx_{n+1})\big)\\ &< \phi\big(p(x_n,x_{n+1})\big)\\ &\leq \phi\big(N_s(x_n,x_{n+1})\big),\end{aligned}\tag{16}$$

where

$$N_s(x_n,x_{n+1}) = \max\big\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\big\} = d(x_n,x_{n+1}).\tag{17}$$

Thus, from (16), (17) and the monotonicity of ϕ, for all $n \in \mathbb{N}\cup\{0\}$, we have

$$d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1}),$$

that is, $\{d(x_n,x_{n+1})\}$ is nonincreasing sequence of nonnegative real numbers. Consequently, there exists a real number $r \geq 0$ such that $d(x_n,x_{n+1}) \to r$ as $n \to \infty$. Suppose that $r = \epsilon > 0$. First, we note that $r = \epsilon < d(x_n,x_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$. On the other hand, from (16), there exists $\delta > 0$ such that

$$\begin{aligned}p(x_n,x_{n+1}) &\leq N_s(x_n,x_{n+1})\\ &= d(x_n,x_{n+1})\\ &< \epsilon + \delta(\epsilon),\end{aligned}$$

for n sufficiently large. Keeping the observations above, Definition 11(a) yields that

$$d(Tx_n, Tx_{n+1}) \leq \frac{\epsilon}{s}.$$

Thus, we have

$$\epsilon < d(x_{n+1},x_{n+2}) = d(Tx_n, Tx_{n+1}) \leq \frac{\epsilon}{s},$$

which is a contradiction. So, we derive that $\epsilon = 0$, that is, $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$. In what follows, we show that the sequence $\{x_n\}$ is b-Cauchy. For this aim, let $m \in \mathbb{N}$ be large enough to satisfy

$$d(x_m,x_{m+1}) < \frac{\delta_1}{s}.$$

Now, we show by induction that

$$d(x_m,x_{m+k}) < \epsilon_1 + \delta_1 \quad \text{for all } k \in \mathbb{N}.\tag{18}$$

Without loss of generality, we assume that $\delta_1 = \delta_1(\epsilon) < \epsilon$. We have already proved the claim for $k = 1$. Now, we consider the following two cases:

Case (i). If $d(x_{m+k},x_{m+k+1}) \leq d(x_m,x_{m+k})$, then we get

$$\frac{d(x_{m+k},x_{m+k+1})}{1+d(x_m,x_{m+k})} \leq d(x_{m+k},x_{m+k+1})$$

and

$$\frac{d(x_{m+k},x_{m+k+1})d(x_m,x_{m+1})}{1+d(x_m,x_{m+k})} < d(x_m,x_{m+1}).$$

Hence, we have

$$\begin{aligned} p(x_m,x_{m+k}) &\leq N_s(x_m,x_{m+k}) \\ &= \max\Bigg\{d(x_m,x_{m+k}), d(x_m,Tx_m), d(x_{m+k},Tx_{m+k}), \\ &\quad \frac{d(x_{m+k},Tx_{m+k})[1+d(x_m,Tx_m)]}{1+d(x_m,x_{m+k})}, \frac{d(x_m,Tx_m)[1+d(x_{m+k},Tx_{m+k})]}{1+d(Tx_m,Tx_{m+k})}\Bigg\} \\ &= \max\Bigg\{d(x_m,x_{m+k}), d(x_m,x_{m+1}), d(x_{m+k},x_{m+k+1}), \\ &\quad \frac{d(x_{m+k},x_{m+k+1})[1+d(x_m,x_{m+1})]}{1+d(x_m,x_{m+k})}, \frac{d(x_m,x_{m+1})[1+d(x_{m+k},x_{m+k+1})]}{1+d(x_{m+1},x_{m+k+1})}\Bigg\} \\ &< \max\{\epsilon_1+\delta_1, 2\delta_1, \delta_1\} = \epsilon_1+\delta_1, \end{aligned}$$

and so it follows from Definition 11(a) that

$$d(Tx_m,Tx_{m+k}) \leq \frac{\epsilon_1}{s}.$$

Thus, by the b-triangle inequality, we have

$$\begin{aligned} d(x_m,x_{m+k+1}) &\leq sd(x_m,x_{m+1}) + sd(x_{m+1},x_{m+k+1}) \\ &= sd(x_m,x_{m+1}) + sd(Tx_m,Tx_{m+k}) \\ &< \epsilon_1+\delta_1. \end{aligned}$$

Case (ii). If $d(x_{m+k},x_{m+k+1}) > d(x_m,x_{m+k})$, then we get

$$\begin{aligned} d(x_m,x_{m+k+1}) &\leq sd(x_m,x_{m+k}) + sd(x_{m+k},x_{m+k+1}) \\ &< 2sd(x_{m+k},x_{m+k+1}) \\ &< 2s\frac{\delta_1}{s} \\ &= 2\delta_1 < \epsilon_1+\delta_1. \end{aligned}$$

Thus, by induction, (18) holds for every $k \in \mathbb{N}$. Since $\epsilon_1 > 0$ is arbitrary, we get

$$\limsup_{k\to\infty} d(x_m,x_{m+k}) = 0,$$

which implies that $\{x_n\}$ is a b-Cauchy sequence in a complete b-metric space (X, d). Hence, $\{x_n\}$ b-converges to some $z \in X$.

Next, we show that z is a fixed point of T. If T is continuous, then we have

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tz,$$

that is, z is a fixed point of T.

If T^2 is continuous, since $x_n \to z$, we get that any subsequence of $\{x_n\}$ converges to the same limit point z, so

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = z \quad \text{and} \quad \lim_{n\to\infty} x_{n+2} = \lim_{n\to\infty} T^2 x_n = z.$$

On the other hand, due to the continuity of T^2,

$$T^2 z = \lim_{n\to\infty} T^2 x_n = z.$$

We claim that $Tz = z$. To the contrary, if $Tz \neq z$, then we have $p(z, Tz) > 0$ and

$$\begin{aligned}
p(z, Tz) &\leq N_s(z, Tz) \\
&= \max\biggl\{ d(z, Tz), d(z, Tz), d\bigl(Tz, T^2 z\bigr), \\
&\quad \frac{d(Tz, T^z)[1 + d(z, Tz)]}{1 + d(z, Tz)}, \frac{d(z, Tz)[1 + d(Tz, T^2 z)]}{1 + d(Tz, T^2 z)} \biggr\} \\
&= \max\biggl\{ d(z, Tz), d(z, Tz), d\bigl(Tz, T^2 z\bigr), \\
&\quad \frac{d(Tz, T^2 z)[1 + d(z, Tz)]}{1 + d(z, Tz)}, \frac{d(z, Tz)[1 + d(Tz, T^2 z)]}{1 + d(Tz, T^2 z)} \biggr\} \\
&= d(z, Tz).
\end{aligned}$$

Therefore, together with the supplementary hypothesis $\alpha(z, Tz) \geq 1$, we have

$$\begin{aligned}
0 &\leq \zeta\bigl(\alpha(z, Tz)\phi\bigl(d\bigl(Tz, T^2 z\bigr)\bigr), \phi\bigl(p(z, Tz)\bigr)\bigr) \\
&< \phi\bigl(p(z, Tz)\bigr) - \alpha(z, Tz)\phi\bigl(d\bigl(Tz, T^2 z\bigr)\bigr).
\end{aligned}$$

From the above inequality, we obtain

$$\begin{aligned}
0 &< \phi\bigl(d(Tz, z)\bigr) \\
&= \phi\bigl(d\bigl(Tz, T^2 z\bigr)\bigr) \\
&\leq \alpha(z, Tz)\phi\bigl(d\bigl(Tz, T^2 z\bigr)\bigr) \\
&< \phi\bigl(p(z, Tz)\bigr) \\
&\leq \phi\bigl(N_s(z, Tz)\bigr) \\
&= \phi\bigl(d(z, Tz)\bigr),
\end{aligned}$$

which is a contradiction. Hence, z is a fixed point of T.

If X is regular, we deduce that $d(z,Tz)=0$, using the same arguments as in the proof of Theorem 6. That is, z is a fixed point of T. □

The uniqueness of fixed point of T can be deduced as in Theorem 7.

Now, we give an example to illustrate Theorem 8.

Example 3 Let $X=[0,4]$ and $d:X\times X\to\mathbb{R}^+$ be defined by $d(x,y)=|x-y|^2$ for all $x,y\in X$. Then (X,d) is a complete b-metric space with $s=2$ which is not a metric space. Let $T: X\to X$ be defined by

$$T(x)=\begin{cases}1 & \text{if } x\in[0,2),\\ \frac{x}{2} & \text{if } x\in[2,4].\end{cases}$$

Also, we define $\alpha:X\times X\to\mathbb{R}^+$, $q:X\times X\to\mathbb{R}^+$ and $\phi:\mathbb{R}^+\to\mathbb{R}^+$ as follows:

$$\alpha(x,y)=\begin{cases}2 & \text{if } x,y\in[0,2),\\ 1 & \text{if } x,y\in[2,4],\\ 0 & \text{otherwise,}\end{cases}$$

$q(x,y)=\max\{d(x,y),\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)},\frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\}$ and $\phi(t)=\frac{t^2}{2}$. First, we note that q satisfies condition $(P_q^1:N_s)$ and $q(x,y)>0$ for all $x\neq y$. Since, for $x=0$ we have $T0=1$ and $\alpha(0,T0)=\alpha(0,1)=2>1$, assumption (ii) of Theorem 8 is satisfied. Also, it is easy to see that T is triangular α-orbital admissible. Let $\zeta\in Z_w$ be is given by $\zeta(t,s)=\frac{2}{3}s-t$. Now, we consider the following cases:

Case 1. For $x,y\in[0,2)$, $x\neq y$, we have $d(Tx,Ty)=0$, so

$$\zeta\big(\alpha(x,y)\phi\big(d(Tx,Ty)\big),\phi\big(q(x,y)\big)\big)=\frac{2\phi(q(x,y))}{3}=\frac{(q(x,y))^2}{3}>0.$$

Case 2. For $x,y\in[2,4]$, $x\neq y$, we have

$$d(Tx,Ty)=\frac{|x-y|}{2},\qquad q(x,y)=\max\left\{|x-y|,\frac{\frac{x}{2}\cdot\frac{y}{2}}{1+|x-y|},\frac{\frac{x}{2}\cdot\frac{y}{2}}{1+|\frac{x-y}{2}|}\right\},$$

so

$$\begin{aligned}\zeta\big(\alpha(x,y)\phi\big(d(Tx,Ty)\big),\phi\big(q(x,y)\big)\big)&=\frac{2\phi(q(x,y))}{3}-\phi\left(\frac{|x-y|}{2}\right)\\&=\frac{(q(x,y))^2}{3}-\frac{(|x-y|)^2}{8}\geq 0.\end{aligned}$$

Case 3. For $x\in[0,2)$ and $y\in[2,4]$, we have $\alpha(x,y)=0$ and

$$\zeta\big(\alpha(x,y)\phi\big(d(Tx,Ty)\big),\phi\big(q(x,y)\big)\big)=\frac{2\phi(q(x,y))}{3}=\frac{(q(x,y))^2}{3}>0.$$

Thus, due to the cases considered above, T satisfies all the conditions of Theorem 8 and has a unique fixed point $x=1$.

Now, we give some corollaries to our main findings.

Corollary 1 *Let (X,d) be a complete b-metric space with $s \geq 1$ and let $T : X \to X$ be a (α,ϕ)-Meir–Keeler hybrid contractive mapping of type I with $p(x,y) = d(x,y)$. Assume that the following conditions are satisfied*:

(i) *T is triangular α-orbital admissible*;

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;

(iii) *either T is continuous or T^2 is continuous and $\alpha(u, Tu) \geq 1$ or (X,d) is regular.*

Then T has a fixed point u. Moreover, $\{T^n x\}$ converges to u for all $x \in X$.

Remark 2 Under the conditions of Corollary 1, since $x \neq y$ implies $d(x,y) > 0$, it is obvious that (b) from Definition 9 is equivalent to the following:

(b′) $d(x,y) > 0$ implies $\zeta(\alpha(x,y)\phi(d(Tx,Ty)), \phi(d(x,y))) \geq 0$.

Proof It is clear that d satisfies the conditions $(P^1_d : M_s)$, respectively $(P^2_d : 0)$, and so all the assumptions of Theorems 4, 5, and 6 are also satisfied. Thus, T has a fixed point. □

Corollary 2 *Let (X,d) be a complete b-metric space with $s \geq 1$, and let $T : X \to X$ be a (α,ϕ)-Meir–Keeler hybrid contractive mapping of type I. Let $\rho : X \times X \to \mathbb{R}^+$ be defined by*

$$\rho(x,y) = a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty),$$

where $a_1, a_2, a_3 \in [0, \frac{1}{s})$, $a_1 + a_2 \leq \frac{1}{2s}$ and $a_3 \leq \frac{1}{2s}$. Assume also that:

(i) *T is triangular α-orbital admissible*;

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;

(iii) *either T is continuous or T^2 is continuous and $\alpha(u, Tu) \geq 1$ or (X,d) is regular.*

Then T has a fixed point u. Moreover, $\{T^n x\}$ converges to u for all $x \in X$.

Proof Let $x, y \in X$ be such that $x \neq y$ and $d(x,Tx) \leq d(x,y)$. Then,

$$\begin{aligned}\rho(x,y) &= a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) \\ &\leq (a_1 + a_2) d(x,y) + a_3 d(y,Ty) \\ &\leq \frac{d(x,y) + d(y,Ty)}{2s} \\ &\leq M_s(x,y),\end{aligned}$$

which shows that $(P^1_\rho : M_s)$ holds. On the other hand, if $x_n \neq y$, then

$$\lim_{n\to\infty} d(x_n,y) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_n,x_{n+1}) = 0$$

hold, and then we have

$$\limsup_{n\to\infty} \rho(x_n,y) = \limsup_{n\to\infty} \big[a_1 d(x_n,y) + a_2 d(x_n,x_{n+1}) + a_3 d(y,Ty)\big] = a_3 d(y,Ty).$$

Thus, $(P^2_\rho : a_3)$ holds. Hence, T has a fixed point. □

4 Conclusion

In 2020, Karapinar et al. [43] introduced a generalized Meir–Keeler contraction via a simulation function and studied fixed point results for the mappings introduced in the setting of metric spaces. In this work, we introduced generalized (α,ϕ)-Meir–Keeler hybrid contractive mappings of type I and II in the setting of b-metric spaces and proved the existence and uniqueness of fixed points for such mappings. Our results extend and generalize the results of Karapinar et al. [43] and many other related fixed point results in the existing literature. Finally, we have also supported the main result of this work by an illustrative example.

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