Fixed Point Theory and Algorithms for Sciences and Engineering

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Solving integral equations via orthogonal hybrid interpolative RI-type contractions

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Abstract

In this paper, we initiate the fixed point theorems for an orthogonal hybrid interpolative Riech Istrastescus type contractions map on orthogonal b-metric spaces to modify this class proficiently. Also, we provide some examples supporting our main results. Finally, we provide an application to solve the existence and uniqueness of an integral equation with numeric results, which is powerful in a greater way.

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1 Introduction

Today, one of the most famous research tools for fixed point (f.p.) results extends and interests mathematics in various fields, and integral equations have powerful applications in this context. Banach [1] initiated the concept of the famous Banach contraction principle in 1922, which is used on complete metric spaces. There are many researchers who have developed the Banach contraction principle in generalized metric f.p. theory. It is suggested that relavent results improved the extensions and established results by referring the reader to to see [2, 3, 32, 37–39]. In addition, during the past few years, f.p. results have played an important role in solving many issues and optimizations [4, 5].

In 1993, Czerwik [6] and Bakhtin [7] introduced the notion of metric spaces, labeled b-metric space (bMS) by changing the triangular inequality of the metric spaces. In this space, some researchers are interested in improving new contraction maps and solving the existence of f.p. results [8, 9, 19–29]. A notion of hybrid interpolative Riech Istrastescus (RI)-type contraction maps in b-metric spaces was recently proposed by Aloqaily et al. [10]. These outcomes extend many existing f.p. theories (see [11, 12, 30, 31]). In 2017, Eshaghi Gordji et al. [13] established the concept of orthogonality and offered a framework to enlarge the results. In the same year, Eshaghi Gordji and Habibi [14] extended this work and proved some f.p. theorems in generalized orthogonal metric spaces. Afterwards, Arul Joseph et al. [15, 16] demonstrated some of the f.p. results with integral equations on orthogonal metric spaces, which have great applications in this field. Recently, many reseachers have improved results related to orthogonal concepts (see [17, 18, 33–36]).

In this paper, we are interested in initiating the concepts in the surrounding area of *bMS* by using an existence and corresponding uniqueness solution on orthogonal *bMS*, and solving an orthogonal hybrid interpolative (*RI*)-type contractions map. Our newly obtained results unify, generalize, and extend many well-known results from the existing literature. An example is provided to demonstrate the utility of our newly proved results. Finally, we show the applicability of our main result to discuss the existence of a solution to the integral equation with the algebraic results.

2 Preliminaries

Now, let us remember some more concepts that will be used for our results.

Definition 2.1 [6] Let $\mathcal{Q}^* \neq \varrho$ and $\mu \geq 1$ be any real number. The function $\eth_\flat : \mathcal{Q}^* \times \mathcal{Q}^* \to \mathbb{R}^+$ fulfill the following axioms on $\mathcal{Q}^*$ is said to be a b-metric on $\mathcal{Q}^*$:

(i) $\eth_\flat(\xi,\eta) = 0 \iff \xi = \eta$;

(ii) $\eth_\flat(\xi,\eta) = \eth_\flat(\eta,\xi)$;

(iii) $\eth_\flat(\xi,\eta) \leq \mu[\eth_\flat(\xi,\kappa) + \eth_\flat(\kappa,\eta)]$;

for all $\xi,\eta,\kappa \in \mathcal{Q}^*$. The pair $(\mathcal{Q}^*,\eth_\flat)$ is called a *bMS*.

Definition 2.2 [11] A map $\mathcal{D} : \mathcal{Q}^* \to \mathcal{Q}^*$ and a function $\pi : \mathcal{Q}^* \times \mathcal{Q}^* \to [0,\infty)$ in a *bMS* is said to be π-orbital admissible if for $\xi \in \mathcal{Q}^*$ it holds

$$\pi(\xi,\mathcal{D}\xi) \geq 1 \text{ implies } \pi\left(\mathcal{D}\xi,\mathcal{D}^2\xi\right) \geq 1.$$

Definition 2.3 [11] Let $(\mathcal{Q}^*,\eth_\flat)$ be a *bMS* and $\pi : \mathcal{Q}^* \times \mathcal{Q}^* \to [0,\infty)$ be a function. A map $\mathcal{D} : \mathcal{Q}^* \to \mathcal{Q}^*$ is said to be a hybrid interpolative RI-type contraction if $\exists\phi \in [0,1)$ such that (s.t.)

$$\pi(\xi,\eta)\eth_\flat\left(\mathcal{D}^2\xi,\mathcal{D}^2\eta\right) \leq \phi\mathbb{M}(\xi,\eta),$$

here,

$$\mathbb{M}(\xi,\eta) = \begin{cases} [\theta_1\eth_\flat(\xi,\eta)^\hbar + \theta_2\eth_\flat(\xi,\mathcal{D}\xi)^\hbar + \theta_3\eth_\flat(\eta,\mathcal{D}\eta)^\hbar & \sum_{\mathrm{i}=1}^5 \theta_\mathrm{i} + \lambda \leq 1, \\ +\theta_4\eth_\flat(\mathcal{D}\xi,\mathcal{D}\eta)^\hbar + \theta_5\eth_\flat(\mathcal{D}\xi,\mathcal{D}^2\eta)^\hbar + \lambda\eth_\flat(\mathcal{D}\eta,\mathcal{D}^2\eta)^\hbar]^{\frac{1}{\hbar}} & \text{if } \hbar > 0, \\ \eth_\flat(\xi,\eta)^{\theta_1}.\eth_\flat(\xi,\mathcal{D}\xi)^{\theta_2}.\eth_\flat(\eta,\mathcal{D}\eta)^{\theta_3} & \sum_{\mathrm{i}=1}^5 \theta_\mathrm{i} + \lambda = 1, \\ \eth_\flat(\mathcal{D}\xi,\mathcal{D}\eta)^{\theta_4}.\eth_\flat(\mathcal{D}\xi,\mathcal{D}^2\eta)^{\theta_5}.\eth_\flat(\mathcal{D}\eta,\mathcal{D}^2\eta)^\lambda & \text{if } \hbar = 0, \end{cases}$$

with $\{\theta_\mathrm{i} : \mathrm{i} = 1,2,\ldots,5 \geq 0\}, \hbar \in \mathbb{R}$ and $\lambda > 0$.

Proposition 2.1 [11] *Let* $\phi \in [0,1)$ *and* $\{\xi_\gamma\} \subset \mathbb{R}^+$ *be any sequence s.t.*

$$\xi_{\gamma+2} \leq \phi \max\{\xi_\gamma,\xi_{\gamma+1}\}, \quad \textit{for all } \gamma \in \mathbb{N} \cup 0,$$

then,

$$\xi_{2\gamma} \leq \phi^\gamma \mathcal{Q}', \qquad \xi_{2\gamma+1} \leq \phi^\gamma \mathcal{Q}', \quad \textit{for all } \gamma \geq 1, \mathcal{Q}' > 0,$$

where $\mathcal{Q}' = \max\{\xi_0,\xi_1\}$.

Lemma 2.2 [11] *Let* $\{\mathfrak{r}_\gamma\}$ *be a sequence in bMS and* $\exists\phi \in [0,1)$ *s.t.*

$$\eth_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_{\gamma+3}) \le \phi \max\{\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1}),\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})\}, \quad \forall\gamma \in \mathbb{N},$$

then $\{\mathfrak{r}_\gamma\}$ *is a Cauchy sequence in* $(\mathcal{Q}^*,\eth_\flat)$.

Corollary 1 *Consider* $\{\mathfrak{r}_\gamma\}$ *as a sequence on bMS and that* $\phi \in [0,1)$ *exists s.t.*

$$\eth_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_{\gamma+3}) \le \phi\big[\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1})^{\rho_1}.\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})^{\rho_2}\big], \quad \forall\gamma \in \mathbb{N},$$

then $\{\mathfrak{r}_\gamma\}$ *is a Cauchy sequence in* $(\mathcal{Q}^*,\eth_\flat)$, *where* $\rho_1,\rho_2 \in [0,1]$ *fulfill* $\rho_1 + \rho_2 = 1$.

The following definition of orthogonality was used as the foundation for the rest of our work.

Definition 2.4 [13] Let $\mathcal{Q}^*$ be a non-void and $\perp\subseteq \mathcal{Q}^* \times \mathcal{Q}^*$ be an binary relation. If $\perp$ fulfills the following condition:

$$\exists\mathfrak{r}_0 : (\forall\eta, \eta \perp \mathfrak{r}_0) \quad \text{or} \quad (\forall\eta, \mathfrak{r}_0 \perp \eta),$$

then $(\mathcal{Q}^*,\perp)$ is called an orthogonal set (O_{set}).

Definition 2.5 [13] Let $(\mathcal{Q}^*,\perp)$ be an O_{set}. A sequence $\{\xi_\sigma\}$ is called an orthogonal sequence (briefly, O-sequence) if

$$(\forall\sigma \in \mathbb{N}, \xi_\sigma \perp \xi_{\sigma+1}) \quad \text{or} \quad (\forall\sigma \in \mathbb{N}, \xi_{\sigma+1} \perp \xi_\sigma).$$

Definition 2.6 [13] Let $(\mathcal{Q}^*,\perp,\eth_\flat)$ be an O-bMS if $(\mathcal{Q}^*,\perp)$ is an O_{set} and $(\mathcal{Q}^*,\eth_\flat)$ is a bMS.

Definition 2.7 [13] Let $(\mathcal{Q}^*,\perp,\eth_\flat)$ be an O-bMS.

(1) A map $\mathcal{D}\colon \mathcal{Q}^* \to \mathcal{Q}^*$ is called an O-continuous in $\xi \in \mathcal{Q}^*$ if for every O-sequence $\{\xi_\sigma\}_{\sigma\in\mathbb{N}}$ in $\mathcal{Q}^*$ with $\xi_\sigma \to \xi$, we obtain $\mathcal{D}(\xi_\sigma) \to \mathcal{D}(\xi)$. Also, $\mathcal{D}$ is called an O-continuous on $\mathcal{Q}^*$ if $\mathcal{D}$ is an O-continuous in each $\xi \in \mathcal{Q}^*$.

(2) A set $\mathcal{Q}^*$ is said to be an orthogonal complete if every Cauchy orthogonal sequence is convergent.

(3) A function $\mathcal{D}\colon \mathcal{Q}^* \to \mathcal{Q}^*$ is called an orthogonal contraction with Lipschitz constant ϕ if, $0 < \phi < 1$ for all $\xi,\eta \in \mathcal{Q}^*$ with $\xi \perp \eta$,

$$\eth_\flat(\mathcal{D}\xi,\mathcal{D}\eta) \le \phi\mathcal{D}(\xi,\eta).$$

(4) A function $\mathcal{D}\colon \mathcal{Q}^* \to \mathcal{Q}^*$ is said to be an O-preserving if $\mathcal{D}(\xi) \perp \mathcal{D}(\eta)$ whenever $\xi \perp \eta$.

3 Main results

In this segment, we improve some f.p results for orthogonal hybrid interpolative RI-type contractions in O-complete *bMS*. Moreover, we provide an illustrative example and application to illustrate our newly obtained results.

Definition 3.1 Let $(\mathcal{Q}^*, \perp, \mathfrak{d}_\flat)$ be an O-complete *bMS* with parameter $\mu \geq 1$ and $\pi : \mathcal{Q}^* \times \mathcal{Q}^* \to [0,\infty)$ be a function. A map $\mathcal{D} : \mathcal{Q}^* \to \mathcal{Q}^*$ is said to be an orthogonal hybrid interpolative RI-type contraction if $\exists \phi \in [0,1)$ s.t. for any $\xi, \eta \in \mathcal{Q}^*$ with $\xi \perp \eta$

$$\pi(\xi,\eta)\mathfrak{d}_\flat\big(\mathcal{D}^2\xi, \mathcal{D}^2\eta\big) \leq \phi \mathbb{M}(\xi,\eta), \tag{1}$$

here,

$$\mathbb{M}(\xi,\eta) = \begin{cases} [\theta_1 \mathfrak{d}_\flat(\xi,\eta)^\hbar + \theta_2 \mathfrak{d}_\flat(\xi,\mathcal{D}\xi)^\hbar + \theta_3 \mathfrak{d}_\flat(\eta,\mathcal{D}\eta)^\hbar \\ \quad \sum_{\mathrm{i}=1}^5 \theta_\mathrm{i} + \lambda \leq 1, \\ + \theta_4 \mathfrak{d}_\flat(\mathcal{D}\xi,\mathcal{D}\eta)^\hbar + \theta_5 \mathfrak{d}_\flat(\mathcal{D}\xi,\mathcal{D}^2\eta)^\hbar + \lambda \mathfrak{d}_\flat(\mathcal{D}\eta,\mathcal{D}^2\eta)^\hbar]^{\frac{1}{\hbar}} \\ \quad \text{if } \hbar > 0, \\ \mathfrak{d}_\flat(\xi,\eta)^{\theta_1}.\mathfrak{d}_\flat(\xi,\mathcal{D}\xi)^{\theta_2}.\mathfrak{d}_\flat(\eta,\mathcal{D}\eta)^{\theta_3} \\ \quad \sum_{\mathrm{i}=1}^5 \theta_\mathrm{i} + \lambda = 1, \\ \mathfrak{d}_\flat(\mathcal{D}\xi,\mathcal{D}\eta)^{\theta_4}.\mathfrak{d}_\flat(\mathcal{D}\xi,\mathcal{D}^2\eta)^{\theta_5}.\mathfrak{d}_\flat(\mathcal{D}\eta,\mathcal{D}^2\eta)^{\lambda} \\ \quad \text{if } \hbar = 0, \end{cases} \tag{2}$$

with $\{\theta_\mathrm{i} : \mathrm{i} = 1,2,\ldots,5 \geq 0\}, \hbar \in \mathbb{R}$ and $\lambda > 0$.

Proposition 3.1 *Let $\phi \in [0,1)$ and $\{\xi_\gamma\} \subset \mathbb{R}^+$ be any O-sequence s.t.*

$$\xi_{\gamma+2} \leq \phi \max\{\xi_\gamma, \xi_{\gamma+1}\}, \quad \textit{for all } \gamma \in \mathbb{N} \cup 0, \tag{3}$$

then

$$\xi_{2\gamma} \leq \phi^\gamma \mathcal{Q}', \xi_{2\gamma+1} \leq \phi^\gamma \mathcal{Q}', \quad \textit{for all } \gamma \geq 1, \tag{4}$$

where $\mathcal{Q}' = \max\{\xi_0, \xi_1\}$.

Proof Letting $\gamma = 0$ in (3), we have

$$\xi_2 \leq \phi \max\{\xi_0, \xi_1\} = \phi \mathcal{Q}',$$

for $\gamma = 1$, we obtain

$$\begin{aligned} \xi_3 &\leq \phi \max\{\xi_1, \xi_2\} \\ &\leq \phi \max\big\{\xi_1, \phi \max\{\xi_0, \xi_1\}\big\} \\ &\leq \phi \max\big\{\xi_1, \phi \mathcal{Q}'\big\} \\ &\leq \phi \mathcal{Q}'. \end{aligned}$$

Suppose that (4) satisfies for some $\gamma \in \mathbb{N}$, then

$$\begin{aligned} \xi_{2\gamma+2} &\leq \phi \max\{\xi_{2\gamma}, \xi_{2\gamma+1}\} \\ &\leq \phi \max\big\{\phi^\gamma \mathcal{Q}', \phi^\gamma \mathcal{Q}'\big\} \end{aligned}$$

$$\leq \phi^{\gamma+1}\mathcal{Q}',$$

similarly, we obtain $\xi_{2\gamma+3} \leq \phi^{\gamma+1}\mathcal{Q}'$.

By using induction method, we complete the proof. □

Lemma 3.2 *Let* $\{\mathfrak{r}_\gamma\}$ *be an O-sequence on O-complete bMS, and* $\exists\phi \in [0,1)$ *s.t.*

$$\eth_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_{\gamma+3}) \leq \phi \max\{\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1}),\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})\}, \quad \forall\gamma \in \mathbb{N},$$

then $\{\mathfrak{r}_\gamma\}$ *is an O-Cauchy sequence in* $(\mathcal{Q}^*,\perp,\eth_\flat)$.

Proof Let $\{\xi_\gamma\}$ be an O-sequence in $\mathcal{Q}^*$ defined as

$$\xi_\gamma = \eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1}), \quad \forall\gamma \in \mathbb{N}.$$

This is an O-sequence that assures the condition (4). We obtain

$$\eth_\flat(\mathfrak{r}_{2\gamma},\mathfrak{r}_{2\gamma+1}) = \xi_{2\gamma} \leq \phi^\gamma\mathcal{Q}', \quad \forall\gamma \geq \mathbb{N}, \tag{5}$$

also,

$$\eth_\flat(\mathfrak{r}_{2\gamma+1},\mathfrak{r}_{2\gamma+2}) = \xi_{2\gamma+1} \leq \phi^\gamma\mathcal{Q}'. \tag{6}$$

Adding (5) and (6), we get

$$\eth_\flat(\mathfrak{r}_{2\gamma},\mathfrak{r}_{2\gamma+1}) + \eth_\flat(\mathfrak{r}_{2\gamma+1},\mathfrak{r}_{2\gamma+2}) \leq 2\phi^\gamma\mathcal{Q}'.$$

Note that $\phi = 0$ or $\mathcal{Q}' = 0$. Hence, an O-sequence is an O-Cauchy sequence. Consider $\phi > 0$, $\mathcal{Q}' > 0$, and $\epsilon > 0$, thus $\frac{\epsilon}{2\mathcal{Q}'\mu^{\eta^*}} > 0$ and $\eta^* > \gamma_0 \geq 1$ s.t.

$$\sum_{\gamma=\gamma_0}^{+\infty}\phi^\gamma < \frac{\epsilon}{2\mathcal{Q}'\mu^{\eta^*}},$$

in particular

$$2\mathcal{Q}'\mu^{\eta^*}\sum_{\gamma=\gamma_0}^{\gamma}\phi^\gamma < 2\mathcal{Q}'\mu^{\eta^*}\sum_{\gamma=\gamma_0}^{+\infty}\phi^\gamma < \epsilon, \quad \forall\eta \in \mathbb{N},$$

s.t. $\eta \geq \gamma$.

Let $\eta^*,\gamma,\ell \in \mathbb{N}$ s.t. $\eta^* > \ell > \gamma \geq 2\gamma_0$, $\eta \geq \gamma_0 + 1$, and $2\eta \geq \ell$; thus, we obtain

$$\begin{aligned}
\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_\ell) &\leq \mu\big[\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1}) + \eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\mathfrak{r}_\ell})\big] \\
&= \mu\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1}) + \mu\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_\ell) \\
&\leq \mu\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1}) + \mu^2\big[\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2}) + \eth_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_\ell)\big] \\
&\leq \mu\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1}) + \mu^2\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2}) + \mu^3\eth_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_{\gamma+3}) + \cdots
\end{aligned}$$

$$
\begin{aligned}
&+\mu^{\ell-\gamma}\mathfrak{d}_\flat(\mathfrak{r}_{\ell-1},\mathfrak{r}_\ell)\\
&=\sum_{\varkappa=\gamma}^{\ell-1}\mu^{\varkappa-\gamma+1}\mathfrak{d}_\flat(\mathfrak{r}_\varkappa,\mathfrak{r}_{\varkappa+1})\\
&\le\sum_{\varkappa=\gamma}^{\ell-1}\mu^{\eta^*}\mathfrak{d}_\flat(\mathfrak{r}_\varkappa,\mathfrak{r}_{\varkappa+1})\\
&\le\sum_{\varkappa=2\gamma_0}^{2\eta-1}\mu^{\eta^*}\mathfrak{d}_\flat(\mathfrak{r}_\varkappa,\mathfrak{r}_{\varkappa+1})\\
&\le\sum_{\gamma=\gamma_0}^{2\eta-1}\mu^{\eta^*}\left\{\mathfrak{d}_\flat(\mathfrak{r}_{2\gamma},\mathfrak{r}_{2\gamma+1})+\mathfrak{d}_\flat(\mathfrak{r}_{2\gamma+1},\mathfrak{r}_{2\gamma+2})\right\}\\
&\le\sum_{\gamma=\gamma_0}^{\gamma-1}\mu^{\eta^*}2\phi^\gamma\mathcal{Q}'\le\sum_{\gamma=\gamma_0}^{\gamma}\mu^{\eta^*}2\phi^\gamma\mathcal{Q}'\\
&\le\sum_{\gamma=\gamma_0}^{+\infty}\mu^{\eta^*}2\phi^\gamma\mathcal{Q}'\\
&<\epsilon,
\end{aligned}
$$

shows that $\{\mathfrak{r}_\gamma\}$ is an *O*-Cauchy sequence in an orthogonal *bMS*. □

Corollary 2 *Let* $\{\mathfrak{r}_\gamma\}$ *be an O-sequence in orthogonal bMS, and* $\phi\in[0,1)$ *exists s.t.*

$$\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_{\gamma+3})\le\phi\left[\mathfrak{d}_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1})^{\rho_1}.\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})^{\rho_2}\right],\quad\forall\gamma\in\mathbb{N},$$

then $\{\mathfrak{r}_\gamma\}$ *is an O-Cauchy sequence in* $(\mathcal{Q}^*,\perp,\mathfrak{d}_\flat)$, *where* $\rho_1,\rho_2\in[0,1]$ *fulfill* $\rho_1+\rho_2=1$.

Proof Consider $\xi_\gamma=\mathfrak{d}_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1})$ and a Proposition 3.1, hence, we get

$$
\begin{aligned}
\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_{\gamma+3})&\le\phi\left[\mathfrak{d}_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1})^{\rho_1}.\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})^{\rho_2}\right]\\
&\le\phi\left[(\xi_\gamma)^{\rho_1}.(\xi_{\gamma+1})^{\rho_2}\right].\\
&\le\phi\Big[\max\left\{\mathfrak{d}_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1}),\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})\right\}^{\rho_1}\\
&\quad.\max\left\{\mathfrak{d}_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1}),\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})\right\}^{\rho_2}\Big]\\
&\le\phi\left[\max\left\{\mathfrak{d}_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1}),\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})\right\}\right]^{\rho_1+\rho_2}\\
&\le\phi\left[\max\left\{\mathfrak{d}_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1}),\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})\right\}\right],
\end{aligned}
$$

then, by using Lemma 3.2, we conclude the results. □

Theorem 3.3 *Let* $(\mathcal{Q}^*,\perp,\mathfrak{d}_\flat)$ *be an O-complete bMS with an orthogonal element* $\mathfrak{r}_0$ *and constant* $\mu\ge1$, *and let* $\mathcal{D}:\mathcal{Q}^*\to\mathcal{Q}^*$ *be an orthogonal hybrid interpolative RI-type contraction map satisfying*:

(i) $\mathcal{D}$ *is an O-preserving*;
(ii) $\mathcal{D}$ *is an O-*π *orbital-admissible mapping*;

(iii) $\mathfrak{r}_0 \in \mathcal{Q}^*$ *exists s.t.* $\pi(\mathfrak{r}_0, \mathcal{D}\mathfrak{r}_0) \geq 1$;
(iv) $\mathcal{D}$ *is an O-continuous*;
Then $\mathcal{D}$ *has a unique f.p.*

Proof By the definition of orthogonality, we see that $(\mathcal{Q}^*, \perp)$ is an O_{set}, then there exists

$$\mathfrak{r}_0 \in \mathcal{Q}^* : \forall \mathfrak{r} \in \mathcal{Q}^*, \mathfrak{r} \perp \mathfrak{r}_0 \quad \text{(or)} \quad \forall \mathfrak{r} \in \mathcal{Q}^*, \mathfrak{r}_0 \perp \mathfrak{r}.$$

It follows that $\mathfrak{r}_0 \perp \mathcal{D}\mathfrak{r}_0$ or $\mathcal{D}\mathfrak{r}_0 \perp \mathfrak{r}_0$. Let

$$\mathfrak{r}_1 = \mathcal{D}\mathfrak{r}_0, \qquad \mathfrak{r}_2 = \mathcal{D}\mathfrak{r}_1 = \mathcal{D}^2\mathfrak{r}_0 \cdots \mathfrak{r}_\gamma = \mathcal{D}\mathfrak{r}_{\gamma-1} = \mathcal{D}^\gamma\mathfrak{r}_0, \quad \forall \gamma \in \mathbb{N}.$$

For any $\mathfrak{r}_0 \in \mathcal{Q}^*$, set $\mathfrak{r}_\gamma = \mathcal{D}\mathfrak{r}_{\gamma-1}$. Now, we consider the following cases:

(i) If there exists $\gamma \in \mathbb{N} \cup \{0\}$ s.t. $\mathfrak{r}_\gamma = \mathfrak{r}_{\gamma+1}$, then we get $\mathcal{D}\mathfrak{r}_\gamma = \mathfrak{r}_\gamma$. It is easy to see that $\mathfrak{r}_\gamma$ is a f.p. of $\mathcal{D}$. Hence, the proof is finished.

(ii) If $\mathfrak{r}_\gamma \neq \mathfrak{r}_{\gamma+1}$, for every $\gamma \in \mathbb{N} \cup \{0\}$, then we obtain $\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+1}, \mathfrak{r}_\gamma) > 0$, for any $\gamma \in \mathbb{N}$.

Since $\mathcal{D}$ is an O-preserving, we have

$$\mathfrak{r}_\gamma \perp \mathfrak{r}_{\gamma+1} \quad \text{(or)} \quad \mathfrak{r}_{\gamma+1} \perp \mathfrak{r}_\gamma.$$

Therefore, $\{\mathfrak{r}_\gamma\}$ is an O-sequence. Since $\mathcal{D}$ is an O-π orbital-admissible, we get

$$\pi\left(\mathcal{D}\mathfrak{r}_0, \mathcal{D}^2\mathfrak{r}_0\right) \geq 1.$$

Now consider,

$$\begin{aligned}
\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+2}, \mathfrak{r}_{\gamma+3}) &\leq \pi(\mathfrak{r}_{\gamma+2}, \mathfrak{r}_{\gamma+3})\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+2}, \mathfrak{r}_{\gamma+3}) \\
&\leq \pi(\mathfrak{r}_{\gamma+2}, \mathfrak{r}_{\gamma+3})\mathfrak{d}_\flat\left(\mathcal{D}^2\mathfrak{r}_\gamma, \mathcal{D}^2\mathfrak{r}_{\gamma+1}\right) \\
&\leq \phi\mathbb{M}(\mathfrak{r}_\gamma, \mathfrak{r}_{\gamma+1}).
\end{aligned} \tag{7}$$

We will next discuss the two possibilities for the way $\hbar$ could be chosen.

Case I: If $\hbar > 0$,

$$\begin{aligned}
\mathbb{M}(\mathfrak{r}_\gamma, \mathfrak{r}_{\gamma+1}) &= \Big[\theta_1\mathfrak{d}_\flat(\mathfrak{r}_\gamma, \mathfrak{r}_{\gamma+1})^\hbar + \theta_2\mathfrak{d}_\flat(\mathfrak{r}_\gamma, \mathcal{D}\mathfrak{r}_\gamma)^\hbar + \theta_3\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+1}, \mathcal{D}\mathfrak{r}_{\gamma+1})^\hbar \\
&\quad + \theta_4\mathfrak{d}_\flat(\mathcal{D}\mathfrak{r}_\gamma, \mathcal{D}\mathfrak{r}_{\gamma+1})^\hbar + \theta_5\mathfrak{d}_\flat\left(\mathcal{D}\mathfrak{r}_\gamma, \mathcal{D}^2\mathfrak{r}_\gamma\right)^\hbar + \lambda\mathfrak{d}_\flat\left(\mathcal{D}\mathfrak{r}_{\gamma+1}, \mathcal{D}^2\mathfrak{r}_{\gamma+1}\right)^\hbar\Big]^{\frac{1}{\hbar}} \\
&= \Big[\theta_1\mathfrak{d}_\flat(\mathfrak{r}_\gamma, \mathfrak{r}_{\gamma+1})^\hbar + \theta_2\mathfrak{d}_\flat(\mathfrak{r}_\gamma, \mathfrak{r}_{\gamma+1})^\hbar + \theta_3\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+1}, \mathfrak{r}_{\gamma+2})^\hbar \\
&\quad + \theta_4\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+1}, \mathfrak{r}_{\gamma+2})^\hbar + \theta_5\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+1}, \mathfrak{r}_{\gamma+2})^\hbar + \lambda\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+2}, \mathfrak{r}_{\gamma+3})^\hbar\Big]^{\frac{1}{\hbar}} \\
&= \Big[(\theta_1 + \theta_2)\mathfrak{d}_\flat(\mathfrak{r}_\gamma, \mathfrak{r}_{\gamma+1})^\hbar + (\theta_3 + \theta_4 + \theta_5)\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+1}, \mathfrak{r}_{\gamma+2})^\hbar \\
&\quad + \lambda\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+2}, \mathfrak{r}_{\gamma+3})^\hbar\Big]^{\frac{1}{\hbar}} \\
&\leq \Big[(\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5)\max\left\{\mathfrak{d}_\flat(\mathfrak{r}_\gamma, \mathfrak{r}_{\gamma+1})^\hbar, \mathfrak{d}_\flat(\mathfrak{r}_{\gamma+1}, \mathfrak{r}_{\gamma+2})^\hbar\right\} \\
&\quad + \lambda\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+2}, \mathfrak{r}_{\gamma+3})^\hbar\Big]^{\frac{1}{\hbar}},
\end{aligned}$$

letting power of $\hbar$ on Equation (7), we get

$$\begin{aligned}\eth_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_{\gamma+3})^\hbar \le \phi^\hbar(\theta_1+\theta_2+\theta_3+\theta_4+\theta_5)\max\{\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1})^\hbar,\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})^\hbar\}\\ +\phi^\hbar\lambda\eth_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_{\gamma+3})^\hbar.\end{aligned}$$

Therefore,

$$\begin{aligned}&(1-\phi^\hbar\lambda)\eth_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_{\gamma+3})^\hbar\\ &\quad\le\phi^\hbar(\theta_1+\theta_2+\theta_3+\theta_4+\theta_5)\max\{\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1})^\hbar,\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})^\hbar\}\\ &\quad\le\phi^\hbar(1-\lambda)\max\{\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1}),\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})\}^\hbar,\\ &\eth_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_{\gamma+3})^\hbar\le\left(\frac{\phi^\hbar(1-\lambda)}{1-\phi^\hbar\lambda}\right)\max\{\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1}),\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})\}^\hbar,\quad\forall\gamma\in\mathbb{N},\end{aligned}$$

or equally

$$\begin{aligned}\eth_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_{\gamma+3})&=\left(\frac{\phi^\hbar(1-\lambda)}{1-\phi^\hbar\lambda}\right)^{\frac{1}{\hbar}}\max\{\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1}),\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})\}\\ &=\phi\max\{\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1}),\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})\},\end{aligned}$$

where

$$\phi=\left(\frac{\phi^\hbar(1-\lambda)}{1-\phi^\hbar\lambda}\right)^{\frac{1}{\hbar}},\quad\phi\in(0,1).$$

Hence, Lemma 3.2 satisfies that $\{\mathfrak{r}_\gamma\}$ is an O-Cauchy sequence.

Case II: If $\hbar=0$,

$$\begin{aligned}\mathbb{M}(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1})&=\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1})^{\theta_1}.\eth_\flat(\mathfrak{r}_\gamma,\mathcal{D}\mathfrak{r}_\gamma)^{\theta_2}.\eth_\flat(\mathfrak{r}_{\gamma+1},\mathcal{D}\mathfrak{r}_{\gamma+1})^{\theta_3}\\ &\quad.\eth_\flat(\mathcal{D}\mathfrak{r}_\gamma,\mathcal{D}\mathfrak{r}_{\gamma+1})^{\theta_4}.\eth_\flat(\mathcal{D}\mathfrak{r}_\gamma,\mathcal{D}^2\mathfrak{r}_\gamma)^{\theta_5}.\eth_\flat(\mathcal{D}\mathfrak{r}_{\gamma+1},\mathcal{D}^2\mathfrak{r}_{\gamma+1})^{\lambda}\\ &=\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1})^{\theta_1}.\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1})^{\theta_2}.\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})^{\theta_3}\\ &\quad.\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})^{\theta_4}.\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})^{\theta_5}.\eth_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_{\gamma+3})^{\lambda}\\ &=\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1})^{\theta_1+\theta_2}.\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})^{\theta_3+\theta_4+\theta_5}\\ &\quad.\eth_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_{\gamma+3})^{\lambda},\end{aligned}$$

from (7), it implies that

$$\begin{aligned}\eth_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_{\gamma+3})\le\phi\eth_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_{\gamma+1})^{\theta_1+\theta_2}.\eth_\flat(\mathfrak{r}_{\gamma+1},\mathfrak{r}_{\gamma+2})^{\theta_3+\theta_4+\theta_5}\\ .\eth_\flat(\mathfrak{r}_{\gamma+2},\mathfrak{r}_{\gamma+3})^{\lambda}.\end{aligned}\tag{8}$$

Our assumption

$$\sum_{\mathrm{i}=1}^{5}\theta_\mathrm{i}+\lambda=1,$$

taking $\lambda = 1$, then the Equation (8) is contradiction. Let $\lambda < 1$, so

$$\sum_{\mathrm{i}=1}^{5} \theta_{\mathrm{i}} = 1 - \lambda > 0.$$

Now consider,

$$\rho_1 = \frac{\theta_1 + \theta_2}{1 - \lambda}, \qquad \rho_2 = \frac{\theta_3 + \theta_4 + \theta_5}{1 - \lambda},$$

by adding ρ_1 and ρ_2, we get

$$\begin{aligned} \rho_1 + \rho_2 &= \frac{\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5}{1 - \lambda} \\ &= \frac{1 - \lambda}{1 - \lambda} = 1. \end{aligned}$$

Therefore, satisfying $\rho_1 + \rho_2 = 1$. Now, setting these in (8), we obtain

$$\begin{aligned} &\eth_\flat(\mathfrak{r}_{\gamma+2}, \mathfrak{r}_{\gamma+3})^{1-\lambda} \leq \phi \eth_\flat(\mathfrak{r}_\gamma, \mathfrak{r}_{\gamma+1})^{\theta_1+\theta_2} . \eth_\flat(\mathfrak{r}_{\gamma+1}, \mathfrak{r}_{\gamma+2})^{\theta_3+\theta_4+\theta_5}. \\ &\quad \Longrightarrow \quad \eth_\flat(\mathfrak{r}_{\gamma+2}, \mathfrak{r}_{\gamma+3}) \leq \phi^{\frac{1}{1-\lambda}} \eth_\flat(\mathfrak{r}_\gamma, \mathfrak{r}_{\gamma+1})^{\rho_1} . \eth_\flat(\mathfrak{r}_{\gamma+1}, \mathfrak{r}_{\gamma+2})^{\rho_2}, \quad \text{as } \phi \in (0, 1). \end{aligned}$$

Therefore,

$$0 < 1 - \lambda \leq 1 \quad \Longrightarrow \quad 1 \leq \frac{1}{1 - \lambda} \quad \Longrightarrow \quad \phi^{\frac{1}{1-\lambda}} \leq \phi < 1.$$

Hence, Corollary 2 concludes that $\{\mathfrak{r}_\gamma\}$ is an O-Cauchy sequence. As $\mathcal{Q}^*$ is an O-complete, $\exists \mathfrak{r}^* \in \mathcal{Q}^*$ s.t.

$$\eth_\flat(\mathfrak{r}^*, \mathcal{D}\mathfrak{r}^*) = \lim_{\gamma \to \infty} \eth_\flat(\mathfrak{r}_{\gamma+1}, \mathcal{D}\mathfrak{r}^*) = \lim_{\gamma \to \infty} \eth_\flat(\mathcal{D}\mathfrak{r}_\gamma, \mathcal{D}\mathfrak{r}^*) = 0.$$

So $\mathcal{D}\mathfrak{r}^* = \mathfrak{r}^*$, that is $\mathfrak{r}^*$ is the f.p. of $\mathcal{D}$.

Now, we show that $\tau \in \mathcal{Q}^*$ is unique.

Suppose that τ and $\mathfrak{v}$ are two different f.p. of $\mathcal{D}$. Assume that $\mathcal{D}^\gamma \tau = \tau \neq \mathfrak{v} = \mathcal{D}^\gamma \mathfrak{v}$ for all $\tau, \mathfrak{v} \in \mathbb{N}$. By choice of $\mathfrak{r}^*$, we obtain

$$(\mathfrak{r}^* \perp \tau, \mathfrak{r}^* \perp \mathfrak{v}) \quad \text{or} \quad (\tau \perp \mathfrak{r}^*, \mathfrak{v} \perp \mathfrak{r}^*).$$

Since $\mathcal{D}$ is $\perp$-preserving, we have

$$(\mathcal{D}^\gamma \mathfrak{r}^* \perp \mathcal{D}^\gamma \tau, \mathcal{D}^\gamma \mathfrak{r}^* \perp \mathcal{D}^\gamma \mathfrak{v}) \quad \text{or} \quad (\mathcal{D}^\gamma \tau \perp \mathcal{D}^\gamma \mathfrak{r}^*, \mathcal{D}^\gamma \mathfrak{v} \perp \mathcal{D}^\gamma \mathfrak{r}^*),$$

for all $\tau, \mathfrak{v} \in \mathbb{N}$. Therefore, by Definition 2.1 of triangle inequality, we get

$$\begin{aligned} \eth_\flat(\tau, \mathfrak{v}) &= \eth_\flat(\mathcal{D}^\gamma \tau, \mathcal{D}^\gamma \mathfrak{v}) \\ &= \mu\big[\eth_\flat(\mathcal{D}^\gamma \tau, \mathcal{D}^\gamma \mathfrak{r}^*) + \eth_\flat(\mathcal{D}^\gamma \mathfrak{r}^*, \mathcal{D}^\gamma \mathfrak{v})\big] \\ &\leq \mu \rho^\gamma \eth_\flat(\tau, \mathfrak{r}^*) + \mu \rho^\gamma \eth_\flat(\mathfrak{r}^*, \mathfrak{v}). \end{aligned}$$

Letting limit as $\gamma \to \infty$ in the above inequality, we have

$$\eth_\flat(\tau, \mathfrak{v}) = 0 \quad \Longrightarrow \quad \tau = \mathfrak{v}.$$

Therefore, our assumption has a contradiction. Then, $\tau = \mathfrak{v}$. Hence, $\mathcal{D}$ has a unique f.p. in $\mathcal{Q}^*$. □

Corollary 3 *Let $(\mathcal{Q}^*, \perp, \eth_\flat)$ be an O-complete bMS and $\eth_\flat$ be an O-continuous; also, $\mathcal{D}: \mathcal{Q}^* \to \mathcal{Q}^*$ is an O-continuous map. Assume that $\theta_1, \theta_2 \in (0,1)$ exists satisfying $\theta_1 + \theta_2 < 1$ s.t. for any $\xi, \eta \in \mathcal{Q}^*$ with $\xi \perp \eta$*

$$\eth_\flat(\mathcal{D}^2\xi, \mathcal{D}^2\eta) \le \theta_1 \eth_\flat(\xi, \eta) + \theta_2 \eth_\flat(\mathcal{D}\xi, \mathcal{D}\eta),$$

then $\mathcal{D}$ has a unique f.p.

Theorem 3.4 *Let $(\mathcal{Q}^*, \perp, \eth_\flat)$ be an O-complete bMS, with an orthogonal element $\mathfrak{r}_0$ and constant $\mu \ge 1$, and let $\mathcal{D}: \mathcal{Q}^* \to \mathcal{Q}^*$ be an orthogonal hybrid interpolative RI-type contraction map. Suppose that*

(a) *$\mathcal{D}^2$ is an O-continuous;*
(b) *$\mathcal{D}$ is an O-π orbital-admissible map;*
(c) *$\xi_0 \in \mathcal{Q}^*$ exists s.t. $\pi(\xi_0, \mathcal{D}\xi_0) \ge 1$;*
(d) *$\pi(\xi, \mathcal{D}\xi) \ge 1$ for all $\xi \in \mathrm{Fix}_{\mathcal{D}^2}(\mathcal{Q}^*)$;*

then $\mathcal{D}$ has a unique f.p.

Proof Let $(\mathcal{Q}^*, \perp)$ is an orthogonal set, there exists

$$\mathfrak{r}_0 \in \mathcal{Q}^* : \forall \mathfrak{r} \in \mathcal{Q}^*, \mathfrak{r} \perp \mathfrak{r}_0 \quad \text{(or)} \quad \forall \mathfrak{r} \in \mathcal{Q}^*, \mathfrak{r}_0 \perp \mathfrak{r}.$$

It follows that $\mathfrak{r}_0 \perp \mathcal{D}\mathfrak{r}_0$ or $\mathcal{D}\mathfrak{r}_0 \perp \mathfrak{r}_0$. Let

$$\mathfrak{r}_1 = \mathcal{D}\mathfrak{r}_0, \qquad \mathfrak{r}_2 = \mathcal{D}\mathfrak{r}_1 = \mathcal{D}^2\mathfrak{r}_0 \cdots \mathfrak{r}_\gamma = \mathcal{D}\mathfrak{r}_{\gamma-1} = \mathcal{D}^\gamma \mathfrak{r}_0, \quad \forall \gamma \in \mathbb{N}.$$

For any $\mathfrak{r}_0 \in \mathcal{Q}^*$, set $\mathfrak{r}_\gamma = \mathcal{D}\mathfrak{r}_{\gamma-1}$. Now, we consider the following two cases:

(i) If $\exists \gamma \in \mathbb{N} \cup \{0\}$ s.t $\mathfrak{r}_\gamma = \mathfrak{r}_{\gamma+1}$, then we have $\mathcal{D}\mathfrak{r}_\gamma = \mathfrak{r}_\gamma$. Obviously, $\mathfrak{r}_\gamma$ is a f.p. of $\mathcal{D}$. Hence, the proof is finished.
(ii) If $\mathfrak{r}_\gamma \neq \mathfrak{r}_{\gamma+1}$, for any $\gamma \in \mathbb{N} \cup \{0\}$, then we obtain $\eth_\flat(\mathfrak{r}_{\gamma+1}, \mathfrak{r}_\gamma) > 0$, for every $\gamma \in \mathbb{N}$.

Since $\mathcal{D}$ is an O-preserving, we have

$$\mathfrak{r}_\gamma \perp \mathfrak{r}_{\gamma+1} \quad \text{(or)} \quad \mathfrak{r}_{\gamma+1} \perp \mathfrak{r}_\gamma.$$

Therefore $\{\mathfrak{r}_\gamma\}$ is an O-sequence.

Let $\{\mathfrak{r}_\gamma\}$ be an O-sequence of $\mathcal{D}$ based on ξ_0 defined by $\xi_\gamma = \mathcal{D}^\gamma \xi_0$. By orthogonal completeness of $\mathcal{D}$, it follows that

$$\eth_\flat(\xi^*, \mathcal{D}^2\xi^*) = \lim_{\gamma\to\infty} \eth_\flat(\xi_{\gamma+1}, \mathcal{D}^2\xi^*) = \lim_{\gamma\to\infty} \eth_\flat(\mathcal{D}^2\xi_\gamma, \mathcal{D}^2\xi^*) = 0,$$

that is $\xi^* = \mathcal{D}^2\xi^*$. Therefore ξ^* is a f.p. of $\mathcal{D}^2$.

Since $\mathcal{D}$ is an O-π orbital-admissible, we get

$$\begin{aligned} 0 \le \mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*) &\le \pi(\xi^*, \mathcal{D}\xi^*)\mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*) \\ &\le \pi(\xi^*, \mathcal{D}\xi^*)\mathfrak{d}_\flat(\mathcal{D}\xi^*, \mathcal{D}^2\xi^*) \\ &\le \phi\mathbb{M}(\xi^*, \mathcal{D}\xi^*). \end{aligned} \tag{9}$$

Now, we choose $\hbar$ to discuss the possible cases.

Case-I: If $\hbar > 0$

$$\begin{aligned} \mathbb{M}(\xi^*, \mathcal{D}\xi^*) &= \big[\theta_1\mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*)^\hbar + \theta_2\mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*)^\hbar + \theta_3\mathfrak{d}_\flat(\mathcal{D}\xi^*, \mathcal{D}^2\xi^*)^\hbar \\ &\quad + \theta_4\mathfrak{d}_\flat(\mathcal{D}\xi^*, \mathcal{D}^2\xi^*)^\hbar + \theta_5\mathfrak{d}_\flat(\mathcal{D}\xi^*, \mathcal{D}^2\xi^*)^\hbar + \lambda\mathfrak{d}_\flat(\mathcal{D}^2\xi^*, \mathcal{D}^3\xi^*)^\hbar\big]^{\frac{1}{\hbar}} \\ &= \big[\theta_1\mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*)^\hbar + \theta_2\mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*)^\hbar + \theta_3\mathfrak{d}_\flat(\mathcal{D}\xi^*, \xi^*)^\hbar \\ &\quad + \theta_4\mathfrak{d}_\flat(\mathcal{D}\xi^*, \xi^*)^\hbar + \theta_5\mathfrak{d}_\flat(\mathcal{D}\xi^*, \xi^*)^\hbar + \lambda\mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*)^\hbar\big]^{\frac{1}{\hbar}} \\ &= \big[(\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \lambda)\mathfrak{d}_\flat(\mathcal{D}\xi^*, \xi^*)^\hbar\big]^{\frac{1}{\hbar}} \\ &\le \big[\mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*)^\hbar\big]^{\frac{1}{\hbar}} \\ &= \mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*). \end{aligned}$$

This implies contradiction in (9).

Case-II: If $\hbar = 0$

$$\begin{aligned} \mathbb{M}(\xi^*, \mathcal{D}\xi^*) &= \mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*)^{\theta_1}.\mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*)^{\theta_2}.\mathfrak{d}_\flat(\mathcal{D}\xi^*, \mathcal{D}^2\xi^*)^{\theta_3} \\ &\quad .\mathfrak{d}_\flat(\mathcal{D}\xi^*, \mathcal{D}^2\xi^*)^{\theta_4}.\mathfrak{d}_\flat(\mathcal{D}\xi^*, \mathcal{D}^2\xi^*)^{\theta_5}.\mathfrak{d}_\flat(\mathcal{D}^2\xi^*, \mathcal{D}^3\xi^*)^{\lambda} \\ &= \mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*)^{\theta_1}.\mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*)^{\theta_2}.\mathfrak{d}_\flat(\mathcal{D}\xi^*, \xi^*)^{\theta_3} \\ &\quad .\mathfrak{d}_\flat(\mathcal{D}\xi^*, \xi^*)^{\theta_4}.\mathfrak{d}_\flat(\mathcal{D}\xi^*, \xi^*)^{\theta_5}.\mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*)^{\lambda} \\ &= \mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*)^{\theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\lambda} \\ &= \mathfrak{d}_\flat(\xi^*, \mathcal{D}\xi^*), \end{aligned}$$

which is again a contradiction to (9). Hence, Corollary 2 concludes that $\{\mathfrak{r}_\gamma\}$ is an O-Cauchy sequence. As $\mathcal{Q}^*$ is an O-complete, $\exists\mathfrak{r}^* \in \mathcal{Q}^*$ s.t.

$$\mathfrak{d}_\flat(\mathfrak{r}^*, \mathcal{D}\mathfrak{r}^*) = \lim_{\gamma\to\infty}\mathfrak{d}_\flat(\mathfrak{r}_{\gamma+1}, \mathcal{D}\mathfrak{r}^*) = \lim_{\gamma\to\infty}\mathfrak{d}_\flat(\mathcal{D}\mathfrak{r}_\gamma, \mathcal{D}\mathfrak{r}^*) = 0,$$

thus $\mathcal{D}\mathfrak{r}^* = \mathfrak{r}^*$ and $\mathcal{D}\mathfrak{r}^* = \mathfrak{r}^*$. Hence, the point $\mathfrak{r}^*$ is a f.p. of $\mathcal{D}$.

Next, we show that τ is a unique f.p. of $\mathcal{D}$.

Suppose that τ and $\mathfrak{v}$ are two different f.p. of $\mathcal{D}$. Consider $\mathcal{D}^\gamma\tau = \tau \neq \mathcal{D}^\gamma\mathfrak{v} = \mathfrak{v}$ for all $\tau, \mathfrak{v} \in \mathbb{N}$. By choosing $\mathfrak{r}^*$, we obtain

$$(\mathfrak{r}^* \perp \tau, \mathfrak{r}^* \perp \mathfrak{v}) \quad \text{or} \quad (\tau \perp \mathfrak{r}^*, \mathfrak{v} \perp \mathfrak{r}^*).$$

Since $\mathcal{D}$ is an $\perp$-preserving, we have

$$\left(\mathcal{D}^{\gamma}\mathfrak{r}^{*} \perp \mathcal{D}^{\gamma}\tau, \mathcal{D}^{\gamma}\mathfrak{r}^{*} \perp \mathcal{D}^{\gamma}\mathfrak{v}\right) \quad \text{or} \quad \left(\mathcal{D}^{\gamma}\tau \perp \mathcal{D}^{\gamma}\mathfrak{r}^{*}, \mathcal{D}^{\gamma}\mathfrak{v} \perp \mathcal{D}^{\gamma}\mathfrak{r}^{*}\right),$$

for all $\tau, \mathfrak{v} \in \mathbb{N}$. Therefore, by using the triangle inequality, we obtain

$$\begin{aligned} \mathfrak{d}_{\flat}(\tau, \mathfrak{v}) &= \mathfrak{d}_{\flat}\left(\mathcal{D}^{\gamma}\tau, \mathcal{D}^{\gamma}\mathfrak{v}\right) \\ &= \mu\left[\mathfrak{d}_{\flat}\left(\mathcal{D}^{\gamma}\tau, \mathcal{D}^{\gamma}\mathfrak{r}^{*}\right) + \mathfrak{d}_{\flat}\left(\mathcal{D}^{\gamma}\mathfrak{r}^{*}, \mathcal{D}^{\gamma}\mathfrak{v}\right)\right] \\ &\leq \mu\rho^{\gamma}\mathfrak{d}_{\flat}\left(\tau, \mathfrak{r}^{*}\right) + \mu\rho^{\gamma}\mathfrak{d}_{\flat}\left(\mathfrak{r}^{*}, \mathfrak{v}\right). \end{aligned} \tag{10}$$

Setting limit as $\gamma \to \infty$ in (10), we have

$$\mathfrak{d}_{\flat}(\tau, \mathfrak{v}) = 0 \quad \Longrightarrow \quad \tau = \mathfrak{v}.$$

Therefore, our assumption has a contradiction. Then, $\tau = \mathfrak{v}$. Hence, $\mathcal{D}$ has a unique f.p. in $\mathcal{Q}^{*}$. □

Example 3.5 Consider the space $\mathcal{Q}^{*} = [-1, 1]$ provided with an orthogonal b-metric $\mathfrak{d}_{\flat}$ on $\mathbb{R}^{+}$. Let the binary relation $\perp$ on $\mathcal{Q}^{*}$ by $\xi \perp \eta$ if $\xi, \eta \geq 0$, for every $\xi, \eta \in \mathcal{Q}^{*}$.

Let $\mathfrak{d}_{\flat} : \mathcal{Q}^{*} \times \mathcal{Q}^{*} \to (0, \infty)$ be defined as

$$\mathfrak{d}_{\flat}(\xi, \eta) = |\xi - \eta|^{2}.$$

Clearly, $(\mathcal{Q}^{*}, \perp, \mathfrak{d}_{\flat})$ be an O-complete *bMS*.

Let $\mathcal{D} : \mathcal{Q}^{*} \to \mathcal{Q}^{*}$ be defined as

$$\mathcal{D}\xi = \begin{cases} \sqrt{1 - \xi^{2}} & \text{if } -1 \leq \xi \leq 0, \\ \frac{\xi^{2}}{2} & \text{if } 0 \leq \xi \leq 1, \end{cases}$$

then

$$\mathcal{D}^{2}\xi = \begin{cases} \frac{1-\xi^{2}}{2} & \text{if } -1 \leq \xi \leq 0, \\ \frac{\xi^{4}}{8} & \text{if } 0 \leq \xi \leq 1. \end{cases}$$

Next, define $\pi : \mathcal{Q}^{*} \times \mathcal{Q}^{*} \to [0, \infty)$, by

$$\pi(\xi, \eta) = \begin{cases} \frac{3}{2} & \text{if } 0 \leq \xi \leq 1, \\ 1 & \text{if } \eta = 1, \xi = -1, \\ 0 & \text{if otherwise.} \end{cases}$$

Clearly, $\mathcal{Q}^{*}$ is an O-preserving.

Now, we verify that orthogonal hybrid interpolative RI-type contractions, for $0 \leq \xi \leq 1$.

$$\pi(\xi, \eta)\mathfrak{d}_{\flat}\left(\mathcal{D}^{2}\xi, \mathcal{D}^{2}\eta\right) = \frac{3}{(2)(8)}\left|\xi^{4} - \eta^{4}\right|$$

$$\begin{aligned}
&= \frac{3}{(2)(8)}\left|\left(\xi^2-\eta^2\right)\left(\xi^2+\eta^2\right)\right| \\
&\le \frac{3}{8}\left|\xi^2-\eta^2\right| \\
&= \frac{3}{8}\sqrt{\left|\xi^2-\eta^2\right|}\sqrt{\left|\xi^2-\eta^2\right|} \\
&= \frac{3}{8}\sqrt{|\xi-\eta|.|\xi+\eta|}\sqrt{\frac{2|\xi^2-\eta^2|}{2}} \\
&\le \frac{3}{4}\mathfrak{d}_\flat(\xi,\eta)^{\frac{1}{4}}\mathfrak{d}_\flat(\mathcal{D}\xi,\mathcal{D}\eta)^{\frac{1}{4}}.
\end{aligned}$$

For $\xi = -1, \eta = 1$, we obtain

$$\begin{aligned}
\pi(\xi,\eta)\mathfrak{d}_\flat\left(\mathcal{D}^2\xi,\mathcal{D}^2\eta\right) &= \frac{1}{8} < \frac{3}{4} \\
&= \frac{3}{4}\mathfrak{d}_\flat(\xi,\eta)^{\frac{1}{4}}\mathfrak{d}_\flat(\mathcal{D}\xi,\mathcal{D}\eta)^{\frac{1}{4}}.
\end{aligned}$$

It is easy to see that $\mathcal{D}$ is an O-continuous with $\delta = 0$. Therefore, all the hypothesis of Theorem 3.3 are fulfilled. Hence, $\mathcal{D}$ has a unique f.p.

Example 3.6 Consider the space $\mathcal{Q}^* = [-1,1]$ provided with an orthogonal b-metric $\mathfrak{d}_\flat$ on $\mathbb{R}^+$. Let the binary relation $\perp$ on $\mathcal{Q}^*$ by $\xi \perp \eta$ if $\xi,\eta \ge 0$, for every $\xi,\eta \in \mathcal{Q}^*$.

Let $\mathfrak{d}_\flat : \mathcal{Q}^* \times \mathcal{Q}^* \to (0,\infty)$ be defined as

$$\mathfrak{d}_\flat(\xi,\eta) = |\xi-\eta|^2.$$

Clearly, $(\mathcal{Q}^*,\perp,\mathfrak{d}_\flat)$ be an O-complete *bMS*.

Let $\mathcal{D} : \mathcal{Q}^* \to \mathcal{Q}^*$ be defined as

$$\mathcal{D}\xi = \begin{cases} 3 & \text{if } -1 \le \xi \le 0, \\ 2 & \text{if } 0 \le \xi \le 1, \end{cases}$$

then

$$\mathcal{D}^2\xi = \begin{cases} 1 & \text{if } -1 \le \xi \le 0, \\ 4 & \text{if } 0 \le \xi \le 1. \end{cases}$$

Next, define $\pi : \mathcal{Q}^* \times \mathcal{Q}^* \to [0,\infty)$, by

$$\pi(\xi,\eta) = \begin{cases} 1.5 & \text{if } 0 \le \xi \le 1, \\ 1 & \text{if } \eta = 1, \xi = -1, \\ 0 & \text{if otherwise.} \end{cases}$$

Clearly, $\mathcal{Q}^*$ is an O-preserving.

Now, we verify that orthogonal hybrid interpolative RI-type contractions, for $0 \le \xi \le 1$, we obtain

$$0 = 1.5\mathfrak{d}_\flat(1,1) = \pi(\xi,\eta)\mathfrak{d}_\flat\left(\mathcal{D}^2\xi,\mathcal{D}^2\eta\right) \le \phi\mathbb{M}(\xi,\eta).$$

For $\xi = -1, \eta = 1$, we obtain

$$9 = (1)\mathfrak{d}_\flat(1,4) = \pi(\xi,\eta)\mathfrak{d}_\flat\big(\mathcal{D}^2\xi, \mathcal{D}^2\eta\big) \le \phi\mathbb{M}(\xi,\eta). \tag{11}$$

Now, we chose $\hbar$ to discuss the possible cases.

Case I: if $\hbar > 0$, we get

$$\begin{aligned}\mathbb{M}(\xi,\eta) = \big[&\theta_1\mathfrak{d}_\flat(\xi,\eta)^\hbar + \theta_2\mathfrak{d}_\flat(\xi,\mathcal{D}\xi)^\hbar + \theta_3\mathfrak{d}_\flat(\eta,\mathcal{D}\eta)^\hbar \\ &+ \theta_4\mathfrak{d}_\flat(\mathcal{D}\xi,\mathcal{D}\eta)^\hbar + \theta_5\mathfrak{d}_\flat\big(\mathcal{D}\xi,\mathcal{D}^2\eta\big)^\hbar + \lambda\mathfrak{d}_\flat\big(\mathcal{D}\eta,\mathcal{D}^2\eta\big)^\hbar\big]^{\frac{1}{\hbar}}.\end{aligned}$$

Taking $\theta_1 = \theta_2 = 0.5, \theta_3 = \theta_4 = \theta_5 = 0.4, \lambda = 0.1, \phi = 0.8$ and $\hbar = 2$ in (11), we obtain

$$\begin{aligned}9 &\le 0.8\big[0.5\mathfrak{d}_\flat(-1,1)^2 + 0.5\mathfrak{d}_\flat(-1,3)^2 + 0.4\mathfrak{d}_\flat(1,2)^2 + 0.4\mathfrak{d}_\flat(3,2)^2 \\ &\quad + 0.4\mathfrak{d}_\flat(3,4)^2 + 0.1\mathfrak{d}_\flat(2,4)^2\big]^{\frac{1}{2}} \\ &\le 0.8\big[(0.5)|-1-1|^4 + (0.5)|-1-3|^4 + (0.4)|1-2|^4 + (0.4)|3-2|^4 \\ &\quad + (0.4)|3-4|^4 + (0.1)|2-4|^4\big]^{\frac{1}{2}} \\ &\le 0.8\big[0.5(16) + 0.5(256) + 0.4(1) + 0.4(1) + 0.4(1) + 0.1(16)\big]^{0.5} \\ &\le 0.8[8 + 128 + 0.4 + 0.4 + 0.4 + 1.6]^{0.5} = 0.8(138.8)^{0.5} \\ 9 &\le 9.42.\end{aligned}$$

Case II: if $\hbar = 0$, we get

$$\begin{aligned}\mathbb{M}(\xi,\eta) = {}&\mathfrak{d}_\flat(\xi,\eta)^{\theta_1}.\mathfrak{d}_\flat(\xi,\mathcal{D}\xi)^{\theta_2}.\mathfrak{d}_\flat(\eta,\mathcal{D}\eta)^{\theta_3}.\mathfrak{d}_\flat(\mathcal{D}\xi,\mathcal{D}\eta)^{\theta_4} \\ &.\mathfrak{d}_\flat\big(\mathcal{D}\xi,\mathcal{D}^2\eta\big)^{\theta_5}.\mathfrak{d}_\flat\big(\mathcal{D}\eta,\mathcal{D}^2\eta\big)^{\lambda}.\end{aligned}$$

Taking $\theta_1 = \theta_2 = 0.6, \theta_3 = \theta_4 = \theta_5 = 0.3, \phi = 0.8$ and $\lambda = 0.1$ in (11), we obtain

$$\begin{aligned}9 &\le 0.8\big[\mathfrak{d}_\flat(-1,1)^{0.6}.\mathfrak{d}_\flat(-1,3)^{0.6}.\mathfrak{d}_\flat(1,2)^{0.3}.\mathfrak{d}_\flat(3,2)^{0.3}.\mathfrak{d}_\flat(3,4)^{0.3}.\mathfrak{d}_\flat(2,4)^{0.1}\big] \\ &\le 0.8\big[\big(4^{0.6}\big).\big(16^{0.6}\big).\big(1^{0.3}\big).\big(1^{0.3}\big).\big(1^{0.3}\big).\big(4^{0.1}\big)\big] \\ &\le 0.8\big[(2.2974).(5.2780).(1).(1).(1).(1.1487)\big] = 0.8(13.9287) \\ 9 &\le 11.14.\end{aligned}$$

Otherwise, we obtain $\pi(\xi,\eta) = 0$.

Clearly, $\mathcal{D}$ is an O-continuous. Therefore, all the hypothesis of Theorem 3.3 are fulfilled. Hence, $\mathcal{D}$ has a unique f.p.

4 Application

In this segment, we find an existence and unique solution for a Fredhlom integral equation.

Consider a Fredholm integral equation

$$\varrho(\aleph) = \mathfrak{f}(\aleph) + \int_0^1 \gamma_\varrho\big(\aleph, \mathfrak{r}^*, \varrho\big(\mathfrak{r}^*\big)\big)\, d\mathfrak{r}^*, \aleph \in [0,1]. \tag{12}$$

4.1 The theorem that follows supports orthogonality

Theorem 4.1 *Let $\mathcal{B}^\infty = [0,1]$ and $\mathcal{Q}^* = \mathcal{C}(\mathcal{B}^\infty, \mathbb{R}^2)$ be the family of all O-continuous functions defined from $\mathcal{B}^\infty$ to $\mathbb{R}^2$, and the given axioms hold*:

(1) *Let $\gamma_\varrho : \mathcal{B}^\infty \times \mathcal{B}^\infty \times \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathfrak{f} : \mathcal{B}^\infty \to \mathbb{R}^2$ be an O-continuous*;

(2) *$\varrho_0 \in \mathcal{Q}^*$ exists s.t. $\varrho_\gamma = \mathcal{D}\varrho_{\gamma-1}$*;

(3) *A O-continuous function $\mathfrak{f} : \mathcal{B}^\infty \times \mathcal{B}^\infty \to \mathcal{B}^\infty$ exists s.t.*

$$\left|\gamma_\varrho\left(\xi, \xi^*, \varrho\left(\xi^*\right)\right) - \gamma_\omega\left(\xi, \xi^*, \omega\left(\xi^*\right)\right)\right|^{\lambda^*} \le \left|\mathfrak{f}\left(\varrho\left(\xi^*\right), \omega\left(\xi^*\right)\right)\right|\left|\varrho\left(\xi^*\right) - \omega\left(\xi^*\right)\right|^{\lambda^*},$$

for each $\xi, \xi^* \in \mathcal{B}^\infty$ and $|\mathfrak{f}(\varrho(\xi^*), \omega(\xi^*))| \le \frac{1}{\nu}$, where $\nu > 0$.

Then, (12) *has a unique solution*.

Proof Define the orthogonal relation $\perp$ on $\mathcal{Q}^*$ by

$$\varrho \perp \omega \quad \Longleftrightarrow \quad \varrho(\aleph)\omega(\aleph) \ge \varrho(\aleph) \quad \text{or} \quad \varrho(\aleph)\omega(\aleph) \ge \omega(\aleph), \quad \forall \aleph \in [0,1].$$

Define a function $\mathfrak{d}_\flat : \mathcal{Q}^* \times \mathcal{Q}^* \to [0, \infty)$ by

$$\mathfrak{d}_\flat(\varrho, \omega) = \|\varrho - \omega\|_\infty = \sup_{\xi \in \mathcal{B}^\infty} \left\{\left|\varrho(\xi) - \omega(\xi)\right|^{\lambda^*}\right\}, \quad \lambda^* > 1,$$

for all $\varrho, \omega \in \mathcal{Q}^*$ with $\varrho \perp \omega$. Clearly, $(\mathcal{Q}^*, \perp, \mathfrak{d}_\flat)$ is an *O*-complete *bMS*.

Define a map $\mathcal{D} : \mathcal{Q}^* \to \mathcal{Q}^*$, as

$$\mathcal{D}\left(\varrho(\aleph)\right) = \mathfrak{f}(\aleph) + \int_0^1 \gamma_\varrho\left(\aleph, \mathfrak{r}^*, \varrho\left(\mathfrak{r}^*\right)\right) d\mathfrak{r}^*.$$

Now, we prove that $\mathcal{D}$ is an *O*-preserving. For every $\varrho, \omega \in \rho$ with $\varrho \perp \omega$ and $\xi \in \mathcal{Q}^*$, we get

$$\mathcal{D}\left(\varrho(\aleph)\right) = \mathfrak{f}(\aleph) + \int_0^1 \gamma_\varrho\left(\aleph, \mathfrak{r}^*, \varrho\left(\mathfrak{r}^*\right)\right) \ge 1.$$

It follows that $[(\mathcal{D}\varrho)(\xi)][(\mathcal{D}\omega)(\xi)] \ge (\mathcal{D}\omega)(\xi)$ and so $(\mathcal{D}\varrho)(\xi) \perp (\mathcal{D}\omega)(\xi)$. Then, $\mathcal{D}$ is an $\perp$-preserving.

Since $(\mathcal{Q}^*, \perp, \mathfrak{d}_\flat)$ be an *O*-complete *bMS* and $\pi(\varrho, \omega) = 1$. Consider $\hbar^* > 1$ s.t. $\frac{1}{\hbar^*} + \frac{1}{\lambda^*} = 1$, then there exists $\varrho^* \in \mathcal{D}(\varrho)$ and we obtain

$$\begin{aligned}
\mathfrak{d}_\flat\left(\mathcal{D}\varrho^*(\xi), \mathcal{D}\omega^*(\xi)\right) &= \sup_{\xi \in \mathcal{B}^\infty} \left|\mathcal{D}\varrho^*(\xi), \mathcal{D}\omega^*(\xi)\right|^{\lambda^*} \\
&= \sup_{\xi \in \mathcal{B}^\infty} \left|\int_0^1 \gamma_\varrho\left(\xi, \xi^*, \varrho\left(\xi^*\right)\right) - \gamma_\omega\left(\xi, \xi^*, \omega\left(\xi^*\right)\right)\right|^{\lambda^*} d\xi^* \\
&\le \sup_{\xi \in \mathcal{B}^\infty} \Bigg[\left(\int_0^1 |1|^{\hbar^*} d\xi^*\right)^{\frac{1}{\hbar^*}} \\
&\quad \times \int_0^1 \left(\left|\gamma_\varrho\left(\xi, \xi^*, \varrho\left(\xi^*\right)\right) - \gamma_\omega\left(\xi, \xi^*, \omega\left(\xi^*\right)\right)\right|^{\lambda^*}\right)^{\frac{1}{\lambda^*}}\Bigg]^{\lambda^*} d\xi^* \\
&= \sup_{\xi \in \mathcal{B}^\infty} \int_0^1 \left|\gamma_\varrho\left(\xi, \xi^*, \varrho\left(\xi^*\right)\right) - \gamma_\omega\left(\xi, \xi^*, \omega\left(\xi^*\right)\right)\right|^{\lambda^*} d\xi^*
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{\xi\in\mathcal{B}^\infty}\int_0^1 \left|\mathfrak{f}\big(\varrho(\xi^*),\omega(\xi^*)\big)\right|\left|\varrho(\xi^*)-\omega(\xi^*)\right|^{\lambda^*} d\xi^* \\
&\leq \frac{1}{\nu}\left\|\varrho(\xi^*)-\omega(\xi^*)\right\|_\infty \\
&\leq \frac{1}{\nu}\mathfrak{d}_\flat\big(\varrho(\xi^*),\omega(\xi^*)\big).
\end{aligned}
$$

Similarly, one can easily obtain

$$
\begin{aligned}
\mathfrak{d}_\flat\big(\mathcal{D}^2\varrho^*(\xi),\mathcal{D}^2\omega^*(\xi)\big) &= \sup_{\xi\in\mathcal{B}^\infty}\left|\mathcal{D}^2\varrho^*(\xi),\mathcal{D}^2\omega^*(\xi)\right|^{\lambda^*} \\
&\leq \left(\frac{1}{\nu}\right)^2\mathfrak{d}_\flat\big(\varrho(\xi^*),\omega(\xi^*)\big) \\
&= \left(\frac{1}{\nu}\right)^2\mathbb{M}\mathfrak{d}_\flat\big(\varrho(\xi^*),\omega(\xi^*)\big).
\end{aligned}
$$

Each and every hypothesis of the Theorem 3.3 are fulfiled by choosing that $\phi(\frac{1}{\nu})^2\in(0,1)$ and

$$
\theta_1 = 1, \qquad \theta_2=\theta_3=\theta_4=\theta_5=\delta=0.
$$

Therefore the Fredholm integral Equation (12) has a unique solution. □

4.2 The theorem that follows does not support orthogonality

Theorem 4.2 *Let $\mathcal{B}^\infty=[0,1]$ and $\mathcal{Q}^*=\mathcal{C}(\mathcal{B}^\infty,\mathbb{R}^2)$ are the family of all continuous functions defined from $\mathcal{B}^\infty$ to $\mathbb{R}^2$, and the given axioms are hold*:

(1) *Let $\gamma_\varrho:\mathcal{B}^\infty\times\mathcal{B}^\infty\times\mathbb{R}^2\to\mathbb{R}^2$ and $\mathfrak{f}:\mathcal{B}^\infty\to\mathbb{R}^2$ be a continuous*;

(2) *$\varrho_0\in\mathcal{Q}^*$ exists s.t. $\varrho_\gamma=\mathcal{D}\varrho_{\gamma-1}$*;

(3) *A continuous function $\mathfrak{f}:\mathcal{B}^\infty\times\mathcal{B}^\infty\to\mathcal{B}^\infty$ exists s.t.*

$$
\left|\gamma_\varrho\big(\xi,\xi^*,\varrho(\xi^*)\big)-\gamma_\omega\big(\xi,\xi^*,\omega(\xi^*)\big)\right|^{\lambda^*}\leq\left|\mathfrak{f}\big(\varrho(\xi^*),\omega(\xi^*)\big)\right|\left|\varrho(\xi^*)-\omega(\xi^*)\right|^{\lambda^*},
$$

for each $\xi,\xi^*\in\mathcal{B}^\infty$ and $|\mathfrak{f}(\varrho(\xi^*),\omega(\xi^*))|\leq\frac{1}{\nu}$, where $\nu>0$.

Then, (12) *has a unique solution.*

Proof Define a function $\mathfrak{d}_\flat:\mathcal{Q}^*\times\mathcal{Q}^*\to[0,\infty)$ by

$$
\mathfrak{d}_\flat(\varrho,\omega)=\|\varrho-\omega\|_\infty=\sup_{\xi\in\mathcal{B}^\infty}\left\{\left|\varrho(\xi)-\omega(\xi)\right|^{\lambda^*}\right\},\lambda^*>1,\quad\forall\varrho,\omega\in\mathcal{Q}^*.
$$

Consider the sequence $\{\mathfrak{r}_\gamma\}$ in $\mathcal{Q}^*$ that converges at a point $\mathfrak{r}$ if

$$
\lim_{\gamma\to\infty}\big(\mathfrak{d}_\flat(\mathfrak{r}_\gamma,\mathfrak{r})\big)=0.
$$

And a sequence $\{\mathfrak{r}_\gamma\},\{\mathfrak{r}_\ell\}$ in $\mathcal{Q}^*$ is a Cauchy sequence if

$$
\lim_{\gamma,\ell\to\infty}\big(\mathfrak{d}_\flat(\mathfrak{r}_\gamma,\mathfrak{r}_\ell)\big)<\infty.
$$

Clearly, $(\mathcal{Q}^*,\mathfrak{d}_\flat)$ be a complete *bMS*.

Define a map $\mathcal{D}: \mathcal{Q}^* \to \mathcal{Q}^*$, as

$$\mathcal{D}(\varrho(\aleph)) = \mathfrak{f}(\aleph) + \int_0^1 \gamma_\varrho(\aleph, \mathfrak{r}^*, \varrho(\mathfrak{r}^*))\, d\mathfrak{r}^*.$$

Since $(\mathcal{Q}^*, \mathfrak{d}_\flat)$ is a complete bMS and $\pi(\varrho, \omega) = 1$. Consider $\hbar^* > 1$ s.t. $\frac{1}{\hbar^*} + \frac{1}{\lambda^*} = 1$, then there exists $\varrho^* \in \mathcal{D}(\varrho)$ and we obtain

$$\begin{aligned}
\mathfrak{d}_\flat(\mathcal{D}\varrho^*(\xi), \mathcal{D}\omega^*(\xi)) &= \sup_{\xi \in \mathcal{B}^\infty} |\mathcal{D}\varrho^*(\xi), \mathcal{D}\omega^*(\xi)|^{\lambda^*} \\
&= \sup_{\xi \in \mathcal{B}^\infty} \left| \int_0^1 \gamma_\varrho(\xi, \xi^*, \varrho(\xi^*)) - \gamma_\omega(\xi, \xi^*, \omega(\xi^*)) \right|^{\lambda^*} d\xi^* \\
&\le \sup_{\xi \in \mathcal{B}^\infty} \left[\left(\int_0^1 |1|^{\hbar^*} d\xi^* \right)^{\frac{1}{\hbar^*}} \right. \\
&\quad \left. \times \int_0^1 \left(|\gamma_\varrho(\xi, \xi^*, \varrho(\xi^*)) - \gamma_\omega(\xi, \xi^*, \omega(\xi^*))|^{\lambda^*} \right)^{\frac{1}{\lambda^*}} \right]^{\lambda^*} d\xi^* \\
&= \sup_{\xi \in \mathcal{B}^\infty} \int_0^1 |\gamma_\varrho(\xi, \xi^*, \varrho(\xi^*)) - \gamma_\omega(\xi, \xi^*, \omega(\xi^*))|^{\lambda^*} d\xi^* \\
&\le \sup_{\xi \in \mathcal{B}^\infty} \int_0^1 |\mathfrak{f}(\varrho(\xi^*), \omega(\xi^*))| |\varrho(\xi^*) - \omega(\xi^*)|^{\lambda^*} d\xi^* \\
&\le \frac{1}{\nu} \|\varrho(\xi^*) - \omega(\xi^*)\|_\infty \\
&\le \frac{1}{\nu} \mathfrak{d}_\flat(\varrho(\xi^*), \omega(\xi^*)).
\end{aligned}$$

Similarly, one can easily obtain

$$\begin{aligned}
\mathfrak{d}_\flat(\mathcal{D}^2\varrho^*(\xi), \mathcal{D}^2\omega^*(\xi)) &= \sup_{\xi \in \mathcal{B}^\infty} |\mathcal{D}^2\varrho^*(\xi), \mathcal{D}^2\omega^*(\xi)|^{\lambda^*} \\
&\le \left(\frac{1}{\nu}\right)^2 \mathfrak{d}_\flat(\varrho(\xi^*), \omega(\xi^*)) \\
&= \left(\frac{1}{\nu}\right)^2 \mathbb{M}\mathfrak{d}_\flat(\varrho(\xi^*), \omega(\xi^*)).
\end{aligned}$$

Each and every hypothesis of the Theorem 3.3 are fulfilled by choose that $\phi(\frac{1}{\nu})^2 \in (0,1)$ and

$$\theta_1 = 1, \qquad \theta_2 = \theta_3 = \theta_4 = \theta_5 = \delta = 0.$$

Therefore the Fredholm integral Equation (12) has a unique solution. □

Example 4.3 Consider the Fredholm integral equation as follows:

$$\upsilon(\aleph) = \mathfrak{f}(\aleph) + \int_0^\eta \mathcal{K}(\aleph, \wp, \upsilon(\wp))\, d\wp, \quad \forall 0 \le \eta \le 1, \tag{13}$$

Table 1 A comparison between approximate and exact numeric solutions

Iteration	Approximate solution	Exact solution	Absolute error
0.1	1.2214	−2.1804	3.4018
0.2	1.4918	−1.3111	2.8029
0.3	1.8221	−0.6573	2.4794
0.4	2.2255	−0.0261	2.2516
0.5	2.7183	0.6660	2.0523
0.6	3.3201	1.4812	1.8389
0.7	4.0552	2.4820	1.5732
0.8	4.9530	3.7393	1.2137
0.9	6.0496	5.3393	0.7103
1.0	7.3891	7.3891	0.0000

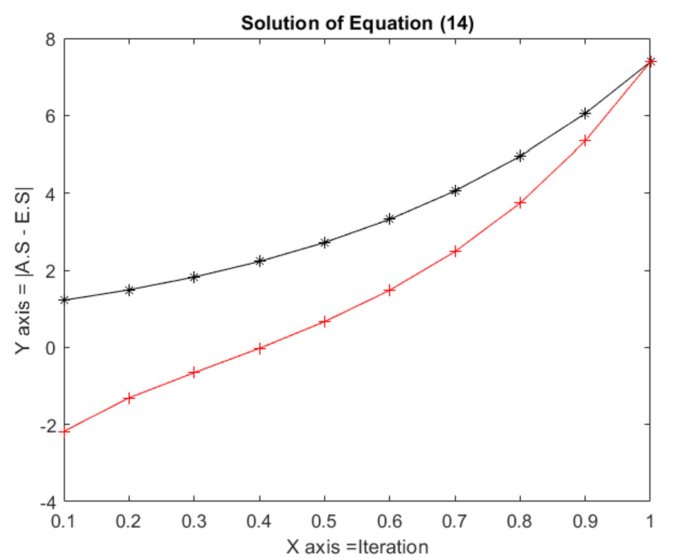

Figure 1 Shows that the f.p. of $\aleph$ is 1 and which is unique

where,

$$\upsilon(\aleph) = In\aleph + \int_0^\eta e^{2\aleph-2\wp}\upsilon(\wp)\,d\wp, \quad \forall 0 \le \eta \le 1. \tag{14}$$

Let us assume $\mathcal{K}(\aleph, \xi, \upsilon(\xi)) = e^{2\aleph}$ to be the exact solution of the Equation (14).

Hence, the absolute solution of given equation is $In\aleph + \aleph e^{2\aleph}$ for $\aleph > 0$. In Table 1 numerical results are given.

In the Figure 1, it is clear that $\mathcal{K}(\aleph, \xi, \upsilon(\xi)) = e^{2\aleph}$ is continuous. Therefore, Equation (14) has a unique solution. From Table 1, we see that the f.p. of $\aleph$ is 1 and it is a unique.

Comparison between approximate solution (A.S) and exact solution (E.S) shown in Figure 1.

5 Conclusions

In this paper, we extend the f.p results for orthogonal hybrid interpolative RI-type contractions in the surrounding area of an O-complete bMS. The non-trivial examples we derived were supported by our results. Finally, we demonstrated an application to prove analytical results for the integral equation with the algebraic result also proposed. It is an open problem to extend the solution to orthogonal metric spaces (Branciari metric spaces, G-metric spaces) by using hybrid interpolative RI-type contractions.

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