https://doi.org/10.1186/s13663-024-00760-7

Fixed Point Theory and Algorithms for Sciences and Engineering

RESEARCH Open Access

Comparative analysis of classical and Caputo models for COVID-19 spread: vaccination and stability assessment

Asifa Tassaddiq[1*], Sania Qureshi[2,3,4], Amanullah Soomro[2], Omar Abu Arqub[5] and Mehmet Senol[6]

*Correspondence: a.tassaddiq@mu.edu.sa
[1]Department of Basic Sciences and Humanities, College of Computer and Information Sciences, Majmaah University, Al-Majmaah, 11952, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

Several epidemiological models use the Caputo fractional-order differential operator without establishing its significance. This study verifies a Caputo operator-based fractional-order epidemiological model of the SAIVR type. COVID-19 kills. Infection weakens the immune system. The fractional Caputo operator describes COVID-19 immunization. Fundamental system characteristics are determined using fractional calculus. Our analysis included the fractional system's Hyers–Ulam–Rassias stability and stable states. The uniqueness and existence of fractional Caputo system solutions are explored. The least-squares approach determines system parameters. The Caputo fractional-order α value is optimized to 6.757e–01, indicating that the system best fits real-life medical data for infection. Caputo and classical systems were compared for absolute mean errors. The Box-Whisker chart case summaries show the Caputo operator superiority. When $\alpha \to 1$, the memory traces and hereditary traits are also observed. Finally, the Caputo fractional framework simulates COVID-19 using strong numerical methods.

Keywords: Fractional Caputo operator; Hyers–Ulam–Rassias stability analysis; Fractional differential equations; Approximate solutions

1 Introduction

The World Health Organization (WHO) called COVID-19 a pandemic after it spread to many countries on all continents in the first quarter of 2020. Different mutations of the new coronavirus have been observed in different regions, resulting in a wide range of symptoms. Fever, dry cough, and fatigue are the most common signs of illness, but other signs, such as a sore throat, diarrhea, loss of taste or smell, or a rash, are less prevalent. COVID-19 can spread through direct contact, indirect contact, droplet spray (like when someone sneezes), short-range transmission, airborne transmission (through aerosol), and long-range transmission.

Integer-order differential equations have been extensively explored for their application in epidemiological models of infectious diseases. Evidence from mathematical modeling of epidemics in the literature suggests that nonlinear dynamical equations can shed light on disease transmission dynamics. Constructing realistic nonlinear compartmental mathematical models that are data-driven to better explain the transmission dynamics of epidemics has garnered much attention in the wake of the recent global COVID-19 outbreaks [1–8], including several other important research studies on mathematical modeling with the Caputo and other memory operators, as discussed in [9–12].

Musa et al. [13] propose a novel eight-compartmental deterministic model for the COVID-19 epidemic in Nigeria, which accounts for awareness campaign initiatives and hospitalization control tactics for severe and moderate instances of infections. Infected cumulative instances in Nigeria were used to fit the suggested nonlinear dynamical model from 29 March 2020 to 12 June 2020. Their research shows that an increase in infections could happen in the population if programs to raise awareness are not done well. Some authors in [14] developed and tested a SEQIJR epidemic model that looks at how well quarantine and isolation work to stop COVID-19 outbreaks in Pakistan.

Caputo fractional differential operators are used in mathematical epidemiology to model the dynamics of infectious diseases in populations where memory effects play an important role. Unlike classical differential equations that assume the instantaneous response to perturbations, fractional calculus considers the possibility of memory effects, meaning that the system's response to a perturbation depends on the entire history of the system. In epidemiology, Caputo fractional differential operators have been used to model the spread of infectious diseases that have a long-lasting impact on individuals, such as measles or rubella. The Caputo fractional derivative of a function can be interpreted as the fractional order derivative of the function's memory, which can be used to capture the long-term effects of infection on an individual's immune system. Caputo fractional differential operators have also been used to model the effect of vaccination campaigns, where individuals who have been vaccinated have a lower probability of becoming infected in the future. Using fractional calculus, it is possible to model the long-term effects of vaccination on the dynamics of the disease, which can help inform public health policy decisions. Overall, the use of Caputo fractional differential operators in mathematical epidemiology allows for a more accurate representation of the dynamics of infectious diseases, taking into account the complex interplay between memory effects, disease transmission, and control measures. When modeling biological and engineering processes mathematically, fractional order differential equations are extremely valuable and powerful tools. This is due to the fact that the differential operators in such equations or models are associated with systems with memory dynamics, a property shared by the vast majority of biological and technical systems [15, 16]. To solve the problem of fractional-order models with control functions that change over time, authors in [17] develop a new and effective numerical method based on hybrid Chelyshkov functions.

The fractional calculus Caputo-Fabrizio derivative operator was used by Rezapour and coauthors in [18] to expand the nonlinear integer-order anthrax illness model established and assessed by Githire et al. [19]. The existence criterion of solutions was presented for the suggested fractional-order anthrax disease epidemic model using the Picard-Lindelof technique. Several infectious disease transmission dynamics, including those of HIV/AIDS [20], tuberculosis [21], malaria [22], dengue fever [23], Zika [24], Ebola [25], and hepatitis B [26], have been studied and analyzed using differential equations characterized by the Caputo fractional-order derivative.

In [27], the dynamics of reverse bifurcation are investigated in a simple but realistic model of a vaccine pandemic. To back up the theoretical results, the author provides qualitative and numerical simulations of the formulated mathematical model. The mathematical model of the cholera outbreak that the author developed in [28] was extended and investigated by Javidi et al. [29] to capture the Caputo fractional-order derivative. In [30], the author examined the uniform asymptotic stability of a few simple epidemic models (SIS, SIR, SIRS), as well as the well-known Ross vector-borne diseases in the Caputo sense using Lyapunov functions of the Volterra type.

In [31], the Caputo fractional derivative is used to analyze the fundamental mathematical model of SEIR, predicated on stochastic population dynamics. The authors provided an in-depth qualitative stability investigation of their novel and plausible deterministic model. The authors in [32] used a nonlinear system of differential equations in the sense of the Caputo fractional order derivative operator to form an epidemiological model for Zika virus infection, dividing the total human and mosquito populations into two compartmental classes (susceptible people and infected people; susceptible mosquitoes and infected mosquitoes). The authors of [33] came up with and studied an SEIR-type epidemic model based on classical and Caputo fractional-order differential operators to describe how the Rubella epidemic in Pakistan changed over time. In [34], a fractional system models liquid surface-stripe wave evolution. The Riemann-Liouville fractional derivative solves the fractional system analytically.

Recent research has attempted to mathematically model this devastating COVID-19 pandemic using some of these valuable differentiable operators, which are, in turn, derived from fractional-order differential equations [35]. Parameter estimation and numerical simulations for a nonlinear COVID-19 epidemiological model built with Caputo and Atangana-Baleanu fractional derivative operators were supplied by Naik et al. [36], along with a comprehensive qualitative analysis. Another study [37] used a nonlinear Atangana-Baleanu fractional-order differential equation model to analyze the COVID-19 pandemic in Nigeria. The latest research by Baleanu et al. [38] explored a Caputo-Fabrizio derivative version of the integer-order epidemic model introduced and studied by Chen et al. [39]. The uniqueness of the solution to the nonlinear Caputo-Fabrizio fractional order COVID-19 model was demonstrated using fixed point theory. They used the transform method of homotopy analysis and developed a convergent series approximation to the model problem. In [40], a Caputo fractional order deterministic epidemic model for COVID-19 infection was constructed and analyzed. To prove that there is a unique solution to the mathematical model, they turned to the well-known Banach contraction mapping concept. To add the importance of fractional theory, authors in [41] solve the $(2 + 1)$-dimensional elliptic nonlinear Schrodinger problem with three fractional operators and derive fractional analytical solutions.

It is difficult to predict how the present pandemic may influence people's choices to enroll in a COVID-19 vaccine clinical trial or to get immunized against the COVID-19 vaccine. France, which has the highest number of people who refuse to get vaccinated, is particularly concerned about this. The Centers for Disease Control and Prevention (CDC) classifies four of the top five counties in the United States with the highest percentage of totally vaccinated people (84.3%) as "high" transmission jurisdictions. Many countries consider vaccinated people unlikely to be a source of disease transmission. When thinking about ways to manage public health, it seems like it would be irresponsible not to consider the vaccinated population as a possible and major source of transmission.

In terms of the reasons for the lower COVID-19, the majority of research (25/35 = 71.4%) found that the male gender was an enabling factor in vaccine apprehension and increased desire for COVID-19 vaccinations. When subjects were older or had a PhD or higher education, the acceptance rate of the COVID-19 vaccination was higher in the 23 trials (23/35 = 65.7%). Vaccination was approved in 51.4 percent of the studies (18/35 = 51.4%). In addition, white or Asian ethnicity, higher income/education, history of chronic illness, familiarity with COVID-19 infection and sickness, employment in non-rural regions, and confidence that vaccinations may protect family and community members were all associated with greater acceptance of the COVID-19 immunization.

The present literature's use of fractional-order operators provided the impetus for developing an epidemiological model of the COVID-19 infection, which includes a class for the vaccinated population. This paper proposes an alternate SAIVR model to the model originally proposed in [42] under classical calculus, in which the spread of COVID-19 occurs directly between one class of vulnerable individuals and the infected patients being vaccinated against. The model uses the Caputo fractional-order derivative and is based on the coupling of nonlinear ordinary differential equations with real-world properties.

The following outline constitutes the paper's structure: The mathematical construction of a SAIVR nonlinear dynamical system is carried out in Sect. 2 with some explanations for the usage of the Caputo fractional derivative operator in Sect. 3. In Sect. 4, the biological parameters of the model are estimated. The qualitative analysis of the nonlinear model is discussed in detail in Sect. 5, which includes an examination of the existence, uniqueness, and steady states of the Caputo model. The simulations of the model are included in Sect. 6. Section 7 gives the conclusion and future research directions.

2 Mathematical formulation

Recently, in [42], a standard epidemic model for explaining COVID-19 transmission has been developed using the first-order time derivatives from classical calculus. The vaccinated SAIVR nonlinear dynamical model is a deterministic type of system based on five compartments whose flow chart is in Fig. 1, wherein the dynamics for the movement of individuals can easily be comprehended. The model is presented as follows:

$$\begin{aligned}
S'(t) &= -\beta_1 I(t)\frac{S(t)}{N} - \rho_1 A(t)\frac{S(t)}{N} - \delta\frac{S(t)}{N} + (1-\lambda)\epsilon V(t),\\
A'(t) &= \rho_1 A(t)\frac{S(t)}{N} + \beta_2 I(t)\frac{S(t)}{N} + \eta A(t)\frac{V(t)}{N} - \gamma A(t),\\
I'(t) &= \beta_1 I(t)\frac{S(t)}{N} + \rho_2 A(t)\frac{S(t)}{N} + \zeta I(t)\frac{V(t)}{N} - \gamma I(t),\\
V'(t) &= \delta\frac{S(t)}{N} - \eta A(t)\frac{V(t)}{N} - \zeta I(t)\frac{V(t)}{N} - \epsilon V(t),\\
R'(t) &= \gamma I(t) + \gamma A(t) + \lambda\epsilon V(t).
\end{aligned} \tag{1}$$

The Caputo operator, a typical fractional-order operator, is used to analyze the integer-order deterministic model. The COVID-19 epidemic SAIVR model is fractionally evaluated using the Caputo-type operator under the premise that solutions exist and that the operator utilized for nonlinear differential equations is of this kind. As a result of its widespread application in epidemiological modeling, the Caputo fractional operator has

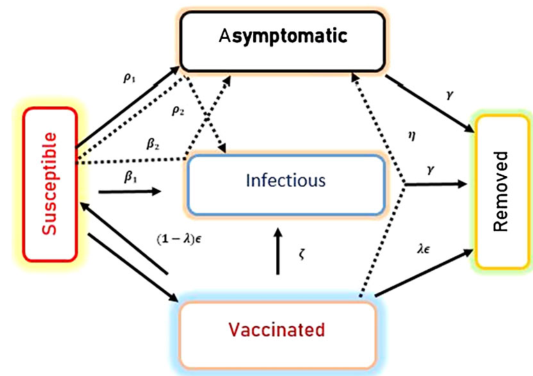


Figure 1 A flow chart for the dynamics of the SAIVR model given in (1)

gained popularity. The SAIVR model in a deterministic system has been separated into five categories: susceptible, asymptomatic/undetected, infected, vaccinated, and recovered individuals at any point in time t. Nonlinear differential equations are shown below under the Caputo operator, with α being the fractional-order operator in the Caputo sense.

$$\begin{aligned}
{}^{C}\mathbb{D}^{\alpha}_{0,t}S(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\kappa)^{-\alpha}S'(\kappa)\,d\kappa \\
&= -\beta_1^{\alpha}I(t)\frac{S(t)}{N} - \rho_1^{\alpha}A(t)\frac{S(t)}{N} - \delta^{\alpha}\frac{S(t)}{N} + \left(1-\lambda^{\alpha}\right)\epsilon^{\alpha}V(t), \\
{}^{C}\mathbb{D}^{\alpha}_{0,t}A(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\kappa)^{-\alpha}A'(\kappa)\,d\kappa \\
&= \rho_1^{\alpha}A(t)\frac{S(t)}{N} + \beta_2^{\alpha}I(t)\frac{S(t)}{N} + \eta^{\alpha}A(t)\frac{V(t)}{N} - \gamma^{\alpha}A(t), \\
{}^{C}\mathbb{D}^{\alpha}_{0,t}I(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\kappa)^{-\alpha}I'(\kappa)\,d\kappa \\
&= \beta_1^{\alpha}I(t)\frac{S(t)}{N} + \rho_2^{\alpha}A(t)\frac{S(t)}{N} + \zeta^{\alpha}I(t)\frac{V(t)}{N} - \gamma^{\alpha}I(t), \\
{}^{C}\mathbb{D}^{\gamma}_{0,t}V(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\kappa)^{-\alpha}V'(\kappa)\,d\kappa \\
&= \delta^{\alpha}\frac{S(t)}{N} - \eta^{\alpha}A(t)\frac{V(t)}{N} - \zeta^{\alpha}I(t)\frac{V(t)}{N} - \epsilon^{\alpha}V(t), \\
{}^{C}\mathbb{D}^{\alpha}_{0,t}R(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\kappa)^{-\alpha}R'(\kappa)\,d\kappa = \gamma^{\alpha}I(t) + \gamma^{\alpha}A(t) + \lambda^{\alpha}\epsilon^{\alpha}V(t).
\end{aligned} \tag{2}$$

In this study, we examine how fractional calculus in mathematical epidemiology can be used to create a model of COVID-19, among other areas. In fact, accurate modeling of those infectious disease systems makes use of fractional derivative formulations. In any case, fractional order differential equation systems provide an interesting modeling technique in the context of epidemiology, as one could expect, given that they permit higher degrees of freedom and incorporate memory effects in the model. In fact, the fractional-order derivative is a good tool for expressing the memory and heredity features of many materials and processes, making fractional differential equations more sufficient for describing phenomena related to non-locality than integer-order derivatives. Therefore, epidemic-based fractional derivatives have also been employed to address some epidemic tendencies. In general, this basic/classical model does not yield adequate or satisfactory results, as demonstrated by the failure of the classical first-order differential equations to accurately reproduce the statistical data collected during an actual epidemic of the disease. In our work, we have explored a more precise and intricate set of fractional differential equations in an effort to achieve better findings that are more in line with reality.

3 Why COVID-19 model with the Caputo operator?

Traditional models employ local differential and integral operators, which ignore the details of the epidemic being studied. Therefore, the memory features of the underlying system are ignored by conventional calculus. Some recent studies have shown that nonlocal operators are superior to classical ones, making them the only choice for incorporating memory effects into the deterministic model of the epidemic [43]. Recent scientific literature has seen the proposal of multiple epidemic models for the spread of the COVID-19 virus. Several have been created with the help of fractional-order differential operators, the most popular of which is called Caputo. However, most research articles practically never explain why the Caputo operator was used. In this paper, we have attempted to describe some of the more salient reasons why the Caputo operator should be considered when modeling an infectious disease with nonlinear differential equations. The following are some of the explanations:

- Because it is common knowledge that infectious diseases contain hereditary components, the most appropriate alternatives for modeling infectious diseases are fractional-order operators that can potentially preserve the memory of the systems being considered [44]. In this way, the Caputo operator is the best choice to replace the integer-order time derivative in the COVID-19 model that is currently being talked about.
- Substituting $\alpha \in \mathbb{C}$ for $n \in \mathbb{N}$ in the Cauchy formula for repeated integration yields the well-known Riemann-Liouville integral formula, which has served as the basis for developing numerous numerical methods for solving fractional ordinary and partial differential equations.
- Recent works have successfully used Caputo's revision of traditional epidemiological models, which are paired with particulars regarding the existence of a single solution and a study of the system's stability [45]. In that part, simulations were used to show why the Caputo variant is better than the standard (classical: integer-order derivatives) one.
- Epidemiological research published recently [46] demonstrates that older models failed to account for the complex and unpredictable dynamics of an infection's spread. Instead, we used actual data regarding the outbreak, largely from reputable sources like the World Health Organization and scientifically published articles, to confirm and corroborate the Caputo accounts. To further clarify the basic reproductive number, which describes the typical number of secondary infected cases produced when an infectious individual enters a completely susceptible class, we can use Caputo's differential operator to analyze the disease's behavior under different values for biological parameters.

- Finally, using a fractional-order operator for any physical or biological model, one can verify that there is no physical or geometrical meaning of the fractional parameter α while keeping in mind that there is no physical or geometrical meaning of even integer-order derivatives, such as the fourth-order ODE for the deflection of a beam [47] wherein the fourth derivative itself does not have any meaning.

4 Best fitting of biological parameters

An epidemic's dynamics can be explained by a system of nonlinear ordinary differential equations, but there are challenges associated with validating the suggested differential equations and achieving appropriate values for the working (or biological) parameters of the proposed equations. Since most parameters are fully indeterminate from the available information about the nonlinear system under study, the models with parameters that need to be optimized and fitted are spread out across several domains. Infectious diseases in mathematical epidemiology are typically represented using a set of nonlinear differential equations with a number of continuous parameters whose fitted values are crucial to get using various methods currently accessible in the literature. However, with the aid of demography and the studied disease's past history, several of the biological parameters connected with the suggested epidemic can be simply computed. It is typical practice to utilize the biological parameters by employing the values discovered through study or making educated assumptions about them. However, this method can sometimes make the disease act in ways that are hard to predict, which means that the proposed model cannot be tested.

Unknown parameters are one of the most challenging aspects of infectious disease mathematics study. In epidemiology, deterministic models may employ several approaches to fit values for unknown parameters, including principle component iterated sorting and nonlinear least-squares curve fitting. In the COVID-19 SAIVR model, scientists need to know eleven epidemiological parameters: the inflow susceptible rate, asymptomatic/undetected, vaccinated rate, and recovery rate. Data from the WHO's official website was used to predict the outbreak of COVID-19 in Turkey (March–April 2022). The Caputo derivative has an initial recovery of $R(0) = 50{,}269$, people are initially asymptomatic/undetected $A(0) = 23{,}954$, people initially infected $I(0) = 59{,}885$, $V(0) = 40{,}658$ is the number of people who are vaccinated, and $S(0) = 8{,}506{,}527$ people who are susceptible to getting infected. Both the standard SAIVR model (1) and the proposed Caputo fractional SAIVR model (2) are used to obtain the COVID-19 pandemic's behavior.

The Caputo fractional derivative operator clearly outperforms the standard approach as shown by the absolute mean errors: $|E_C| = 1.9861\text{e-}01$ for the classical model and $|E_{\mathrm{Cap}}| = 1.0474\text{e-}01$ for the Caputo model. The biological parameters obtained with the technique of least-squares are stored in Table 1 wherein the optimized parameters, including the optimal value of α (the fractional order), are also shown. In addition, Figs. 2 and 3 compare the simulations obtained from the models and the real medical data. It is clear that the Caputo simulations agree better with the real medical data. It is also evident from the Box-Whisker chart in Fig. 4, wherein the Caputo version of the model describes summary statistics that match well enough with the real ones as shown in the first and third boxes in Fig. 4.

Table 1 The biological parameters used in the model

Parameter	Explanation	(Caputo)	(Classical)	Source
β_1	the rate at which individuals are exposed to symptomatic	6.876e-04	1.079e-03	fitted
α	fractional-order	6.757e-01	1	fitted
ρ_1	asymptomatic infection rate	0.002	0.02	fixed
β_2	the rate at which individuals are infected to asymptomatic	0.005	0.01	fixed
ρ_2	symptomatic infection rate	0.005	0.005	fixed
γ	the removal rate	1/9	1/12	fixed
ζ	the rate at which a vaccinated (not immune)	0.02	0.5	fixed
η	the transmission rate at which asymptomatic individual comes into contact and infects vaccinated (but still not immune) individuals	0.05	0.05	fixed
δ	the first shot vaccination rate	0.01	0.01	fixed
λ	the vaccine efficacy	0.05	0.002	fixed
ϵ	immunity and moving to the removed compartment	0.02	0.02	fixed

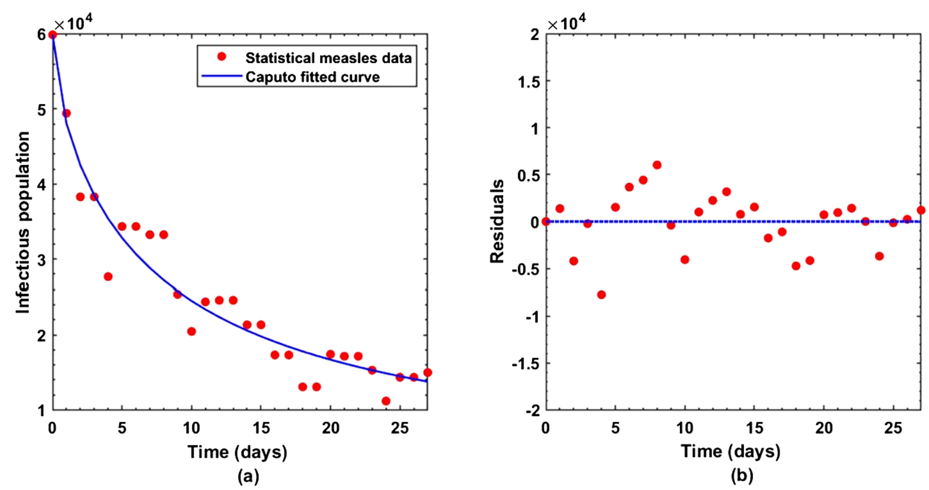


Figure 2 The behavior of the Caputo model in comparison with the statistical cases with the residual plot

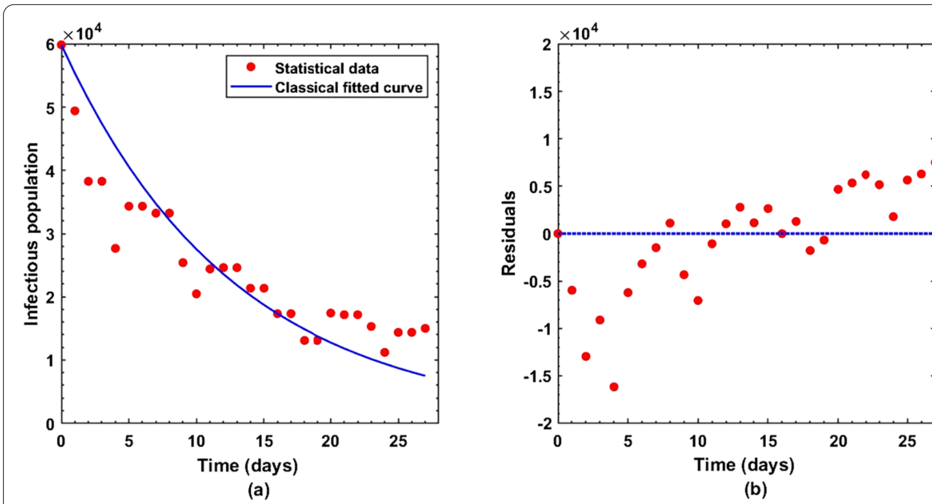
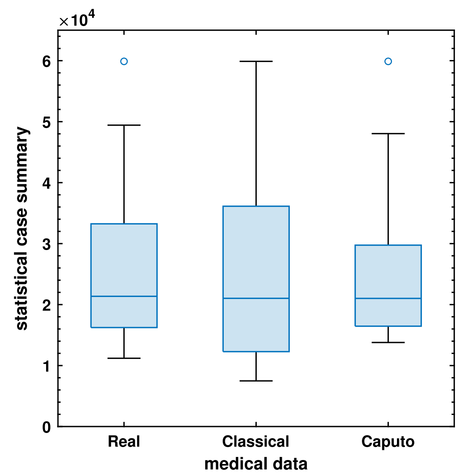


Figure 3 The behavior of the classical model in comparison with the statistical cases with the residual plot

Figure 4 The comparison among real medical data, simulations from classical model (1), and the simulations from the Caputo model given in (2) via Box-Whisker chart



5 Abstract analysis

In this section, a detailed qualitative analysis of the fractional-order Caputo model (2) has been carried out wherein major focus is placed upon the existence and uniqueness of solutions of the system, including its stability analysis via Hyers–Ulam–Rassias stability criterion.

5.1 Existence and uniqueness results

In the context of this study, the Banach contraction principle supports the uniqueness of the solution by providing a mathematical foundation to prove that the proposed Caputo SAIVR model converges to a single, unique solution, indicating the reliability and stability of the model in predicting the dynamics of the COVID-19 pandemic. This section will discuss the Lipchitz condition, existence, uniqueness, and stability of the model (2). As a result, we first assume the following five kernel values for simplicity and clarity:

$$\begin{cases} {}^{C}\mathbb{D}^{\alpha}_{0,t}S(t) = \Delta_1(t,S), \\ {}^{C}\mathbb{D}^{\alpha}_{0,t}A(t) = \Delta_2(t,A), \\ {}^{C}\mathbb{D}^{\alpha}_{0,t}I(t) = \Delta_3(t,I), \\ {}^{C}\mathbb{D}^{\alpha}_{0,t}V(t) = \Delta_4(t,V), \\ {}^{C}\mathbb{D}^{\alpha}_{0,t}R(t) = \Delta_5(t,R). \end{cases} \tag{3}$$

Applying the fractional-order Caputo operator on the above five kernels given in (3),

$$\begin{cases} S(t) - S(0) = \Xi(\alpha)\Delta_1(t,S) + \Omega(\alpha)\int_0^t \Delta_1(\theta,S)\,d\theta, \\ A(t) - A(0) = \Xi(\alpha)\Delta_2(t,A) + \Omega(\alpha)\int_0^t \Delta_2(\theta,A)\,d\theta, \\ I(t) - I(0) = \Xi(\alpha)\Delta_3(t,I) + \Omega(\alpha)\int_0^t \Delta_3(\theta,I)\,d\theta, \\ V(t) - V(0) = \Xi(\alpha)\Delta_4(t,V) + \Omega(\alpha)\int_0^t \Delta_4(\theta,V)\,d\theta, \\ R(t) - R(0) = \Xi(\alpha)\Delta_5(t,R) + \Omega(\alpha)\int_0^t \Delta_5(\theta,R)\,d\theta, \end{cases} \tag{4}$$

where $\Xi(\alpha)$ and $\Omega(\alpha)$ are positive real constants. Now, we will prove the Lipchitz condition for the Caputo system (2).

Theorem 1 *The above five kernels $\Delta_1(t,S)$, $\Delta_2(t,A)$, $\Delta_3(t,I)$, $\Delta_4(t,V)$ and $\Delta_5(t,R)$ satisfy the Lipchitz condition.*

Proof First, the Lipchitz condition is justified for Δ_1 kernel. Take S and S^* as two functions, then the corresponding norm is as follows:

$$\begin{aligned} &\left\|\Delta_1(t,S) - \Delta_1\left(t,S^*\right)\right\| \\ &\quad\leq \left\|\left(-\beta_1^\alpha I(t)\frac{S(t)}{N} - \rho_1^\alpha A(t)\frac{S(t)}{N} - \delta^\alpha\frac{S(t)}{N} + \left(1-\lambda^\alpha\right)\kappa^\alpha V(t)\right)\right. \\ &\qquad \left. - \left(-\beta_1^\alpha I(t)\frac{S^*(t)}{N} - \rho_1^\alpha A(t)\frac{S^*(t)}{N} - \delta^\alpha\frac{S^*t)}{N} + \left(1-\lambda^\alpha\right)\kappa^\alpha V(t)\right)\right\|. \end{aligned} \tag{5}$$

Simplifying and applying the norm property, we get

$$\left\|\Delta_1(t,S) - \Delta_1\left(t,S^*\right)\right\| \leq \left(\frac{\beta_1^\alpha I}{N} + \frac{\rho_1^\alpha A}{N} + \frac{\delta^\alpha}{N}\right)\left\|S - S^*\right\|,$$

$$\left\|\Delta_1(t,S) - \Delta_1\left(t,S^*\right)\right\| \leq \kappa_1\left\|S - S^*\right\|, \tag{6}$$

taking $\kappa_1 = \frac{\beta_1^\alpha I}{N} + \frac{\rho_1^\alpha A}{N} + \frac{\delta^\alpha}{N}$, where $\frac{\beta_1^\alpha I}{N}$, $\frac{\rho_1^\alpha A}{N}$ and $\frac{\delta^\alpha}{N}$ are bounded functions. Similarly, the norms can be created for rest of the model equations. □

5.2 Existence of the solution

In this subsection, we will prove that the Caputo model (2) under consideration has at least one solution. Thus, the recursive formula for (4) becomes the one shown below:

$$\begin{cases} S_n(t) = \Xi(\alpha)\Delta_1(t,S_{n-1}) + \Omega(\alpha)\int_0^t \Delta_1(\theta,S_{n-1})\,d\theta, \\ A_n(t) = \Xi(\alpha)\Delta_2(t,A_{n-1}) + \Omega(\alpha)\int_0^t \Delta_2(\theta,A_{n-1})\,d\theta, \\ I_n(t) = \Xi(\alpha)\Delta_3(t,I_{n-1}) + \Omega(\alpha)\int_0^t \Delta_3(\theta,I_{n-1})\,d\theta, \\ V_n(t) = \Xi(\alpha)\Delta_4(t,V_{n-1}) + \Omega(\alpha)\int_0^t \Delta_4(\theta,V_{n-1})\,d\theta, \\ R_n(t) = \Xi(\alpha)\Delta_5(t,R_{n-1}) + \Omega(\alpha)\int_0^t \Delta_5(\theta,R_{n-1})\,d\theta. \end{cases} \tag{7}$$

The positive initial conditions are first iterative values. The difference between two consecutive terms is as follows

$$\begin{cases} \Theta_{1n} = S_n(t) - S_{n-1}(t) \\ \qquad = \Xi(\alpha)(\Delta_1(t,S_{n-1}) - \Delta_1(t,S_{n-2})) + \Omega(\alpha)\int_0^t(\Delta_1(\theta,S_{n-1}) - \Delta_1(\theta,S_{n-2}))\,d\theta, \\ \Theta_{2n} = A_n(t) - A_{n-1}(t) \\ \qquad = \Xi(\alpha)(\Delta_2(t,A_{n-1}) - \Delta_2(t,A_{n-2})) + \Omega(\alpha)\int_0^t(\Delta_2(\theta,A_{n-1}) - \Delta_2(\theta,A_{n-2}))\,d\theta, \\ \Theta_{3n} = I_n(t) - I_{n-1}(t) \\ \qquad = \Xi(\alpha)(\Delta_3(t,I_{n-1}) - \Delta_3(t,I_{n-2})) + \Omega(\alpha)\int_0^t(\Delta_3(\theta,I_{n-1}) - \Delta_3(\theta,I_{n-2}))\,d\theta, \\ \Theta_{4n} = V_n(t) - V_{n-1}(t) \\ \qquad = \Xi(\alpha)(\Delta_4(t,V_{n-1}) - \Delta_4(t,V_{n-2})) + \Omega(\alpha)\int_0^t(\Delta_4(\theta,V_{n-1}) - \Delta_4(\theta,V_{n-2}))\,d\theta, \\ \Theta_{5n} = R_n(t) - R_{n-1}(t) \\ \qquad = \Xi(\alpha)(\Delta_5(t,R_{n-1}) - \Delta_5(t,R_{n-2})) + \Omega(\alpha)\int_0^t \Delta_5(\theta,R_{n-1}) - \Delta_5(\theta,R_{n-2}))\,d\theta. \end{cases} \tag{8}$$

It is worth noticing that

$$\begin{cases} \sum_{i=0}^{n} \Theta_{1i} = S_n(t), \\ \sum_{i=0}^{n} \Theta_{2i} = A_n(t), \\ \sum_{i=0}^{n} \Theta_{3i} = I_n(t), \\ \sum_{i=0}^{n} \Theta_{4i} = V_n(t), \\ \sum_{i=0}^{n} \Theta_{5i} = R_n(t). \end{cases} \tag{9}$$

Taking first equation of system (8), we assess the following

$$\begin{aligned} \Theta_{1n} &= \left\| S_n(t) - S_{n-1}(t) \right\| \\ &= \left\| \Xi(\alpha)\big(\Delta_1(t,S_{n-1}) - \Delta_1(t,S_{n-2})\big) + \Omega(\alpha) \int_0^t \big(\Delta_1(\theta,S_{n-1}) - \Delta_1(\theta,S_{n-2})\big)\, d\theta \right\|. \end{aligned} \tag{10}$$

Applying the triangular inequality reduces the above equation to

$$\begin{aligned} &\left\| S_n(t) - S_{n-1}(t) \right\| \\ &\quad \leq \Xi(\alpha) \left\| \Delta_1(t,S_{n-1}) - \Delta_1(t,S_{n-2}) \right\| + \Omega(\alpha) \left\| \int_0^t \Delta_1(\theta,S_{n-1}) - \Delta_1(\theta,S_{n-2})\, d\theta \right\|. \end{aligned} \tag{11}$$

The kernel $\Delta_1(t,S)$ satisfies the Lipchitz condition on the evidence of Eq. (6). Therefore, we can write it as follows

$$\left\| S_n(t) - S_{n-1}(t) \right\| \leq \Xi(\alpha)\kappa_1 \|S_{n-1} - S_{n-2}\| + \Omega(\alpha)\kappa_1 \int_0^t \|S_{n-1} - S_{n-2}\|\, d\theta. \tag{12}$$

On the evidence of (9), we can reduce the above inequality in the following manner

$$\left\| \Theta_{1n}(t) \right\| \leq \Xi(\alpha)\kappa_1 \left\| \Theta_{1(n-1)}(t) \right\| + \Omega(\alpha)\kappa_1 \int_0^t \left\| \Theta_{1(n-1)}(\theta) \right\| d\theta. \tag{13}$$

Similarly, we can get the following results:

$$\begin{cases} \|\Theta_{2n}(t)\| \leq \Xi(\alpha)\kappa_2 \|\Theta_{2(n-1)}(t)\| + \Omega(\alpha)\kappa_2 \int_0^t \|\Theta_{2(n-1)}(\theta)\|\, d\theta, \\ \|\Theta_{3n}(t)\| \leq \Xi(\alpha)\kappa_3 \|\Theta_{3(n-1)}(t)\| + \Omega(\alpha)\kappa_3 \int_0^t \|\Theta_{3(n-1)}(\theta)\|\, d\theta, \\ \|\Theta_{4n}(t)\| \leq \Xi(\alpha)\kappa_4 \|\Theta_{4(n-1)}(t)\| + \Omega(\alpha)\kappa_4 \int_0^t \|\Theta_{4(n-1)}(\theta)\|\, d\theta, \\ \|\Theta_{5n}(t)\| \leq \Xi(\alpha)\kappa_5 \|\Theta_{5(n-1)}(t)\| + \Omega(\alpha)\kappa_5 \int_0^t \|\Theta_{5(n-1)}(\theta)\|\, d\theta. \end{cases} \tag{14}$$

Theorem 2 *The analytical solution exists for the fractional-order Caputo model* (2) *under the given condition at* Θ_0:

$$\Xi(\alpha)\kappa_i + \Omega(\alpha)\kappa_i\Theta_{1,0} < 1, \quad \text{for } i = 1,\dots,5. \tag{15}$$

Proof Since the functions $S(t)$, $A(t)$, $I(t)$, $V(t)$, $R(t)$ are bounded and the condition for the Lipschitz is satisfied, using Eq. (13) and the recursive relation yields

$$\|\Theta_{1n}\| \leq \left\| S(0) \right\| \left[\Xi(\alpha)\kappa_1 + \Omega(\alpha)\kappa_1 t \right]^n \tag{16}$$

As a result, the solutions above will continue to exist. On the other hand, to show that the aforementioned functions represent the suggested model's solution, we take into account

$$S(t) - S(0) = S_n(t) - M_{1n}(t).$$

Therefore, we have

$$\begin{aligned}\left\|M_{1n}(t)\right\| = \Bigg\| &\Xi(\alpha)\big(\Delta_1(t,S) - \Delta_1(t,S_{n-1})\big) \\ &+ \Omega(\alpha)\int_0^t \big(\Delta_1(\theta,S) - \Delta_1(\theta,S_{n-1})\big)\,d\theta\Bigg\|.\end{aligned}$$

Using the Lipchitz condition,

$$\left\|M_{1n}(t)\right\| \le \Xi(\alpha)\kappa_1\|S - S_{n-1}\| + \Omega(\alpha)\kappa_1\|S - S_{n-1}\|t. \tag{17}$$

This gives,

$$\left\|M_{1n}(t)\right\| \le \big(\Xi(\alpha) + \Omega(\alpha)t\big)^{n+1}\kappa_1^{n+1}\upsilon. \tag{18}$$

Then, at t_0, we have

$$\left\|M_{1n}(t)\right\| \le \big(\Xi(\alpha) + \Omega(\alpha)t_0\big)^{n+1}\kappa_1^{n+1}\upsilon. \tag{19}$$

As n tends to ∞, we attain

$$\left\|M_{1n}(t)\right\| \to 0. \tag{20}$$

Similarly, we can derive

$$\begin{cases}\|M_{2n}(t)\| \to 0,\\ \|M_{3n}(t)\| \to 0,\\ \|M_{4n}(t)\| \to 0,\\ \|M_{5n}(t)\| \to 0.\end{cases} \tag{21}$$

□

This justifies the existence of the solution.

5.3 Uniqueness of the solution

Now, we must demonstrate whether or not the answer is unique. So, suppose that another solution to the suggested model exists and it is $S(t)$,

$$S(t) - S^*(t) = \Xi(\alpha)\big(\Delta_1(t,S) - \Delta_1(t,S^*)\big) + \Omega(\alpha)\int_0^t \big(\Delta_1(\theta,S) - \Delta_1(\theta,S^*)\big)\,d\theta. \tag{22}$$

Equation (22) with norm,

$$\left\|S(t) - S^*(t)\right\|\big(1 - \Xi(\alpha)\kappa_1 + \Omega(\alpha)\kappa_1 t\big) \le 0. \tag{23}$$

Theorem 3 *The analytical solution is unique for the Caputo fractional model under the following condition that is*

$$\big(1 - \Xi(\alpha)\kappa_1 + \Omega(\alpha)\kappa_1 t\big) > 0. \tag{24}$$

Proof Take into account that (24) holds so that from (23)

$$\big\|S(t) - S^*(t)\big\|\big(1 - \Xi(\alpha)\kappa_1 + \Omega(\alpha)\kappa_1 t\big) \leq 0. \tag{25}$$

Hence, we can say that $\|S(t) - S^*(t)\| = 0$. It implies that $S(t) = S^*(t)$ and the solution is unique. We now apply the same approach to another function yielding the following results

$$A = A_1, \qquad I = I_1, \qquad V = V_1, \qquad R = R_1. \tag{26}$$

Thus, this proof shows that the proposed version of model (2) in the sense of the Caputo operator has a unique solution. □

5.4 Hyers–Ulam–Rassias stability

The Hyers–Ulam–Rassias stability is a concept in functional analysis, particularly in the theory of functional equations. It refers to the stability of a functional equation under small perturbations of its arguments. In other words, it measures how close a solution of a functional equation is to be a solution of a perturbed version of that equation. The stability of a functional equation is a useful property for proving the existence and uniqueness of solutions, and it has important applications in a wide range of fields, including mathematics, physics, engineering, and economics. This subsection deals with stability analysis of fractional model (2) under the concepts of Hyers–Ulam–Rassias stability analysis. Let us rewrite model (3) as follows:

$$\begin{cases} {}^{C}\mathbb{D}_t^{\alpha}[\varphi(t)] = \Delta(t,\varphi(t)), \\ \varphi(0) = \varphi_0, \quad 0 < t < T < \infty, \end{cases} \tag{27}$$

where, the vector $\varphi = \{S, A, I, V, R\}$ and $\Delta = (\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5)$ is a continuous vector function.

Definition 1 Assume that the fractional-order α is $0 < \alpha < 1$ and $\Delta : [0,T] \times \mathbb{R}^5 \to \mathbb{R}^5$ is a continuous mapping. Then, model (27) is Hyers–Ulam stable if $\exists \varepsilon > 0$ and $N > 0$, such that for each solution $\varphi \in P([0,T],\mathbb{R}^5)$, the following inequality exists:

$$\big\|{}^{C}D_t^{\alpha}[\varphi] - \Delta(t,\varphi)\big\| \leq N, \quad \forall t \in [0,T], \tag{28}$$

$\exists$ solution $\varphi' \in P([0,T],\mathbb{R}^5)$ of model (27), such as

$$\big\|\varphi - \varphi'\big\| \leq \varepsilon N, \quad \forall t \in [0,T]. \tag{29}$$

Definition 2 Assume that the fractional order α is $0 < \alpha < 1$. The function $\Delta : [0, T] \times \mathbb{R}^5 \to \mathbb{R}^5$ and $\Pi : [0, T] \to \mathbb{R}^+$ are continuous mappings. Then, model (27) is generalized Hyers–Ulam–Rassias stable regarding to Π if $\exists P_{\Delta,\Pi} > 0$, such that for each solution $\varphi \in P([0, T], \mathbb{R}^5)$, the following inequality exists:

$$\left\| {}^{C}D_t^{\alpha}\left[\varphi(t)\right] - \Delta\left(t, \varphi(t)\right)\right\| \leq \Pi(t), \quad \forall t \in [0, T], \tag{30}$$

$\exists$ a solution $\varphi' \in P([0, T], \mathbb{R}^5)$ of model (27), such as

$$\left\|\varphi - \varphi'\right\| \leq P_{\Delta,\Pi}\Pi(t), \quad \forall t \in [0, T]. \tag{31}$$

Now, to prove that model (27) is the Hyers–Ulam–Rassias stable, we assume that:

- $[Q_1]\Delta : [0, T] \times \mathbb{R}^5 \to \mathbb{R}^5$ is a continuous mapping.
- $[Q_2]\exists P_{\Delta} > 0$ such that for each solution $\varphi, \varphi' \in P([0, T], \mathbb{R}^5)$,

$$\left\|\varphi - \varphi'\right\| \leq P_{\Delta}\left\|\varphi - \varphi'\right\|, \quad \forall t \in [0, T].$$

- $[Q_3]$ Let $\Pi \in ([0, T], \mathbb{R}^+)$ be an increasing mapping, and let $W_{\Pi>0}$, such that

$$\int_0^t \Pi(\theta)\, d\theta \leq W_{\Pi}\Pi(t), \quad \forall \varphi \in [0, T].$$

Theorem 4 *Assuming that* $[Q_1]$–$[Q_3]$ *exist and model* (27) *is generalized Hyers–Ulam–Rassias stable with respect to* Π *on the interval provided that* $\Xi(\alpha)P_{\Delta} < 1$.

Proof Let $\varphi' \in P([0, T], \mathbb{R}^5)$ be a solution of model (27). Then, the unique solution of model (27) from Theorem (3) is

$$\varphi = \varphi(0) + \Xi(\alpha)\Delta(t, \Xi) + \Omega(\alpha)\int_0^t \Delta\left(\varphi, \varphi(\theta)\right) d\theta. \tag{32}$$

On the evidence of (30), we can say that

$$\begin{aligned}
&\left\|\varphi' - \varphi(0) + \Xi(\alpha)\Delta\left(t, \Xi'\right) + \Omega(\alpha)\int_0^t \Delta\left(\varphi, \varphi'(\theta)\right) d\theta\right\| \\
&\quad \leq \Xi(\alpha)\Pi(t) + \Omega(\alpha)\int_0^t \Pi(\theta)\, d\theta \\
&\quad \leq \left(\Xi(\alpha) + \Omega(\alpha)W_{\Pi}\right)\Pi(t).
\end{aligned}$$

So,

$$\begin{aligned}
\left\|\varphi - \varphi'\right\| &\leq \left\|\varphi' - \varphi(0) - \Xi(\alpha)\Delta\left(t, \Xi'\right) - \Omega(\alpha)\int_0^t \Delta\left(\theta, \varphi'(\theta)\right) d\theta\right\| \\
&\leq \left\|\varphi' - \varphi(0) - \Xi(\alpha)\Delta(t, \varphi) - \Omega(\alpha)\int_0^t \Delta\left(\theta, \varphi(\theta)\right) d\theta\right. \\
&\quad - \Xi(\alpha)\Delta\left(t, \varphi'\right) - \Omega(\alpha)\int_0^t \Delta\left(\theta, \varphi'(\theta)\right) d\theta
\end{aligned}$$

$$
\begin{aligned}
&+ \Xi(\alpha)\Delta\big(t,\varphi'\big) + \Omega(\alpha)\int_0^t \Delta\big(\theta,\varphi'(\theta)\big)\,d\theta\bigg\| \\
&\leq \bigg\|\varphi' - \varphi(0) - \Xi(\alpha)\Delta\big(t,\varphi'\big) - \Omega(\alpha)\int_0^t \Delta\big(\theta,\varphi'(\theta)\big)\,d\theta\bigg\| \\
&+ \Xi(\alpha)\big\|\Delta(t,\varphi) - \Delta\big(t,\varphi'\big)\big\| + \Omega(\alpha)\int_0^t \big\|\Delta\big(\theta,\varphi(\theta)\big) - \Delta\big(\theta,\varphi'(\theta)\big)\big\|\,d\theta \\
&\leq \big(\Xi(\alpha) + \Omega(\alpha)W_\Pi\big)\Pi(t) + \Xi(\alpha)P_\Delta\big\|\varphi - \varphi'\big)\big\| \\
&+ \Pi(\alpha)P_\Delta\int_0^t \big\|\varphi(\theta) - \varphi'(\theta)\big\|\,d\theta.
\end{aligned}
$$

Now, $\varphi(\alpha)P_\Delta < 1$, so

$$
\big\|\varphi - \varphi'\big\| \leq \frac{(\Xi(\alpha) + \Omega(\alpha)W_\Pi)\Pi(t)}{1 - \Xi(\alpha)P_\Delta} + \frac{\Omega(\alpha)P_\Delta}{1 - \Xi(\alpha)P_\Delta}\int_0^t \big\|\varphi(\theta) - \varphi'(\theta)\big\|\,d\theta. \tag{33}
$$

The Gronwall's inequality yields

$$
\big\|\varphi - \varphi'\big\| \leq \left[\frac{\Xi(\alpha) + \Omega(\alpha)W_\Pi}{1 - \Xi(\alpha)P_\Delta}\exp(t)\right]\Omega(t). \tag{34}
$$

On setting $P_{\Delta,\Pi} = [\frac{\Xi(\alpha)+\Omega(\alpha)W_\Pi}{1-\Xi(\alpha)P_\Delta}\exp(t)]$, we have

$$
\big\|\varphi - \varphi'\big\| \leq P_{\Delta,\Pi}\Pi(t). \tag{35}
$$

Inequality (35) authenticated that model (27) is generalized Hyers–Ulam–Rassias stable with respect to Π. □

6 Numerical results from simulations

This section analyzes the numerical dynamics of the COVID-19 epidemic as modeled with the Caputo fractional operator. When doing so, it is important to keep in mind that the fractional order parameter that is used for simulations is the one that was optimized in the Sect. 4, while the rest of the parameters are taken from the third column of the Table 1. An explicit numerical method of the predictor-corrector type that was developed [48] specifically for use in simulations of fractional Caputo types of ordinary differential equations is used in this particular instance. One may find the method itself as well as a comprehensive study of it based upon convergence and error analysis in [49]. The numerical method developed specifically for this objective has garnered much praise in the most recent body of research because of its ease of use and adaptability. Furthermore, the author of [50] has thoroughly discussed the numerical approach in MATLAB implementation, including predictor-corrector capabilities. The simulations for the present research work for the Caputo COVID-19 model (2) were significantly simplified by utilizing such readily available routines on MathWorks. It should be noted that we run the MATLAB software on Windows with an Intel(R) Core(TM) i7-1065G7 CPU running at 1.30 GHz, 1.50 GHz, and 24 GB of RAM. The version number of this software is "9.8.0.1323502 (R2020a)."

For the COVID-19 model selected to be operated on by the Caputo operator, there are many crucial parameters whose values need careful attention to determine whether they

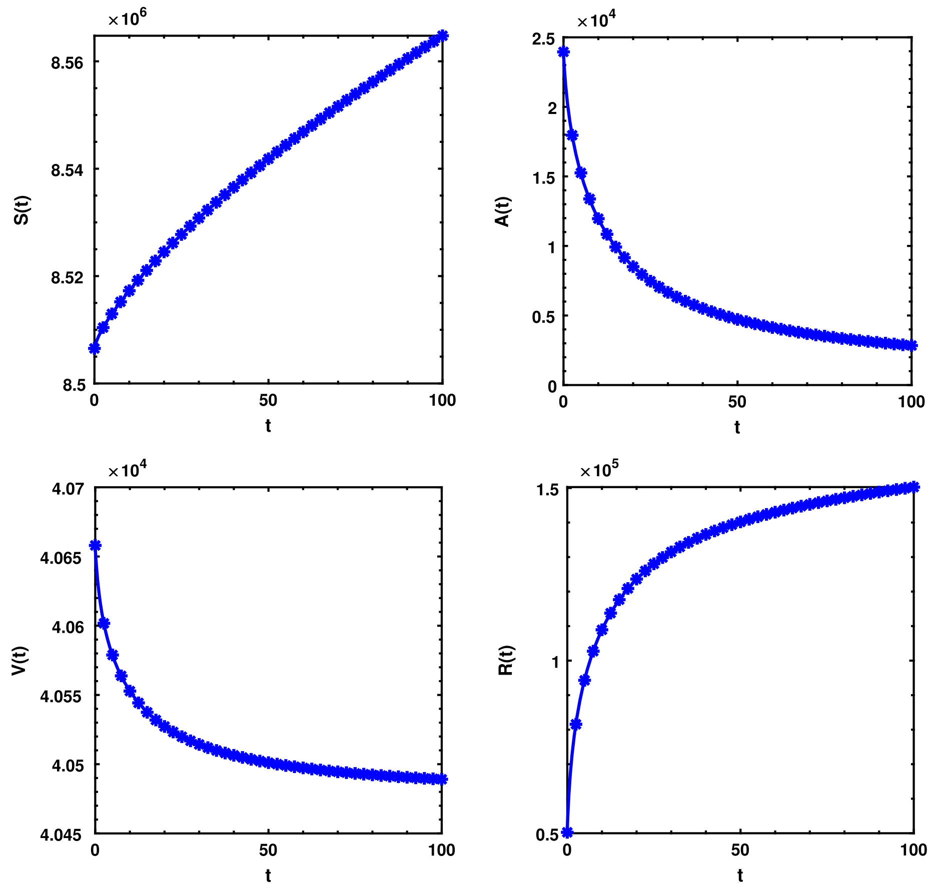


Figure 5 The dynamical behavior of the susceptible, asymptomatic, vaccinated, and removed individuals included in model (2) over a period [0, 100]

are growing or shrinking. In this regard, we have selected a few factors, such as the individuals' exposure rate β_1, the asymptomatic infection rate β_2, and the Caputo fractional order optimized parameter α. We have also shown the numerical dynamics of each of the five state variables using the Caputo operator, taking into account the fitted parameters and the value of α that has been optimized with the least-squares technique.

Several observations have been produced as a consequence of running simulations using the Caputo version of the COVID-19 model. Figure 5 shows that the dynamical behavior of the four classes (susceptible, asymptotic, vaccinated, and recovered) reflects the theoretical analysis that is being observed in the community after it was simulated using the four state variables (susceptible, asymptotic, vaccinated, and recovered). The simulations have been carried out with the parameters that best match the Caputo model. If the parameters shown in the table are used, the number of individuals who are susceptible to the disease and those who have recovered from it will both increase with the passage of time, while the number of individuals in the asymptotic and vaccinated groups will begin to decrease. In addition, it can be seen from Fig. 6 that if we slightly increase the exposure rate, then the infection substantially rises, demonstrating the sensitivity of this fundamental parameter. This is shown to be the case. This demonstrates that if one can manage the number of people exposed to the infection, one can also control the infection. The behav-

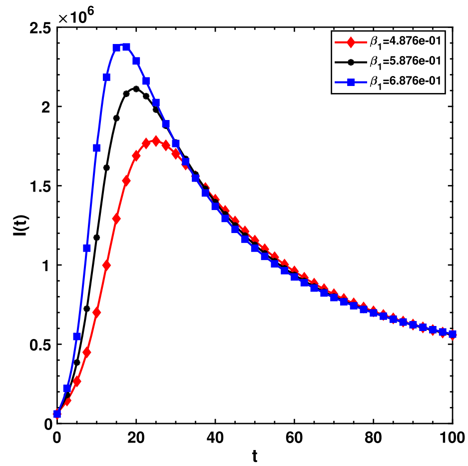


Figure 6 The dynamical behavior of the infectious individuals included in model (2) over a period [0, 100] under the effects of increasing exposure rate β_1

Figure 7 The dynamical behavior of the infectious individuals included in the model (2) over a period [0, 100] under the effects of increasing asymptotic infection rate β_2

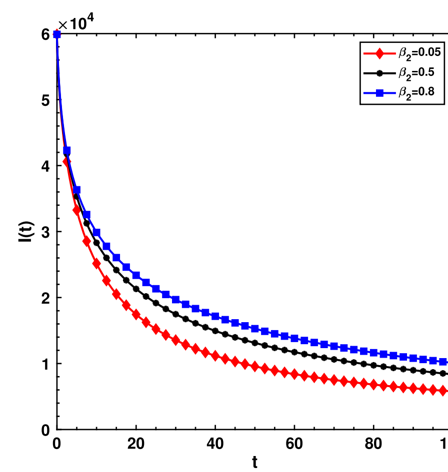
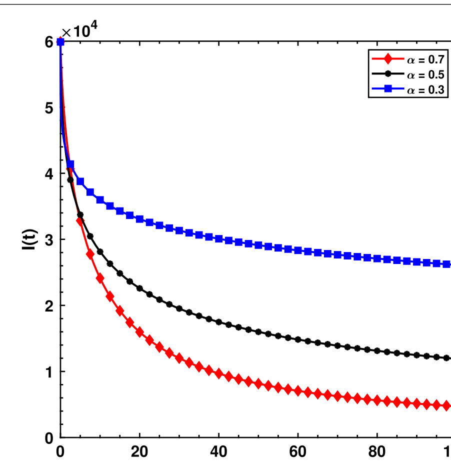


Figure 8 The dynamical behavior of the infectious individuals included in model (2) over a period [0, 100] under the effects of decreasing fractional order α



ior of a comparable nature is seen to occur with regard to the infection rate, as shown in Fig. 7.

Last, the fractional-order parameter is important to consider while comprehending the disease transmission pattern. As can be seen in Fig. 8, it is not advised to use values of this parameter that are lower than those shown, and determining the parameter's optimal value is of the utmost significance. In the current research investigation, this particular procedure was carried out. It should also be mentioned that quite a few studies have been conducted that have carried out this kind of analysis, which finds the optimal value of the fractional-order parameter in the Caputo sense. This is something to consider.

6.1 Memory trace and hereditary traits

Using fractional-order differentiation to represent memory effects in the system, the Caputo fractional-order epidemic model is a mathematical model used to predict the transmission of infectious illnesses [51]. The term "memory trace" is used here to describe how previous infections have affected the current pandemic. To account for genetic factors that may influence the dynamics of illness transmission, the model can be expanded to include hereditary features. Fractional-order differential equations (FODE), in which the derivative order is not an integer but a fractional value, are used in epidemic models to add memory effects. This helps the model more accurately reflect the course of the pandemic. Fractional derivatives capture the effect of prior infections in the setting of the Caputo fractional-order epidemic model, where fractional-order differential equations characterize the dynamics of the epidemic. Therefore, the current status of the disease is dependent not only on the current affected population but also on its past levels. This model's memory trace can be used to capture phenomena like the population's slow but steady development of immunity, the disease's tenacity, and the lasting effects of therapies.

Traits that can be passed down from one generation to the next are said to be hereditary. These characteristics are relevant to epidemiology because they can influence an individual's vulnerability to infection, transmission rate, and prognosis. Incorporating genetic elements into the differential equations that regulate the dynamics of the epidemic is what is meant by including hereditary qualities in the Caputo fractional-order epidemic model. Transmission rates, recovery rates, and contact rates are just some of the model parameters that might be affected by genetic factors. Disease dynamics may be drastically altered by inherited characteristics. Whether an individual contracts a disease, is able to transfer it to others, or benefits from a vaccination program may all depend on their genetic makeup. Integrating genetic data and population genetics principles into an epidemiological model to account for heritable features is a hard endeavor. In conclusion, the Caputo fractional-order epidemic model is an effective tool for modeling the role of memory in the spread of disease. It takes into account the impact of previous infections on the current epidemic by employing fractional-order derivatives. Genetic factors influencing illness transmission and susceptibility can also be accounted for by including hereditary features in the model. These two features can be used to develop more precise and all-encompassing models of infectious illness dynamics in communities.

To delve into the behavior of model (2), we utilize the Caputo operator defined in [52] for our analysis. For $\alpha, 0 < \alpha \leq 1$ derivative, let the fractional derivative of variable $\chi(t)$ be

$$ {}^{C}\mathbb{D}^{\alpha}_{0,t}\chi(t) = G\big(\chi(t), t\big). \tag{36} $$

Using one of the most widely used numerical methods, namely, the L1 scheme [52], the numerical approximation of the fractional-order derivative (FOD) of $\chi(t)$ is as follows:

$$ {}^{C}\mathbb{D}_{0,t}^{\alpha}\chi(t) \approx \frac{dt^{-\alpha}}{\Gamma(2-\alpha)}\left(\sum_{k=1}^{T-1}\big(\chi(t_{k+1})-\chi(t_k)\big)\big((T-k)^{1-\alpha}-(T-k-1)^{1-\alpha}\big)\right), \tag{37} $$

where $dt = h$ stands for the stepsize. One of the most effective numerical methods for discretizing the Caputo fractional-order derivative (CFOD) in the time domain is the L1 scheme. Although the memory component is present in other numerical approaches, its integration is more clearly represented in the L1 scheme. Taking (36) and (37), we get the following numerical solution to (36):

$$ \begin{aligned} \chi(t_T) \approx {}& {}^{C}\mathbb{D}_{0,t}^{\alpha}\Gamma(2-\alpha)H\big(\chi(t),t\big) + \chi(t_{T-1}) \\ & - \left(\sum_{k=1}^{T-2}\big(\chi(t_{k+1})-\chi(t_k)\big)\big((T-k)^{1-\alpha}-(T-k-1)^{1-\alpha}\big)\right). \end{aligned} \tag{38} $$

Therefore, the difference between the Markov term and the memory trace [53, 54] can be thought of as the FODE solution. Here is how the Gamma function affects the Markov term:

$$ \text{Markov Term} = {}^{C}\mathbb{D}_{0,t}^{\alpha}\Gamma(2-\alpha)H\big(\chi(t),t\big) + \chi(t_{T-1}). \tag{39} $$

The memory trace (MT) (χ-memory trace since it is related to variable $\chi(t)$) is

$$ \text{Memory Term} = \sum_{k=1}^{T-2}\big(\chi(t_{k+1})-\chi(t_k)\big)\big((T-k)^{1-\alpha}-(T-k-1)^{1-\alpha}\big). \tag{40} $$

The memory is adept at combining all prior acts, and this includes the system's tremendous historical evolution. When $\alpha = 1$, the memory trace is zero at all times t. The behavior of memory traces changes significantly over time. Nonlinearly increasing from zero, the memory trace increases when α is decreased. Therefore, fractional-order systems behave considerably differently from integer systems. We now present numerical simulations and elaborate biological interpretations of memory traces. To achieve this, the numerical approximation of the fractional-order derivative of $S(t)$ is as follows:

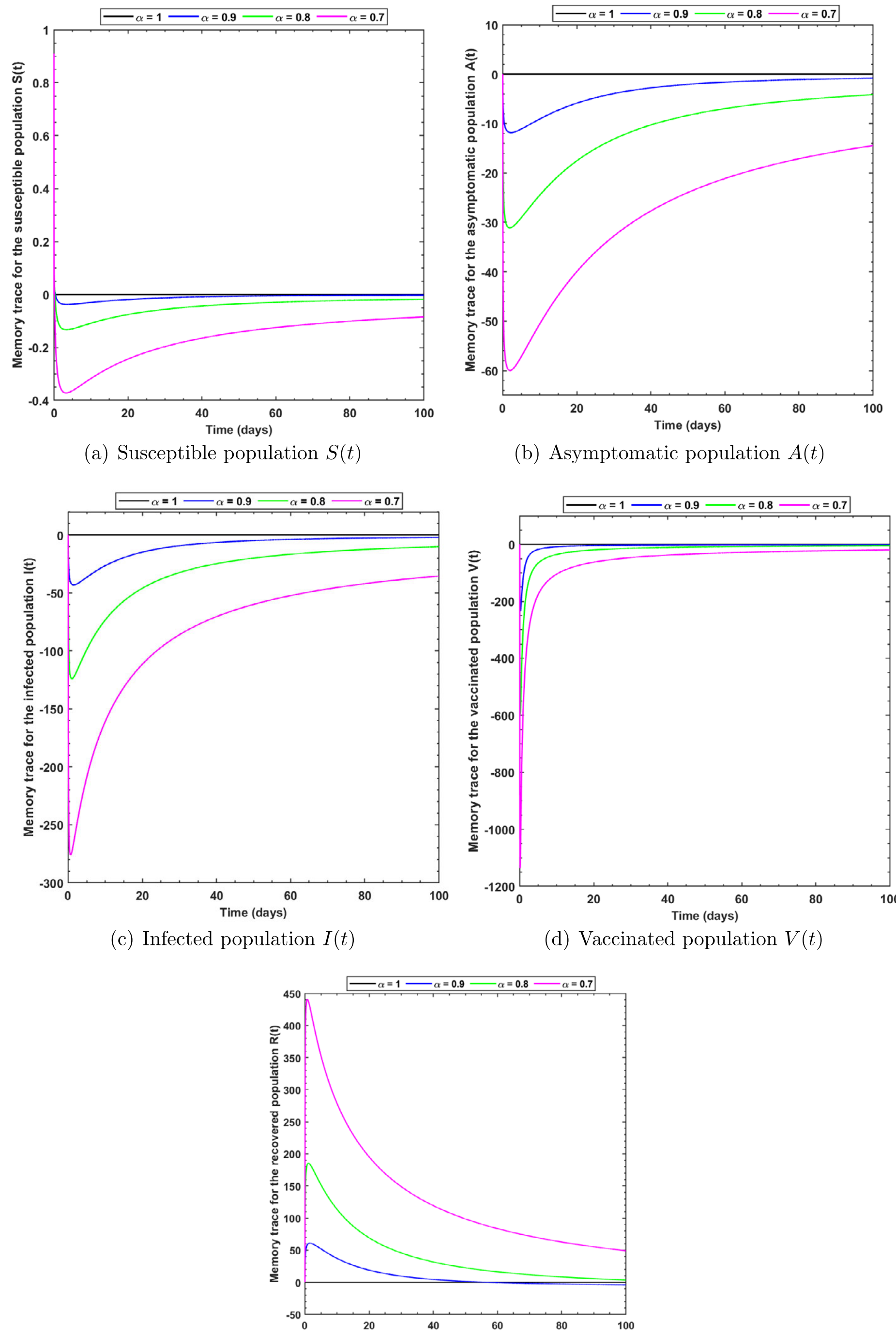
$$ {}^{C}\mathbb{D}_{0,t}^{\alpha}S(t) \approx \frac{dt^{-\alpha}}{\Gamma(2-\alpha)}\left(\sum_{k=1}^{T-1}\big(S(t_{k+1})-S(t_k)\big)\big((T-k)^{1-\alpha}-(T-k-1)^{1-\alpha}\big)\right). \tag{41} $$

Using (41) and the first compartment of the Caputo fractional system (2), the numerical solution of Susceptible individuals $S(t)$ is given by:

$$ S(t_T) \approx \text{Markov term of } S(t) - \text{Memory trace of } S(t), $$

where

$$ \text{Markov Term} = {}^{C}\mathbb{D}_{0,t}^{\alpha}\Gamma(2-\alpha)H\big(S(t),t\big) + S(t_{T-1}), \tag{42} $$



(a) Susceptible population $S(t)$ (b) Asymptomatic population $A(t)$

(c) Infected population $I(t)$ (d) Vaccinated population $V(t)$

(e) Recovered population $R(t)$

Figure 9 The effects of memory trace on each population of model (2)

and the memory trace (MT) is given by:

$$\text{Memory Term} = \sum_{k=1}^{T-2} \big(S(t_{k+1}) - S(t_k)\big)\big((T-k)^{1-\alpha} - (T-k-1)^{1-\alpha}\big). \tag{43}$$

By following the same steps, the numerical approximations of the fractional-order derivative of $A(t)$, $I(t)$, $V(t)$, and $R(t)$ can be achieved. Numerical simulations were conducted to visually illustrate the influence of memory trace on specific sub-populations inside the Caputo fractional system, as defined by Equation (2), employing the aforementioned methodology. Figure 9 illustrates the impact of memory trace on population dynamics across a range of fractional-order α values. Based on the observations depicted in these plots, it can be inferred that the absence of a memory effect is evident when the value of α is set to 1. As the value of α declines from 1 to 0.7, it becomes evident that the fractional order and the presence of a memory effect have discernible effects. The phenomenon of the memory effect has the potential to yield accurate outcomes and predictions pertaining to the COVID-19 pandemic. Therefore, the influence of memory plays a crucial role in epidemiological models. After doing an analysis in Fig. 9, it becomes evident that the memory effect tends to approach zero after a certain period of time. The observed results are consistent with the expected outcomes of real biological mechanisms occurring within the human organism. The results obtained from the graphical representations suggest that FODEs successfully capture the memory impact of the system, eliminating the need for additional components. It is widely recognized that fractional-order derivatives are favored due to their inherent memory effect. The activation of the memory effect plays a crucial role in the system effectiveness.

7 Conclusion

We have fractionalized the SAIVR epidemic model of order α using the well-known nonlocal character of the Caputo differential operator, which is ideal for studying the dynamics of disease transmission. The proposed Caputo SAIVR model is used for the study of the COVID-19 pandemic. Based on the Banach contraction principle, it is proven that the model has a unique solution. The stability of the Caputo model is established using Ulam-Hyers and its generalized form. One of the main contributions of this work is the least-squares method used to acquire the model's fitted parameters from clinical samples of the virus (March–April, 2022); this method also optimizes the fractional order α (6.757e–01). It has been shown that the Caputo model performs better than its classical counterpart. According to the results of a number of numerical simulations, the exposure rate must be reduced to control the pandemic effectively, and this is attainable when people practice social distancing and use protective masks. However, if such regulations are not carefully adhered to, a semi-developed country like Turkey will face serious difficulties. To effectively mitigate the spread of the pandemic, it is imperative to decrease the rate of exposure. This can be achieved by implementing social distancing measures and utilizing protective masks by individuals. Memory traces and hereditary traits are used to show the vanishing behavior of the model when $\alpha \to 1$. In future work, we plan to look into how non-singular differential operators change the classic SAIVR framework.

Acknowledgements
The author (AT) would like to thank DSR Majmaah University for providing the research environment and support.

Funding
No specific external funding was received for this article.

Data availability
Not applicable.

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
All authors have read and agreed to the submitted version of the manuscript.

Competing interests
The authors declare no competing interests.

Author contributions
The authors confirm their contribution to the paper as follows: AT and SQ perceived the Concept, did a Formal Analysis, and wrote the original draft. AS Wrote the original draft, did Investigation, Methodology, and Software; O.A. A and M. S did Validation, and Visualization, Supervision & Final proofreading. All authors reviewed the results and approved the final version of the manuscript.

Author details
[1]Department of Basic Sciences and Humanities, College of Computer and Information Sciences, Majmaah University, Al-Majmaah, 11952, Saudi Arabia. [2]Department of Basic Sciences and Related Studies, Mehran University of Engineering & Technology, Jamshoro, 76062, Pakistan. [3]Department of Mathematics, Near East University, 99138 Mersin, Turkey. [4]Department of Computer Science and Mathematics, Lebanese American University, Beirut, P.O. Box 13-5053, Lebanon. [5]Department of Mathematics, Faculty of Science, Al-Balqa' Applied University, Salt 19117, Jordan. [6]Department of Mathematics, Nevşehir Hacı Bektaş Veli University, Nevşehir, Turkey.

Received: 22 August 2023 Accepted: 3 January 2024 Published online: 22 January 2024

References
1. Al-Shomrani, M.M., Musa, S.S., Yusuf, A.: Unfolding the transmission dynamics of monkeypox virus: an epidemiological modelling analysis. Mathematics **11**(5), 1121 (2023)
2. Jamil, S., Farman, M., Akgül, A., Saleem, M.U., Hincal, E., El Din, S.M.: Fractional order age dependent Covid-19 model: an equilibria and quantitative analysis with modeling. Results Phys. **53**, 106928 (2023)
3. Yao, S.W., Farman, M., Akgül, A., Nisar, K.S., Amin, M., Saleem, M.U., Inc, M.: Simulations and analysis of COVID-19 as a fractional model with different kernels. Fractals **31**, 2340051 (2023)
4. Zarin, R., Khan, M., Khan, A., Yusuf, A.: Deterministic and fractional analysis of a newly developed dengue epidemic model. Waves Random Complex Media, 1–34 (2023). https://doi.org/10.1080/17455030.2023.2226765
5. Partohaghighi, M., Yusuf, A., Alshomrani, A.S., Sulaiman, T.A., Baleanu, D.: Fractional hyper-chaotic system with complex dynamics and high sensitivity: applications in engineering. Int. J. Mod. Phys. B, 2450012 (2023). https://doi.org/10.1142/S0217979224500127
6. Liu, P., Rahman, M.U., Din, A.: Fractal fractional based transmission dynamics of COVID-19 epidemic model. Comput. Methods Biomech. Biomed. Eng. **25**(16), 1852–1869 (2022)
7. Liu, P., Huang, X., Zarin, R., Cui, T., Din, A.: Modeling and numerical analysis of a fractional order model for dual variants of SARS-CoV-2. Alex. Eng. J. **65**, 427–442 (2023)
8. Atede, A.O., Omame, A., Inyama, S.C.: A fractional order vaccination model for COVID-19 incorporating environmental transmission: a case study using Nigerian data. Bull. Biomath. **1**(1), 78–110 (2023)
9. Din, A., Li, Y., Khan, F.M., Khan, Z.U., Liu, P.: On analysis of fractional order mathematical model of hepatitis B using Atangana–Baleanu Caputo (ABC) derivative. Fractals **30**(01), 2240017 (2022)
10. Din, A., Li, Y., Yusuf, A., Ali, A.I.: Caputo type fractional operator applied to hepatitis B system. Fractals **30**(01), 2240023 (2022)
11. Liu, P., Din, A., Zarin, R.: Numerical dynamics and fractional modeling of hepatitis B virus model with non-singular and non-local kernels. Results Phys. **39**, 105757 (2022)
12. Arif, M., Di Persio, L., Kumam, P., Watthayu, W., Akgül, A.: Heat transfer analysis of fractional model of couple stress Casson tri-hybrid nanofluid using dissimilar shape nanoparticles in blood with biomedical applications. Sci. Rep. **13**(1), 4596 (2023)
13. Xu, C., Liu, Z., Pang, Y., Akgül, A.: Stochastic analysis of a COVID-19 model with effects of vaccination and different transition rates: real data approach. Chaos Solitons Fractals **170**, 113395 (2023)
14. Memon, Z., Qureshi, S., Memon, B.R.: Assessing the role of quarantine and isolation as control strategies for COVID-19 outbreak: a case study. Chaos Solitons Fractals **144**, 110655 (2021)
15. Du, M., Wang, Z., Hu, H.: Measuring memory with the order of fractional derivative. Sci. Rep. **3**(1), 3431 (2013)
16. Petráš, I., Terpák, J.: Fractional calculus as a simple tool for modeling and analysis of long memory process in industry. Mathematics **7**(6), 511 (2019)
17. Mohammadi, F., Moradi, L., Baleanu, D., Jajarmi, A.: A hybrid functions numerical scheme for fractional optimal control problems: application to nonanalytic dynamic systems. J. Vib. Control **24**(21), 5030–5043 (2018)
18. Rezapour, S., Etemad, S., Mohammadi, H.: A mathematical analysis of a system of Caputo–Fabrizio fractional differential equations for the anthrax disease model in animals. Adv. Differ. Equ. **2020**(1), 481 (2020)
19. Githire, G.T.O., Kimathi, G., Wainaina, M.: Analysis of transmission dynamics of anthrax in animals: a modeling approach. J. Sci. Res. Rep. **23**(1), 1–9 (2019)
20. Naik, P.A., Owolabi, K.M., Yavuz, M., Zu, J.: Chaotic dynamics of a fractional order HIV-1 model involving AIDS-related cancer cells. Chaos Solitons Fractals **140**, 110272 (2020)
21. Ullah, I., Ahmad, S., ur Rahman, M., Arfan, M.: Investigation of fractional order tuberculosis (TB) model via Caputo derivative. Chaos Solitons Fractals **142**, 110479 (2021)

22. Atangana, A., Qureshi, S.: Mathematical modeling of an autonomous nonlinear dynamical system for malaria transmission using Caputo derivative. In: Fractional Order Analysis: Theory, Methods and Applications, pp. 225–252 (2020)
23. Diethelm, K.: A fractional calculus based model for the simulation of an outbreak of dengue fever. Nonlinear Dyn. **71**(4), 613–619 (2013)
24. Khan, M.A., Ullah, S., Farhan, M.: The dynamics of Zika virus with Caputo fractional derivative. AIMS Math. **4**(1), 134–146 (2019)
25. Tulu, T.W., Tian, B., Wu, Z.: Modeling the effect of quarantine and vaccination on Ebola disease. Adv. Differ. Equ. **2017**(1), 178 (2017)
26. Danane, J., Allali, K., Hammouch, Z.: Mathematical analysis of a fractional differential model of HBV infection with antibody immune response. Chaos Solitons Fractals **136**, 109787 (2020)
27. El-Saka, H.A.A.: Backward bifurcations in fractional-order vaccination models. J. Egypt. Math. Soc. **23**(1), 49–55 (2015)
28. Codeço, C.T.: Endemic and epidemic dynamics of cholera: the role of the aquatic reservoir. BMC Infect. Dis. **1**(1), 1–14 (2001)
29. Javidi, M., Ahmad, B.: A study of a fractional-order cholera model. Appl. Math. Inf. Sci. **8**(5), 2195 (2014)
30. Vargas-De-León, C.: Volterra-type Lyapunov functions for fractional-order epidemic systems. Commun. Nonlinear Sci. Numer. Simul. **24**(1–3), 75–85 (2015)
31. Özalp, N., Demirci, E.: A fractional order SEIR model with vertical transmission. Math. Comput. Model. **54**(1–2), 1–6 (2011)
32. Rezapour, S., Mohammadi, H., Jajarmi, A.: A new mathematical model for Zika virus transmission. Adv. Differ. Equ. **2020**(1), 589 (2020)
33. Qureshi, S.: Periodic dynamics of rubella epidemic under standard and fractional Caputo operator with real data from Pakistan. Math. Comput. Simul. **178**, 151–165 (2020)
34. Al-Deiakeh, R., Al-Smadi, M., Yusuf, A., Al-Omari, S., Momani, S.: Explicit solutions for fractional Chaffee–Infante reaction–diffusion coupled hierarchy system with conservation laws. Math. Methods Appl. Sci. **46**, 12777–12793 (2023)
35. Pinto, C.M., Carvalho, A.R.: The role of synaptic transmission in a HIV model with memory. Appl. Math. Comput. **292**, 76–95 (2017)
36. Naik, P.A., Yavuz, M., Qureshi, S., Zu, J., Townley, S.: Modeling and analysis of COVID-19 epidemics with treatment in fractional derivatives using real data from Pakistan. Eur. Phys. J. Plus **135**(10), 1–42 (2020)
37. Peter, O.J., Shaikh, A.S., Ibrahim, M.O., Nisar, K.S., Baleanu, D., Khan, I., Abioye, A.I.: Analysis and dynamics of fractional order mathematical model of COVID-19 in Nigeria using Atangana-Baleanu operator. Comput. Mater. Continua **66**(2), 1823–1848 (2021)
38. Baleanu, D., Mohammadi, H., Rezapour, S.: A fractional differential equation model for the COVID-19 transmission by using the Caputo–Fabrizio derivative. Adv. Differ. Equ. **2020**(1), 299 (2020)
39. Chen, T.M., Rui, J., Wang, Q.P., Zhao, Z.Y., Cui, J.A., Yin, L.: A mathematical model for simulating the phase-based transmissibility of a novel coronavirus. Infect. Dis. Poverty **9**(1), 1–8 (2020)
40. Baba, I.A., Nasidi, B.A.: Fractional order epidemic model for the dynamics of novel COVID-19. Alex. Eng. J. **60**(1), 537–548 (2021)
41. Faridi, W.A., Asjad, M.I., Jhangeer, A., Yusuf, A., Sulaiman, T.A.: The weakly non-linear waves propagation for Kelvin–Helmholtz instability in the magnetohydrodynamics flow impelled by fractional theory. Opt. Quantum Electron. **55**(2), 172 (2023)
42. Angeli, M., Neofotistos, G., Mattheakis, M., Kaxiras, E.: Modeling the effect of the vaccination campaign on the COVID-19 pandemic. Chaos Solitons Fractals **154**, 111621 (2022)
43. Coronel-Escamilla, A., Gomez-Aguilar, J.F., Stamova, I., Santamaria, F.: Fractional order controllers increase the robustness of closed-loop deep brain stimulation systems. Chaos Solitons Fractals **140**, 110149 (2020)
44. Pinto, C.M., Carvalho, A.R.: Diabetes mellitus and TB co-existence: clinical implications from a fractional order modelling. Appl. Math. Model. **68**, 219–243 (2019)
45. Zeb, A., Atangana, A., Khan, Z.A., Djillali, S.: A robust study of a piecewise fractional order COVID-19 mathematical model. Alex. Eng. J. **61**(7), 5649–5665 (2022)
46. Baba, I.A., Rihan, F.A.: A fractional–order model with different strains of COVID-19. Phys. A, Stat. Mech. Appl. **603**, 127813 (2022)
47. Bonanno, G., Di Bella, B., O'Regan, D.: Non-trivial solutions for nonlinear fourth-order elastic beam equations. Comput. Math. Appl. **62**(4), 1862–1869 (2011)
48. Diethelm, K., Ford, N.J., Freed, A.D.: A predictor-corrector approach for the numerical solution of fractional differential equations. Nonlinear Dyn. **29**(1), 3–22 (2002)
49. Diethelm, K., Ford, N.J., Freed, A.D.: Detailed error analysis for a fractional Adams method. Numer. Algorithms **36**(1), 31–52 (2004)
50. Garrappa, R.: Numerical solution of fractional differential equations: a survey and a software tutorial. Mathematics **6**(2), 16 (2018)
51. Ahmed, I., Akgül, A., Jarad, F., Kumam, P., Nonlaopon, K.: A Caputo-Fabrizio fractional-order cholera model and its sensitivity analysis. Math. Model. Numer. Simul. Appl. **3**(2), 170–187 (2023)
52. Liu, F., Zhuang, P., Liu, Q.: Numerical methods of fractional partial differential equations and applications (2015)
53. Özköse, F.: Long-term side effects: a mathematical modeling of COVID-19 and stroke with real data. Fractal Fract. **7**(10), 719 (2023)
54. Yavuz, M., Özköse, F., Susam, M., Kalidass, M.: A new modeling of fractional-order and sensitivity analysis for hepatitis-B disease with real data. Fractal Fract. **7**(2), 165 (2023)

Publisher's Note