Fixed Point Theory and Algorithms for Sciences and Engineering

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Fixed point theorem and iterated function system in φ-metric modular space

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Abstract

We introduce and study the concept of φ-metric modular space and, then define φ-α-Meir-Keeler contraction on it and explore its fixed point. Further, we define the Hausdorff distance between two non-empty compact subsets of the considered space. Some topological properties of φ-metric modular space are also explored. Additionally, we prove the existence of the attractor (fractal) of the IFS consisting of φ-α-Meir-Keeler contractions.

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1 Introduction

Hutchinson [1] introduced the term iterated function system (IFS) and was extensively explored by Barnsley [2]. The IFS is an effective tool for generating fractals. The IFS is used to construct various fractal interpolation functions. Hutchinson's IFS theory has been extended in numerous aspects, for instance, to more general space, infinite IFS, and generalized contractions. IFS based on condition ϕ-function was constructed by Hata [3], and Fernau [4] introduced the concept of infinite IFS.

In various forms of IFS, the existence of the attractor is fundamentally assured by the fixed point theory. In wide areas of mathematics, for the existence of a solution, we generally look for a fixed point for an appropriate map. It is found that the fixed points are indispensable in wide branches of mathematics. One of the most significant, useful, and well-celebrated findings in the fixed point theory is the Banach fixed point theorem [5]. Extensive research has been done to extend the Banach fixed point theorem in various aspects. One of the techniques used to extend the Banach fixed point theorem is generalizing the underlying space. Chistyakov [6] defined and studied the concept of modular metric space. Many researchers proved fixed point results by generalizing the modular metric spaces. Some of the recent works can be found in [7–9].

In this paper, we define a generalization of modular metric space by relaxing the triangle inequality, namely φ-metric modular space. We also assume that φ-metric modular takes on real values, unlike metric modular. In the defined space, we study the φ-α-Meir-Keeler contraction by invoking the concepts of Meir-Keeler [10] and explore its fixed point. Further, we define the Hausdorff distance on the compact subsets of the φ-metric modular space. Moreover, we consider the IFS consisting of φ-α-Meir-Keeler contractions and prove that the attractor exists uniquely.

1.1 Delineation

We note all the necessary preludes required throughout the study in Sect. 2. We define the φ-metric modular space and provide examples. We also show that a large class of φ-metric modular space can be generated from the given metric modular space. In Sect. 3, we study the φ-α-Meir Keeler contraction and prove our main result regarding fixed point. To validate our result, we provide an example. In Sect. 4, we observe that, in general, φ-metric modular ν need not be continuous on $\mathbb{R}_+ \times \mathcal{X} \times \mathcal{X}$ and give an example to support this. We define the Hausdorff distance between two non-empty compact subsets of $\mathcal{X}$ and show that the defined Hausdorff distance is also a φ-metric modular. Moreover, we explore some topological properties of φ-metric modular space. In Sect. 5, we define an iterated function system on a φ-metric modular space and prove that the attractor exists uniquely. Finally, Sect. 6 is dedicated to the conclusions of our findings and possible future works.

2 Preliminaries

The following notations are used in this paper:

- $\mathcal{X}$:= non-empty set,
- $\mathbb{R}_+ := (0,\infty)$,
- $\mathbb{R}_+^0 := [0,\infty)$,
- $\overline{\mathbb{R}}_+ := [0,\infty]$,
- $\mathbb{N}_N$:= set of first N natural numbers,
- $\mathbb{N}^* := \mathbb{N} \cup \{0\}$,
- $\mathrm{Fix}(g)$:= collection of all fixed points of g.

Here, ν will denote a function from $\mathbb{R}_+ \times \mathcal{X} \times \mathcal{X}$ to $\overline{\mathbb{R}}_+$. With the abuse of notation, we will denote each function $\nu\colon \mathbb{R}_+ \times \mathcal{X} \times \mathcal{X} \to \overline{\mathbb{R}}_+$ as $\nu_\lambda(x,y)$, $\forall\lambda \in \mathbb{R}_+$ and $x,y \in \mathcal{X}$. We will abbreviate the value of a function g at x as gx rather than $g(x)$ for the purpose of our own convenience.

The notion of metric modular on a non-empty set $\mathcal{X}$ can be found in [6]. Consider the following simple example:

Define $\nu\colon \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \overline{\mathbb{R}}_+$ by

$$\nu_\lambda(x,y) = \frac{|x-y|^2}{\lambda}, \quad \text{for } x,y \in \mathbb{R} \text{ and } \lambda \in \mathbb{R}_+.$$

Then, it is easy to verify that ν is not a metric modular space. To be precise, $\forall x,y,z \in \mathbb{R}$ and $\lambda,\mu \in \mathbb{R}_+$

$$\nu_{\lambda+\mu}(x,y) \le 2\left\{\nu_\lambda(x,z) + \nu_\mu(z,y)\right\}.$$

Motivated by this example, we define φ-metric modular on $\mathcal{X}$, which is a generalization of metric modular, by relaxing the triangle inequality as follows:

Definition 2.1 The function ν is called a φ-metric modular on a non-empty set $\mathcal{X}$ if $\forall a,b,c \in \mathcal{X}$ and $\lambda,\mu > 0$ we have:

1. $a = b \iff \nu_\lambda(a,b) = 0$;
2. $\nu_\lambda(a,b) = \nu_\lambda(b,a)$;
3. $\nu_{\lambda+\mu}(a,b) \leq \varphi(\lambda+\mu)[\nu_\lambda(a,c) + \nu_\mu(c,b)]$,

where $\varphi: \mathbb{R}_+ \to [1,\infty)$.

$(\mathcal{X}, \nu)$ is then called a φ-metric modular space.

Remark 2.2 For our own convenience, we abbreviate φ-metric modular as φ-MM.

Next, we define a regular φ-MM by having a weaker assumption on the first condition of Definition 2.1 as follows:

Definition 2.3 The function ν is called a regular φ-MM if $\forall a,b,c \in \mathcal{X}$ we have:
1. $a = b \iff \nu_\lambda(a,b) = 0$ for some $\lambda > 0$;
2. $\forall \lambda > 0$, $\nu_\lambda(a,b) = \nu_\lambda(b,a)$;
3. $\forall \lambda, \mu > 0$, $\nu_{\lambda+\mu}(a,b) \leq \varphi(\lambda+\mu)[\nu_\lambda(a,c) + \nu_\mu(c,b)]$,

where $\varphi: \mathbb{R}_+ \to [1,\infty)$.

$(\mathcal{X}, \nu)$ is then called a regular φ-MM space.

Example Consider $\nu: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \overline{\mathbb{R}}_+$ defined by

$$\nu_\lambda(a,b) = \begin{cases} \infty & \text{if } \lambda < 1; \\ (1 + |\cos\lambda|)|a-b| & \text{if } \lambda \geq 1. \end{cases}$$

The function ν is a regular φ-MM but not a φ-MM.

Remark 2.4 It is worth noting that compared to the class of metric modular spaces, the class of φ-MM spaces is significantly larger. Clearly, every metric modular is a φ-MM on $\mathcal{X}$ for $\varphi(\lambda) = 1$, $\forall \lambda \in \mathbb{R}_+$.

A large class of φ-MM spaces can be generated from a metric modular space, as evident from the following proposition:

Proposition 2.5 *Let $(\mathcal{X}, \nu^*)$ be a metric modular and define $\nu: \mathbb{R}_+ \times \mathcal{X} \times \mathcal{X} \to \overline{\mathbb{R}}_+$ by*

$$\nu_\lambda(a,b) = \varphi(\lambda)\nu^*_\lambda(a,b),$$

where $\varphi: \mathbb{R}_+ \to [1,\infty)$ is an arbitrary function. Then, $(\mathcal{X}, \nu)$ is a φ-MM space.

Proof Since ν^* is a metric modular, so $\forall a,b \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+$, $\nu_\lambda(a,b) \geq 0$ and $\nu_\lambda(a,b) = 0 \iff a = b$. Also, $\nu_\lambda(a,b) = \nu_\lambda(b,a)$. For $a,b,c \in \mathcal{X}$ and $\lambda, \mu \in \mathbb{R}_+$, we have

$$\begin{aligned} \nu_{\lambda+\mu}(a,b) &= \varphi(\lambda+\mu)\nu^*_{\lambda+\mu}(a,b) \\ &\leq \varphi(\lambda+\mu)\big\{\big(\varphi(\lambda)\big)\nu^*_\lambda(a,c) + \big(\varphi(\mu)\big)\nu^*_\lambda(c,b)\big\} \\ &= \varphi(\lambda+\mu)\big(\nu_\lambda(a,c) + \nu_\lambda(c,b)\big) \\ &= \varphi(\lambda+\mu)\big(\nu_\lambda(a,c) + \nu_\lambda(c,b)\big). \end{aligned}$$

Thus, ν is a φ-MM, and hence $(\mathcal{X}, \nu)$ is a φ-MM space. □

Example Consider $\nu\colon \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \overline{\mathbb{R}}_+$, where

$$\nu_\lambda(a,b) = \bigl(1 + |\cos\lambda|\bigr)|a-b| \quad \forall a,b \in \mathbb{R} \text{ and } \lambda \in \mathbb{R}_+.$$

Then, ν is a φ-MM with $\varphi(\lambda) = 1 + |\cos\lambda|$.

At this stage, it would be worth introducing an example that is a φ-Metric Modular Space but fails to be a Metric Modular Space.

Example Consider $\nu\colon \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \overline{\mathbb{R}}_+$, where

$$\nu_\lambda(a,b) = \exp(\lambda)|a-b| \quad \forall a,b \in \mathbb{R} \text{ and } \lambda \in \mathbb{R}_+.$$

Then, ν is a φ-MM with $\varphi(\lambda) = \exp(\lambda)$. However, it fails to be a Metric Modular. To be precise, triangular inequality fails here. For instance, $\nu_{1+1}(0,2) = 2\exp(2)$ but $\nu_1(0,1) + \nu_1(1,2) = 2\exp(1)$.

Definition 2.6 A sequence $\{a_n\}$ in a φ-MM space $(\mathcal{X},\nu)$ is called

1. ν-Cauchy, or simply Cauchy, if for a given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n,m > N$ and $\lambda \in \mathbb{R}_+$ we have $\nu_\lambda(a_n,a_m) < \epsilon$.
2. ν-convergent, or simply convergent to $a \in \mathcal{X}$, if $\nu_\lambda(a_n,a) \to 0$ as $n \to \infty$ $\forall\lambda \in \mathbb{R}_+$.

Definition 2.7 Let $(\mathcal{X},\nu)$ be a φ-MM space. Then,

1. $\mathcal{X}$ is ν-complete, or simply complete, if every Cauchy sequence converges in $\mathcal{X}$.
2. $U \subseteq \mathcal{X}$ is compact if every sequence in U has a convergent subsequence.

Definition 2.8 A self map f on a φ-MM space $(\mathcal{X},\nu)$ is called ν-continuous, or simply continuous, if for every sequence $\{a_n\}$ in $\mathcal{X}$ converging to a, we get $\{f(a_n)\}$ is convergent to $f(a)$.

Definition 2.9 [11] Let $\alpha\colon \mathcal{X} \times \mathcal{X} \to [0,\infty)$ be a function. We say that a self-mapping $T\colon \mathcal{X} \to \mathcal{X}$ is triangular α-admissible if

1. $x,y \in X$, $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$.
2. $x,y,z \in X$, $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$ implies $\alpha(x,y) \geq 1$.

Lemma 2.10 [11] *Let f be a triangular α-admissible mapping. Assume that there exists $x_0 \in \mathcal{X}$ such that $\alpha(x_0,fx_0) \geq 1$. Define the sequence $\{x_n\}$ by $x_n = f^n x_0$. Then,*

$$\alpha(x_m,x_n) \geq 1 \quad \textit{for all } m,n \in \mathbb{N} \textit{ with } m < n.$$

3 Main result

We define the φ-α-Meir-Keeler contraction on the φ-MM space as follows:

Definition 3.1 Let $\alpha\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+^0$ and $g\colon \mathcal{X} \to \mathcal{X}$ be a self map on a φ-MM space $(\mathcal{X},\nu)$. Then, g is called a φ-α-Meir-Keeler contraction if for a given $\epsilon > 0$, $\exists\delta > 0$ such that

$\forall a,b \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+$,

$$\epsilon \leq \nu_\lambda(a,b) < \varphi(\lambda)(\epsilon+\delta) \quad \Longrightarrow \quad \alpha(a,b)\nu_\lambda(ga,gb) < \epsilon. \tag{3.1}$$

If $\alpha(a,b) = 1$ $\forall a,b \in \mathcal{X}$, then g is called a φ-Meir-Keeler contraction.

Remark 3.2 For our own sake of convenience, we shall use φ-α-MK contraction for φ-α-Meir Keeler contraction and φ-MK contraction for φ-Meir-Keeler contraction.

Proposition 3.3 *Let $g\colon \mathcal{X} \to \mathcal{X}$ be a φ-α-MK contraction on a regular φ-MM space $(\mathcal{X},\nu)$ with $\nu_\lambda(x,y) < \infty$, $\forall x,y \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+$. Then, for every $a \neq b \in \mathcal{X}$, $\lambda \in \mathbb{R}_+$ and $\alpha(a,b) \geq 1$,*

$$\nu_\lambda(ga,gb) < \nu_\lambda(a,b).$$

Proof By regularity of ν, $\forall \lambda \in \mathbb{R}_+$ we have $\nu_\lambda(a,b) > 0$, since $a \neq b$. For a given $\epsilon > 0$, $\exists \delta > 0$ such that (3.1) holds. Choose $\epsilon = \nu_\lambda(a,b)$. Then, $\epsilon \leq \nu_\lambda(a,b) < \varphi(\lambda)(\epsilon+\delta)$, and hence by (3.1) we get

$$\nu_\lambda(ga,gb) \leq \alpha(a,b)\nu_\lambda(ga,gb) < \epsilon = \nu_\lambda(a,b).$$

Thus, $\nu_\lambda(ga,gb) < \nu_\lambda(a,b)$. □

Theorem 3.4 *Let $g\colon \mathcal{X} \to \mathcal{X}$ be a self map on a complete regular φ-MM space $(\mathcal{X},\nu)$. Let $\alpha\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+^0$ be such that g is a*:

1. *φ-α-MK contraction*;
2. *triangular α-admissible mapping.*

Also, let $\nu_\lambda(x,y) < \infty$, $\forall x,y \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+$. If $\alpha(a_0,ga_0) \geq 1$ for some $a_0 \in \mathcal{X}$, then g has a fixed point.

Proof Let $a_0 \in \mathcal{X}$ with $\alpha(a_0,ga_0) \geq 1$. Consider the sequence $\{a_n\}$ in $\mathcal{X}$ defined by $a_n = g^n a_0$. Using Lemma 2.10, we get

$$\alpha(a_m,a_n) \geq 1 \quad \forall m,n \in \mathbb{N}, m < n.$$

Clearly a fixed point exists if for some $k \in \mathbb{N}^*$, $a_k = a_{k+1}$.

Now let $a_n \neq a_{n+1}$ $\forall n \in \mathbb{N}^*$. By regularity of $(\mathcal{X},\nu)$, we have

$$\nu_\lambda(a_n,a_{n+1}) > 0 \quad \forall n \in \mathbb{N}^*.$$

By Proposition 3.3,

$$\nu_\lambda(a_n,a_{n+1}) < \nu_\lambda(a_{n-1},a_n) < \cdots < \nu_\lambda(a_0,a_1).$$

Define $\sigma_n = \nu_\lambda(a_n,a_{n+1})$ $\forall n \in \mathbb{N}^*$. Then, $\{\sigma_n\}$ is a strictly decreasing sequence with $\sigma_n > 0$. Clearly $\sigma_n \to \sigma \geq 0$ for some $\sigma \in \mathbb{R}$. Let, if possible, $\sigma > 0$, then $0 < \sigma < \sigma_n$ $\forall n \in \mathbb{N}^*$.

For $\epsilon = \sigma > 0$, $\exists \delta > 0$ such that (3.1) holds. Also, $\exists n_0 \in \mathbb{N}$ such that

$$\epsilon = \sigma < \sigma_{n_0} = \nu_\lambda(a_{n_0},a_{n_0+1}) < \varphi(\lambda)(\epsilon+\delta).$$

Using (3.1), we have

$$\begin{aligned}\sigma_{n_0+1} &= \nu_\lambda(a_{n_0+1}, a_{n_0+2})\\ &\le \alpha(a_{n_0}, a_{n_0+1})\nu_\lambda(a_{n_0+1}, a_{n_0+2})\\ &= \alpha(a_{n_0}, a_{n_0+1})\nu_\lambda(ga_{n_0}, ga_{n_0+1})\\ &< \epsilon\\ &= \sigma\end{aligned}$$

which is a contradiction. Therefore, $\sigma = 0$ and hence $\lim_{n\to\infty} \nu_\lambda(a_n, a_{n+1}) = 0$, $\forall\lambda > 0$. Let $\epsilon > 0$ be given and $\delta < \epsilon$ be such that (3.1) holds. Since $\sigma = 0$, $\exists N \in \mathbb{N}$ such that

$$\sigma_n = \nu_\lambda(a_n, a_{n+1}) < \delta, \quad \forall n \ge N \text{ and } \lambda \in \mathbb{R}_+. \tag{3.2}$$

Claim: For arbitrary fixed $m \ge N + 1$

$$\nu_\lambda(a_m, a_{m+l}) \le \epsilon \quad \forall l \in \mathbb{N}. \tag{3.3}$$

For $l = 1$, (3.3) holds by using (3.2).

Let (3.3) hold for $l = p$. Then, $\nu_\lambda(a_m, a_{m+p}) \le \epsilon$.

Now for $l = p + 1$, we have

$$\begin{aligned}\nu_\lambda(a_{m-1}, a_{m+p}) &\le \varphi(\lambda)\big(\nu_{\frac{\lambda}{2}}(a_{m-1}, a_m) + \nu_{\frac{\lambda}{2}}(a_m, a_{m+p})\big)\\ &\le \varphi(\lambda)(\delta + \epsilon).\end{aligned}$$

If $\nu_\lambda(a_{m-1}, a_{m+p}) \ge \epsilon$, then by (3.1), we have

$$\begin{aligned}\nu_\lambda(a_m, a_{m+p+1}) &\le \alpha(a_{m-1}, a_{m+p})\nu_\lambda(a_m, a_{m+p+1})\\ &= \alpha(a_{m-1}, a_{m+p})\nu_\lambda(ga_{m-1}, ga_{m+p})\\ &< \epsilon.\end{aligned}$$

If $\nu_\lambda(a_{m-1}, a_{m+p}) < \epsilon$, then

$$\nu_\lambda(a_m, a_{m+p+1}) = \nu_\lambda(ga_{m-1}, ga_{m+p}) \le \nu_\lambda(a_{m-1}, a_{m+p}) < \epsilon.$$

Thus, in any case, (3.3) holds. Hence $\{a_n\}$ is a ν-cauchy sequence. By completeness of $\mathcal{X}$, $\exists a^* \in \mathcal{X}$ such that $\nu_1(a_n, a^*) \to 0$ as $n \to \infty$.

$$\nu_1\big(a_{n+1}, ga^*\big) = \nu_1\big(ga_n, ga^*\big) \le \nu_1\big(a_{n+1}, a^*\big) \to 0 \quad \text{as } n \to \infty.$$

Now,

$$\nu_2\big(a^*, ga^*\big) \le \varphi(2)\big(\nu_1\big(a^*, a_{n+1}\big) + \nu_1\big(a_{n+1}, ga^*\big)\big) \to 0 \quad \text{as } n \to \infty.$$

Therefore, $\nu_2(a^*, ga^*) = 0$. By regularity of $\mathcal{X}$, we have $a^* = ga^*$. □

Remark 3.5 The above theorem ensures that the fixed point of the function exists. The following result ensures the uniqueness.

Proposition 3.6 *Let $g\colon \mathcal{X} \to \mathcal{X}$ be a self map on a complete regular φ-MM space $(\mathcal{X}, \nu)$. Let $\alpha\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+^0$ be such that g is a:*

1. *φ-α-MK contraction;*
2. *triangular α-admissible mapping.*

Also, let $\nu_\lambda(x,y) < \infty$, $\forall x,y \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+$. If $\alpha(x,y) \geq 1$, $\forall x,y \in \mathcal{X}$, then g has a unique fixed point.

Proof By Theorem 3.4, g has a fixed point. Let x_1 and x_2 be two fixed points of g. If $x_1 \neq x_2$, using Proposition 3.3, we get,

$$\nu_\lambda(x_1,x_2) = \nu_\lambda(gx_1,gx_2) < \nu_\lambda(x_1,x_2)$$

which is a contradiction. Thus, the fixed point of g is unique. □

Corollary 3.7 *Let $g\colon \mathcal{X} \to \mathcal{X}$ be a φ-MK contraction map on a complete regular φ-MM space $(\mathcal{X}, \nu)$ with $\nu_\lambda(x,y) < \infty$, $\forall x,y \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+$. Then, the fixed point of g is unique.*

Remark 3.8 It is a special case of Theorem 3.4, where $\alpha(x_1,x_2) = 1$, $\forall x_1,x_2 \in \mathcal{X}$.

Example Endow $\mathbb{R}$ with the φ-MM defined by

$$\nu_\lambda(x_1,x_2) = \bigl(1 + |\cos\lambda|\bigr)|x_1 - x_2|, \quad \forall x_1,x_2 \in \mathbb{R} \text{ and } \lambda \in \mathbb{R}_+.$$

Define $g\colon \mathbb{R} \to \mathbb{R}$ and $\alpha\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+^0$ by

$$g(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0); \\ \frac{a}{4} & \text{if } a \in [0,1]; \\ \frac{1}{4} & \text{if } a \in (1,\infty) \end{cases}$$

and

$$\alpha(a,b) = \begin{cases} 1 & \text{if } a,b \in [0,1]; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly g is a triangular α-admissible mapping. Also, $\mathbb{R}$ is ν-complete. Further, there exists $z_0 \in \mathbb{R}$ such that $\alpha(z_0, gz_0) \geq 1$. Next we shall show that g is φ-α-MK contraction.

Let $\epsilon > 0$ be given. Choose any $\delta > 0$ with $\delta < \epsilon$.

Let also $\epsilon \leq \nu_\lambda(a,b) < \varphi(\lambda)(\epsilon + \delta)$.

If a or $b \notin [0,1]$, then obviously $\alpha(a,b)\nu_\lambda(ga,gb) < \epsilon$.

Let $0 \leq a,b \leq 1$. Then by definition $\alpha(a,b) = 1$ and

$$\begin{aligned} \nu_\lambda(ga,gb) &= \bigl(1 + |\cos\lambda|\bigr)\left|\frac{a}{4} - \frac{b}{4}\right| \\ &= \frac{1}{4}\bigl(\bigl(1 + |\cos\lambda|\bigr)\bigr)|a - b| \end{aligned}$$

$$< \frac{1}{4}\big(1 + |\cos\lambda|\big)(\epsilon + \delta)$$
$$< \epsilon.$$

Hence for any given $\epsilon > 0$, $\exists\delta > 0$ such that whenever $\epsilon \le \nu_\lambda(a,b) < \varphi(\lambda)(\epsilon + \delta)$, then $\alpha(a,b)\nu_\lambda(ga,gb) < \epsilon$. Therefore, g is a φ-α-MK contraction. By Theorem 3.4, g has a fixed point. In fact, 0 is the fixed point of g.

4 Hausdorff distance on $K(\mathcal{X})$

In general, the φ-MM ν need not be continuous on $\mathbb{R}_+ \times \mathcal{X} \times \mathcal{X}$. For instance,

Example Let $\mathcal{X} = \mathbb{N} \cup \{\infty\}$ and define $\nu\colon \mathbb{R}_+ \times \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+^0$ by

$$\nu_\lambda(x,y) = \begin{cases} 0 & \text{if } x = y; \\ |\frac{1}{x} - \frac{1}{y}| & \text{if one of } x \neq y \text{ is odd, and the other is odd or } \infty; \\ 5 & \text{if one of } x \neq y \text{ is even, and the other is even or } \infty; \\ 2 & \text{otherwise.} \end{cases}$$

Then, ν_λ is a φ-MM with $\varphi(\lambda) = 3$.

Let $x_n = 2n+1$. Then, $\nu_\lambda(x_n,\infty) = \nu_\lambda(2n+1,\infty) = |\frac{1}{2n+1} - \frac{1}{\infty}| \to 0$ as $n \to \infty$. So, $x_n \to \infty$. Now, $\nu_\lambda(x_n,2) = \nu_\lambda(2n+1,2) = 2$ and $\nu_\lambda(\infty,2) = 5$.

Thus, $\nu_\lambda(x_n,2) \not\to \nu_\lambda(\infty,2)$ as $n \to \infty$, and hence ν_λ is not continuous on $\mathbb{R}_+ \times \mathcal{X} \times \mathcal{X}$.

For the rest of the sections, we will assume that ν is a continuous mapping. From here on, $(\mathcal{X},\nu)$ will denote φ-MM space, where ν is a continuous mapping and $\nu_\lambda(x,y) < \infty$, $\forall x,y \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+$.

Some more notations:

- $K(\mathcal{X}) := \{U \subseteq \mathcal{X} : U \text{ is non-empty and compact}\}$.
- $\nu_\lambda(x,U) := \inf\{\nu_\lambda(x,u)\colon u \in U\}$ for $x \in \mathcal{X}$ and $U \subseteq \mathcal{X}$.
- $\nu_\lambda(U,W) := \inf\{\nu_\lambda(u,w) : u \in U, w \in W\}$ for $U, W \in K(\mathcal{X})$.
- $B_\lambda(x,\epsilon) := \{y \in \mathcal{X} : \nu_\lambda(x,y) < \epsilon\}$.

Definition 4.1 Let $(\mathcal{X},\nu)$ be a φ-MM space. A set $U \subseteq \mathcal{X}$ is said to be totally bounded if for any given $\epsilon > 0$, $\exists$ finite collection $\{u_i; 1 \le i \le k\} \subseteq U$ for some $k \in \mathbb{N}$ such that $U \subseteq \bigcup_{i=1}^n B_\lambda(u_i,\epsilon)$, $\forall\lambda \in \mathbb{R}_+$.

Proposition 4.2 *Let $(\mathcal{X},\nu)$ be a (regular) φ-MM space. Then, for each $x \in \mathcal{X}$, $U \in K(\mathcal{X})$ and $\lambda \in \mathbb{R}_+$,*

$$\nu_\lambda(x,U) = \nu_\lambda(x,u_0) \quad \textit{for some } u_0 \in U.$$

Proof Since $u \mapsto \nu_\lambda(x,u)$ is a continuous function, so by compactness of U, $\exists u_0 \in U$ such that $\inf\{\nu_\lambda(x,u)\colon u \in T\} = \nu_\lambda(x,u_0)$. Thus, $\nu_\lambda(x,U) = \nu_\lambda(x,u_0)$. □

Proposition 4.3 *Let $(\mathcal{X},\nu)$ be a (regular) φ-MM space. Then, for every $U, W \in K(\mathcal{X})$ and $\lambda \in \mathbb{R}_+$, $\exists u_0 \in U$ such that* $\sup_{u\in U}\{\nu_\lambda(u,W)\} = \nu_\lambda(u_0,W)$.

Proof Let $\epsilon = \sup_{u\in U} \nu_\lambda(u, W)$. Then, $\exists u_n \in U$ such that $\epsilon - \frac{1}{n} < \nu_\lambda(u_n, W)$. Since $U \in K(\mathcal{X})$, $\exists$ subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u_0 \in U$ such that $u_{n_k} \to u_0$. Let $w \in W$ be such that $\nu_\lambda(u_0, w) = \nu_\lambda(u_0, W)$. Then, $\lim_{k\to\infty} \nu_\lambda(u_{n_k}, w) = \nu_\lambda(u_0, w)$.

Since for each $k \in \mathbb{N}$, $\epsilon - \frac{1}{n_k} < \nu_\lambda(u_{n_k}, W)$.

We have, $\epsilon \le \nu_\lambda(u_0, w) = \nu_\lambda(u_0, W)$ and obviously $\epsilon \ge \nu_\lambda(u_0, W)$.

Therefore, $\sup_{u\in U}\{\nu_\lambda(U, W)\} = \nu_\lambda(u_0, W)$. □

Consider a (regular) φ-MM space $(\mathcal{X}, \nu)$. We define a function $H_\nu : \mathbb{R}_+ \times K(\mathcal{X}) \times K(\mathcal{X}) \to \mathbb{R}_+^0$ by

$$H_\nu(\lambda, U, W) = \max\Big\{\sup_{u\in U} \nu_\lambda(u, W), \sup_{w\in W} \nu_\lambda(U, w)\Big\}$$

or, equivalently,

$$H_\nu(\lambda, U, W) = \inf\{\epsilon \ge 0 : U \subseteq W + \epsilon, W \subseteq U + \epsilon\},$$

where $U + \epsilon = \{x \in \mathcal{X} : \nu_\lambda(x, u) < \epsilon \text{ for some } u \in U\}$.

Proposition 4.4 *Let $(\mathcal{X}, \nu)$ be a (regular) φ-MM space. Let $U, W \in K(\mathcal{X})$ and $\lambda, \mu \in (0, \infty)$. Then*

$$\nu_{\lambda+\mu}(s, U) \le \varphi(\lambda + \mu)\big\{\nu_\lambda(s, W) + \nu_\mu(w_s, U)\big\},$$

where $w_s \in W$ such that $\nu_\lambda(s, w_s) = \nu_\lambda(s, W)$ and $s \in \mathcal{X}$.

Proof By Proposition 4.2, $\exists w_s \in W$ such that $\nu_\lambda(s, W) = \nu_\lambda(s, w_s)$.

For each $u \in U$, we have

$$\nu_{\lambda+\mu}(s, U) \le \nu_{\lambda+\mu}(s, u) \le \varphi(\lambda + \mu)\big\{\nu_\lambda(s, w_s) + \nu_\mu(w_s, u)\big\}.$$

Hence,

$$\nu_{\lambda+\mu}(s, U) \le \varphi(\lambda + \mu)\big\{\nu_\lambda(s, W) + \nu_\mu(w_s, U)\big\},$$

which completes the proof. □

Theorem 4.5 *Consider the (regular) φ-MM space $(\mathcal{X}, \nu)$. Then, $(K(\mathcal{X}), H_\nu)$ is also a (regular) φ-MM space with the same φ.*

Proof Let $S, U, W \in K(\mathcal{X})$ and $\lambda, \mu \in \mathbb{R}_+$.

Then, $H_\nu(\lambda, S, U) \ge 0$ and $S = U$ if and only if $H_\nu(\lambda, S, U) = 0$.

Now, we shall show the triangle inequality. By Proposition 4.4, for $s \in S$, we have

$$\nu_{\lambda+\mu}(s, U) \le \varphi(\lambda + \mu)\big\{\nu_\lambda(s, W) + \nu_\lambda(w_s, U)\big\},$$

where $w_s \in W$ such that $\nu_\lambda(s, w_s) = \nu_\lambda(s, W)$.

$$\begin{aligned}\sup_{s\in S} \nu_{\lambda+\mu}(s, U) &\leq \varphi(\lambda+\mu)\Big\{\sup_{s\in S} \nu_\lambda(s, W) + \sup_{s\in S} \nu_\mu(w_s, U)\Big\}\\ &\leq \varphi(\lambda+\mu)\Big\{\sup_{s\in S} \nu_\lambda(s, W) + \sup_{w\in W} \nu_\mu(w, U)\Big\}.\end{aligned}$$

Similarly,

$$\sup_{u\in U} \nu_{\lambda+\mu}(S, u) \leq \varphi(\lambda+\mu)\Big\{\sup_{w\in W} \nu_\lambda(S, w) + \sup_{u\in U} \nu_\mu(W, u)\Big\}.$$

Now,

$$\begin{aligned}H_\nu(\lambda+\mu, S, U) &= \max\Big\{\sup_{s\in S} \nu_{\lambda+\mu}(s, U), \sup_{u\in U} \nu_{\lambda+\mu}(S, u)\Big\}\\ &\leq \varphi(\lambda+\mu)\max\Big\{\sup_{s\in S} \nu_\lambda(s, W) + \sup_{w\in W} \nu_\mu(w, U),\\ &\quad \sup_{w\in W} \nu_\lambda(S, w) + \sup_{u\in U} \nu_\mu(W, u)\Big\}\\ &\leq \varphi(\lambda+\mu)\Big\{\max\Big\{\sup_{s\in S} \nu_\lambda(s, W), \sup_{w\in W} \nu_\lambda(S, w)\Big\}\\ &\quad + \max\Big\{\sup_{w\in W} \nu_\mu(w, U), \sup_{u\in U} \nu_\mu(W, u)\Big\}\Big\}\\ &\leq \varphi(\lambda+\mu)\big\{H_\nu(\lambda, S, W) + H_\nu(\mu, W, U)\big\}.\end{aligned}$$

Hence $(K(\mathcal{X}), H_\nu)$ is a φ-MM space with the same φ. □

Proposition 4.6 *Let $U, W \in K(\mathcal{X})$. For each $u \in U$, $\exists w \in W$ such that $\nu_\lambda(u, w) \leq H_\nu(\lambda, U, W)$.*

Proof Let $u \in U$. Then, by Proposition 4.2, $\exists w \in W$ such that $\nu_\lambda(u, w) = \nu_\lambda(u, W) \leq \sup_{u\in U} \nu_\lambda(u, W) \leq H_\nu(\lambda, U, W)$. □

Proposition 4.7 *Consider a complete (regular) φ-MM space $(\mathcal{X}, \nu)$. Every closed subset W of $\mathcal{X}$ is complete.*

Proof It is straightforward. □

Proposition 4.8 *A (regular) φ-MM space $(\mathcal{X}, \nu)$, with $\varphi(\lambda) \leq \Lambda$ for some $\Lambda \geq 1$, is compact iff it is totally bounded and complete.*

Proof We omit the proof as it is analogous to the case where $(\mathcal{X}, d)$ is a metric space. □

Proposition 4.9 *Consider a (regular) φ-MM space $(\mathcal{X}, \nu)$ such that $\varphi(\lambda) \leq \Lambda$ for some $\Lambda \geq 1$. If $\{a_n\}$ is a sequence in $\mathcal{X}$ such that*

$$\nu_\lambda(a_n, a_{n+1}) < \frac{1}{(\Lambda+1)^n}, \quad \forall \lambda \in \mathbb{R}_+ \text{ and } n \in \mathbb{N},$$

then $\{a_n\}$ is a Cauchy sequence.

Proof Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $\{\frac{\Lambda}{\Lambda+1}\}^{N-1} < \frac{\epsilon}{2}$.

For $n > m \geq N$, we have

$$\begin{aligned}
\nu_\lambda(a_m, a_n) &\leq \sum_{k=1}^{n-m-1} \nu_{\frac{\lambda}{2^k}}(a_{m+k-1}, a_{m+k}) \prod_{j=0}^{k-1} \varphi\left(\frac{\lambda}{2^j}\right) \\
&\quad + \prod_{j=0}^{n-m-2} \varphi\left(\frac{\lambda}{2^j}\right) \nu_{\frac{\lambda}{2^{n-m-1}}}(a_{n-1}, a_n) \\
&\leq \frac{\Lambda}{(\Lambda+1)^{m-1}} + \left\{\frac{\Lambda}{\Lambda+1}\right\}^{n-1} \frac{1}{\Lambda^m} \\
&\leq \left\{\frac{\Lambda}{\Lambda+1}\right\}^{m-1} + \left\{\frac{\Lambda}{\Lambda+1}\right\}^{n-1} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, $\{a_n\}$ is a Cauchy sequence. □

Proposition 4.10 *Let $(\mathcal{X}, \nu)$ be a (regular) φ-MM space. For each $U \in K(\mathcal{X})$, the set $U + \epsilon$ is a closed set.*

Proof Let $U \in K(\mathcal{X})$ and u be any limit point of $U + \epsilon$. Then, $\exists \{u_n\}$ such that $u_n \to u$ as $n \to \infty$, where $u_n \in U + \epsilon$, $\forall n \in \mathbb{N}$.

Clearly, $\nu_\lambda(u_n, U) \leq \epsilon$, $\forall n \in \mathbb{N}$. By Proposition 4.2, $\exists x_n \in U$ such that $\nu_\lambda(u_n, U) = \nu_\lambda(u_n, x_n)$ and hence $\nu_\lambda(u_n, x_n) \leq \epsilon$, $\forall n \in \mathbb{N}$. U being a compact subset, $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some point of U, say, x. By continuity of ν, we have $\nu_\lambda(u_{n_k}, x_{n_k}) \to \nu_\lambda(u, x)$ as $k \to \infty$. So, $\nu_\lambda(u, x) \leq \epsilon$. Therefore, $u \in U + \epsilon$ and hence $U + \epsilon$ is a closed set. □

Proposition 4.11 *Consider a (regular) φ-MM space $(\mathcal{X}, \nu)$ such that $\varphi(\lambda) \leq \Lambda$ for some $\Lambda \geq 1$. For a Cauchy sequence $\{U_n\}$ in $K(\mathcal{X})$, let $\{u_{n_k}\}$ be a Cauchy sequence in $\mathcal{X}$ such that $u_{n_k} \in U_{n_k}$, $\forall k \in \mathbb{N}$, for some increasing sequence of natural numbers. Then, $\exists$ a Cauchy sequence $\{x_n\}$ such that $x_n \in U_n$ and $x_{n_k} = u_{n_k}$, $\forall k \in \mathbb{N}$.*

Proof Let $\{u_{n_k}\}$ be a Cauchy sequence in $\mathcal{X}$ such that $u_{n_k} \in U_{n_k}$, $\forall k \in \mathbb{N}$. For $n_{k-1} < n \leq n_k$, where $n_0 = 0$, using Proposition 4.2, choose $x_n \in U_n$ such that $\nu_\lambda(u_{n_k}, U_n) = \nu_\lambda(u_{n_k}, x_n)$. Then,

$$\nu_\lambda(u_{n_k}, x_n) = \nu_\lambda(u_{n_k}, U_n) \leq \sup_{x \in U_{n_k}} \{\nu_\lambda(x, U_n)\} \leq H_\nu(\lambda, U_{n_k}, U_n).$$

Clearly, $u_{n_k} = x_{n_k}$, $\forall k \in \mathbb{N}$. Let $\epsilon > 0$ be given. Since $\{U_n\}$ is a Cauchy sequence in $K(\mathcal{X})$, $\exists N_1 \in \mathbb{N}$ such that $H_\nu(\lambda, U_n, U_m) < \frac{\epsilon}{\Lambda+2\Lambda^2}$, $\forall n, m \geq N_1$ and $\lambda \in \mathbb{R}_+$. Also, since $\{u_{n_k}\}$ is a Cauchy sequence in $\mathcal{X}$, $\exists N_2 \in \mathbb{N}$ such that $\nu_\lambda(u_{n_k}, u_{n_j}) < \frac{\epsilon}{\Lambda+2\Lambda^2}$, $\forall n_k, n_j \geq N_2$ and $\lambda \in \mathbb{R}_+$.

Now for $n, m \geq N = \max\{N_1, N_2\}$,

$$\begin{aligned}
\nu_\lambda(x_n, x_m) &\leq \varphi(\lambda)\left\{\nu_{\frac{\lambda}{2}}(x_n, u_{n_k}) + \nu_{\frac{\lambda}{2}}(u_{n_k}, x_m)\right\} \\
&\leq \varphi(\lambda)\nu_{\frac{\lambda}{2}}(x_n, u_{n_k}) + \varphi(\lambda)\varphi\left(\frac{\lambda}{2}\right)\nu_{\frac{\lambda}{4}}(u_{n_k}, u_{n_j}) \\
&\quad + \varphi(\lambda)\varphi\left(\frac{\lambda}{2}\right)\nu_{\frac{\lambda}{4}}(u_{n_j}, x_m) \\
&= \varphi(\lambda)\nu_{\frac{\lambda}{2}}(u_{n_k}, U_n) + \varphi(\lambda)\varphi\left(\frac{\lambda}{2}\right)\nu_{\frac{\lambda}{4}}(u_{n_k}, u_{n_j}) \\
&\quad + \varphi(\lambda)\varphi\left(\frac{\lambda}{2}\right)\nu_{\frac{\lambda}{4}}(u_{n_j}, U_m) \\
&\leq \varphi(\lambda)H_\nu\left(\frac{\lambda}{2}, U_{n_k}, U_n\right) + \varphi(\lambda)\varphi\left(\frac{\lambda}{2}\right)\nu_{\frac{\lambda}{4}}(u_{n_k}, u_{n_j}) \\
&\quad + \varphi(\lambda)\varphi\left(\frac{\lambda}{2}\right)H_\nu\left(\frac{\lambda}{4}, U_{n_j}, U_m\right) \\
&\leq \Lambda\frac{\epsilon}{\Lambda + 2\Lambda^2} + \Lambda^2\frac{\epsilon}{\Lambda + 2\Lambda^2} + \Lambda^2\frac{\epsilon}{\Lambda + 2\Lambda^2} \\
&= \epsilon.
\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence such that $x_n \in U_n$ and $x_{n_k} = u_{n_k}$, $\forall k \in \mathbb{N}$. □

Proposition 4.12 *Consider a complete (regular) φ-MM space $(\mathcal{X}, \nu)$ such that $\varphi(\lambda) \leq \Lambda$ for some $\Lambda \geq 1$, and let $\{U_n\} \in K(\mathcal{X})$ be a Cauchy sequence. Define $U = \{u \in \mathcal{X} : u_n \to u, \text{where } u_n \in U_n\}$. Then, the set U is non-empty and closed.*

Proof Given that $\{U_n\}$ is a Cauchy sequence in $K(\mathcal{X})$, choose $n_1 \in \mathbb{N}$ such that $H_\nu(\lambda, U_m, U_n) < \frac{1}{\Lambda+1}$, $\forall n, m \geq n_1$ and $\lambda \in \mathbb{R}_+$. Again choose $n_2 > n_1$ such that $H_\nu(\lambda, U_m, U_n) < \frac{1}{(\Lambda+1)^2}$, $\forall n, m \geq n_2$ and $\lambda \in \mathbb{R}_+$. Continuing the process, we get an increasing sequence $\{n_k\}$ such that $H_\nu(\lambda, U_m, U_n) < \frac{1}{(\Lambda+1)^k}$, $\forall n, m \geq n_k$ and $\lambda \in \mathbb{R}_+$. Let us fix an element $u_{n_1} \in U_{n_1}$. Using Proposition 4.2, $\exists u_{n_2} \in U_{n_2}$ such that $\nu_\lambda(u_{n_1}, u_{n_2}) = \nu_\lambda(u_{n_1}, U_{n_2})$. Now,

$\nu_\lambda(u_{n_1}, u_{n_2}) = \nu_\lambda(u_{n_1}, U_{n_2}) \leq \sup_{u \in U_{n_1}}\{\nu_\lambda(u, U_{n_2})\} \leq H_\nu(\lambda, U_{n_1}, U_{n_2}) < \frac{1}{(\Lambda+1)}$. Similarly, we choose $u_{n_3} \in U_{n_3}$ such that $\nu_\lambda(u_{n_2}, u_{n_3}) = \nu_\lambda(u_{n_2}, U_{n_3}) \leq H_\nu(\lambda, U_{n_2}, U_{n_3}) < \frac{1}{(\Lambda+1)^2}$. Continuing the process, we get a sequence $\{u_{n_k}\}$, where $u_{n_k} \in U_{n_k}$, $\forall k \in \mathbb{N}$ such that

$$\nu_\lambda(u_{n_k}, u_{n_{k+1}}) \leq H_\nu(\lambda, U_{n_k}, U_{n_{k+1}}) < \frac{1}{(\Lambda + 1)^k}, \quad \forall \lambda \in \mathbb{R}_+.$$

Using Proposition 4.9, we get $\{u_{n_k}\}$ is a Cauchy sequence. Again, by Proposition 4.11, $\exists$ a Cauchy sequence $\{x_n\}$ in $\mathcal{X}$ such that $x_n \in U_n$ and $x_{n_k} = u_{n_k}$, $\forall k \in \mathbb{N}$. $\mathcal{X}$ being complete, $\{x_n\}$ converges to x (say) $\in \mathcal{X}$. Thus, U is a non-empty set.

Let z be any limit point of U. Then, $\exists$ a sequence $\{z_k\} \in U \setminus \{z\}$ such that $z_k \to z$ as $k \to \infty$. Since each $z_k \in U$, $\exists$ a sequence $\{a_n^k\}$ such that $a_n^k \to z_k$ as $n \to \infty$ and $a_n^k \in U_n$ for each $n \in \mathbb{N}$. It follows that $\exists n_1$ such that $a_{n_1}^1 \in U_{n_1}$ and $\nu_\lambda(a_{n_1}^1, z_1) < 1$. Similarly, $\exists n_2 > n_1$ such that $a_{n_2}^2 \in U_{n_2}$ and $\nu_\lambda(a_{n_2}^2, z_2) < \frac{1}{2}$. Continuing the process, we get an increasing sequence $\{n_k\}$ such that $\nu_\lambda(a_{n_k}^k, z_k) < \frac{1}{k}$, $\forall k \in \mathbb{N}$ and $\lambda \in \mathbb{R}_+$. Now, $\nu_\lambda(a_{n_k}^k, z) \leq \varphi(\lambda)\{\nu_{\frac{\lambda}{2}}(a_{n_k}^k, z_k) + \nu_{\frac{\lambda}{2}}(z_k, z)\}$. So, $a_{n_k}^k \to z$ as $k \to \infty$. Thus, $\{a_{n_k}\}$ is a Cauchy sequence such that $a^k_{n_k} \in U_{n_k}$, $\forall k \in \mathbb{N}$. Using Proposition 4.11, $\exists$ a Cauchy sequence $\{y_n\}$ in $\mathcal{X}$ such that $y_n \in U_n$ and $y_{n_k} = a^k_{n_k}$. Thus, $z \in U$, and, hence, U is a closed set. □

Proposition 4.13 *Consider a complete (regular) φ-MM space $(\mathcal{X}, \nu)$ such that $\varphi(\lambda) \le \Lambda$ for some $\Lambda \ge 1$, and let $\{U_n\}$ be a sequence of totally bounded subsets of $\mathcal{X}$. Also, let $U \subseteq \mathcal{X}$ be such that for each $\epsilon > 0$, $U \subseteq U_N + \epsilon$ for some $N \in \mathbb{N}$. Then, U is also a totally bounded set.*

Proof Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $U \subseteq U_N + \frac{\epsilon}{4\Lambda^2}$. Since U_N is a totally bounded set, there exists a finite set $\{u_i \in U_N; 1 \le i \le k\}$ such that $U_N \subseteq \bigcup_{i=1}^k B_\lambda(u_i, \frac{\epsilon}{4\Lambda^2})$, $\forall \lambda \in \mathbb{R}_+$. For each $u \in U$, $\exists x \in U_N$ such that $\nu_\lambda(x, u) \le \frac{\epsilon}{4\Lambda^2}$, $\forall \lambda \in \mathbb{R}_+$. Moreover, $\exists u_i \in U_N$ such that $\nu_\lambda(x, u_i) \le \frac{\epsilon}{4\Lambda^2}$, $\forall \lambda \in \mathbb{R}_+$. Now,

$$\begin{aligned}
\nu_\lambda(u, u_i) &\le \varphi(\lambda)\left\{\nu_{\frac{\lambda}{2}}(u, x) + \nu_{\frac{\lambda}{2}}(x, u_i)\right\} \\
&\le \Lambda\left\{\frac{\epsilon}{4\Lambda^2} + \frac{\epsilon}{4\Lambda^2}\right\} \\
&= \frac{\epsilon}{2\Lambda}.
\end{aligned}$$

Therefore, for some $1 \le i \le k$, $B_\lambda(u_i, \frac{\epsilon}{2\Lambda}) \cap U \neq \emptyset$, $\forall \lambda \in \mathbb{R}_+$. By reordering u_i's, if required, we may assume that

$B_\lambda(u_i, \frac{\epsilon}{2\Lambda}) \cap U \neq \emptyset$ for $1 \le i \le p$ and $B_\lambda(u_i, \frac{\epsilon}{2\Lambda}) \cap U = \emptyset$ for $p < i \le k$.

Now, for each $1 \le i \le p$, let $y_i \in B_\lambda(u_i, \frac{\epsilon}{2\Lambda}) \cap U$. Let $u \in U$. Then,

$$\begin{aligned}
\nu_\lambda(u, y_i) &\le \varphi(\lambda)\left\{\nu_{\frac{\lambda}{2}}(u, u_i) + \nu_{\frac{\lambda}{2}}(u_i, y_i)\right\} \\
&\le \Lambda\left\{\frac{\epsilon}{2\Lambda} + \frac{\epsilon}{2\Lambda}\right\} \\
&= \epsilon.
\end{aligned}$$

Thus, for each $u \in U$, $\exists y_i, 1 \le i \le p$ such that $u \in B_\lambda(y_i, \epsilon)$, $\forall \lambda \in \mathbb{R}_+$. Hence, U is totally bounded. □

Proposition 4.14 *Consider a complete (regular) φ-MM space $(\mathcal{X}, \nu)$ such that $\varphi(\lambda) \le \Lambda$ for some $\Lambda \ge 1$. Then, $(K(\mathcal{X}), H_\nu)$ is also a complete (regular) φ-MM space.*

Proof Since $(\mathcal{X}, \nu)$ is a φ-MM space, by Theorem 4.5, $(K(\mathcal{X}), H_\nu)$ is also a φ-MM space. Let $\{U_n\}$ be a Cauchy sequence in $K(\mathcal{X})$. Then, each U_n is totally bounded and complete. Define $U = \{x \in \mathcal{X} : x_n \to x, \text{where } x_n \in U_n\}$. We shall show that $U \in K(\mathcal{X})$ and $\{U_n\}$ converges to U. By Proposition 4.12, U is non-empty and closed. Let $\epsilon > 0$ be given. Since $\{U_n\}$ is a Cauchy sequence $\exists N \in \mathbb{N}$ such that $H_\nu(\lambda, U_m, U_n) < \epsilon$, $\forall m, n \ge N$ and $\lambda \in \mathbb{R}_+$. Then, $U_m \subseteq U_n + \epsilon$, $\forall m, n \ge N$. Let $u \in U$ and fix $n \ge N$. Then, $\exists$ a sequence $\{u_i\}$ such that $\{u_i\}$ converges to u and $u_i \in U_i$, $\forall i \in \mathbb{N}$. By Proposition 4.10, $U_n + \epsilon$ is closed, and since $u_i \in U_n + \epsilon$, $\forall i \ge N$, we get that $u \in U_n + \epsilon$. Hence $U \subseteq U_n + \epsilon$. By Proposition 4.13, U is totally bounded. Also, the set U is complete. Since U is totally bounded and complete, U is compact. Thus, $U \in K(\mathcal{X})$.

Let $\epsilon > 0$ be given. We shall prove that $\exists N_1 \in \mathbb{N}$ such that $H_\nu(\lambda, U_n, U) < \epsilon$, $\forall n \ge N_1$ and $\lambda \in \mathbb{R}_+$. It is sufficient to show that $U \subseteq U_n + \epsilon$ and $U_n \subseteq U + \epsilon$. From the first part of the proof, it is already known that $\exists N_1$ such that $U \subseteq U_n + \epsilon$ $\forall n \geq N_1$. Now, we shall show that $U_n \subseteq U + \epsilon$. Let $y \in U_n$. Since $\{U_n\}$ is a Cauchy sequence, $\exists N_2 \in \mathbb{N}$ such that $H_\nu(\lambda, U_n, U_m) < \frac{\epsilon}{2(\Lambda+1)} = \epsilon_1$, $\forall n \geq N_2$. Moreover, $\exists$ a strictly increasing sequence of natural numbers $\{n_k\}$ such that $H_\nu(\lambda, U_m, U_n) < \frac{\epsilon_1}{(1+\Lambda)^{k+1}}$, $\forall m, n \geq n_k$ and $n_1 > N_2$. Since $U_n \subseteq U_{n_1} + \frac{\epsilon_1}{(1+\Lambda)}$, $\exists x_{n_1} \in U_{n_1}$ such that $\nu_\lambda(y, x_{n_1}) < \frac{\epsilon_1}{(1+\Lambda)}$. Again, since $U_{n_1} \subseteq U_{n_2} + \frac{\epsilon_1}{(1+\Lambda)^2}$, $\exists x_{n_2} \in U_{n_2}$ such that $\nu_\lambda(x_{n_1}, x_{n_2}) < \frac{\epsilon_1}{(1+\Lambda)^2}$. Continuing the process, we get a sequence $\{x_{n_i}\}$ such that $\nu_\lambda(x_{n_i}, x_{n_{i+1}}) < \frac{\epsilon_1}{(1+\Lambda)^{i+1}}$ for $i \in \mathbb{N}^*$, where $y = x_{n_0}$. Since $\{x_{n_i}\}$ is a Cauchy sequence, by Proposition 4.11, $\exists$ a Cauchy sequence $\{a_n\}$ such that $a_n \in U_n$, $\forall n \in \mathbb{N}$ and $a_{n_i} = x_{n_i}$, $\forall i \in \mathbb{N}$. Let $\{a_n\}$ converges to a. Then,

$$\begin{aligned}
\nu_\lambda(y, x_{n_i}) &\leq \left\{ \sum_{k=1}^{i-1} \nu_{\frac{\lambda}{2^k}}(x_{n_{k-1}}, x_{n_k}) \prod_{j=0}^{k-1} \varphi\left(\frac{\lambda}{2^j}\right) \right\} + \nu_{\frac{\lambda}{2^{i-1}}}(x_{n_{i-1}}, x_{n_i}) \prod_{j=0}^{i-2} \varphi\left(\frac{\lambda}{2^j}\right) \\
&\leq \left\{ \sum_{k=1}^{i-1} \left\{ \frac{\Lambda}{1+\Lambda} \right\}^k \epsilon_1 \right\} + \frac{\Lambda^{i-1}}{(1+\Lambda)^i} \epsilon_1 \\
&< \epsilon_1(1+\Lambda) \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

Since ν is a continuous function, we have $\nu_\lambda(y, a) < \epsilon$ and hence $U_n \subseteq U + \epsilon$. Thus, $\exists N_1 \in \mathbb{N}$ such that $U_n \subseteq U + \epsilon$, $\forall n \geq N_1$. Therefore, $H_\nu(\lambda, U_n, U) < \epsilon$ $\forall n \geq N_1$ and $\lambda \in \mathbb{R}_+$. Thus, $\{U_n\}$ converges to $U \in K(\mathcal{X})$. This completes the proof. □

5 Iterated function system

Definition 5.1 A mapping $g\colon Y \to Y$ on a complete metric space (Y, d) is called a contraction mapping if

$$d\big(g(a), g(b)\big) \leq r d(a, b), \quad \forall a, b \in Y \text{ and for some constant } r \in [0, 1).$$

The constant r is said to be the contractivity factor for g.

Definition 5.2 A complete metric space (Y, d) together with a finite collection of contraction mappings $g_n\colon Y \to Y$; $n \in \mathbb{N}_m$ is called an iterated function system (IFS).

Define $F\colon K(Y) \to K(Y)$, known as the Hutchinson operator, by $F(A) = \bigcup_{n=1}^m g_n(A)$ for each $A \in K(Y)$, where $g_n(A) = \{g_n(x) \colon x \in A\}$. Any set $G \in K(Y)$ such that $F(G) = G$ is called an attractor of the IFS.

Similarly, we define the iterated function system (IFS) on the φ-MM space consisting of φ-α-MK contractions as follows:

Definition 5.3 A complete (regular) φ-MM space, $(\mathcal{X}, \nu)$, together with a finite collection $g_n\colon \mathcal{X} \to \mathcal{X}$; $n \in \mathbb{N}_N$ of φ-α-MK contractions is called φ-α-MK contractive iterated function system in $(\mathcal{X}, \nu)$ and will be denoted as $\{\mathcal{X}; g_n, n \in \mathbb{N}_N\}$.

Define $F\colon K(\mathcal{X}) \to K(\mathcal{X})$ by $F(A) = \bigcup_{n=1}^N g_n(A)$ for each $A \in K(\mathcal{X})$, where $g_n(A) = \{g_n(x) \colon x \in A\}$. Any set $G \in K(\mathcal{X})$ such that $F(G) = G$ is called an attractor for the IFS.

Remark 5.4 We call the set valued map F on $K(\mathcal{X})$ as the Hutchinson operator.

Let $\{\mathcal{X}; g_k, k \in \mathbb{N}_N\}$ be a φ-α-MK contractive IFS on a (regular) φ-MM space $(\mathcal{X}, \nu)$. We associate a multivalued function on $K(\mathcal{X})$ using g_k; $k \in \mathbb{N}_N$ as follows:

$$\psi : \mathcal{X} \to K(\mathcal{X}) \quad \text{defined by} \quad \psi(x) = \{g_k(x) : k = 1, 2, \dots, N\}.$$

Then, the operator $F : K(\mathcal{X}) \to K(\mathcal{X})$ can also be written as

$$F(B) = \bigcup_{b \in B} \psi(b) = \bigcup_{n=1}^{N} g_k(B).$$

Proposition 5.5 *Let ψ be defined as above. Then, ψ is a continuous map.*

Proof Let $\epsilon > 0$ be given. Also, let $x_n \to x$ in $\mathcal{X}$. Define,

$$B_n = \psi(x_n) = \{g_1(x_n), g_2(x_n), \dots, g_N(x_n)\},$$
$$B = \psi(x) = \{g_1(x), g_2(x), \dots, g_N(x)\}.$$

As each g_k; $k \in \mathbb{N}_N$ is a continuous map, so $g_k(x_n) \to g_k(x)$ for each $k \in \mathbb{N}_N$.

So, for $\frac{\epsilon}{2} > 0$, $\exists m \in \mathbb{N}$ such that for all $\lambda \in \mathbb{R}_+$,

$$\nu_\lambda\big(g_k(x_n), g_k(x)\big) < \frac{\epsilon}{2} \quad \text{for all } n \geq m \text{ and for all } k \in \mathbb{N}_N.$$

Then, $B_n \subseteq B + \frac{\epsilon}{2}$ and $B \subseteq B_n + \frac{\epsilon}{2}$ for all $n \geq m$.

So, $H_\nu(\lambda, B_n, B) = \inf\{\delta \geq 0; B_n \subseteq B + \delta, B \subseteq B_n + \delta\} < \epsilon$.

Therefore, $\psi(x_n) \to \psi(x)$, and hence ψ is a continuous map. □

Proposition 5.6 *The Hutchinson operator for $\{\mathcal{X}; g_n, n \in \mathbb{N}_N\}$ is a continuous map on $K(\mathcal{X})$.*

Proof Let $S_n, S \in K(\mathcal{X})$ such that $S_n \to S$ with respect to H_ν.

Setting $S^* = \bigcup_{n=1}^{\infty}\{S_n\} \cup S$ we get $S^* \in K(\mathcal{X})$. By Proposition 5.5, ψ is a continuous map and S^* being compact, it is uniformly continuous on S^*. Let $\epsilon > 0$ be given. Then, for $\frac{\epsilon}{2} > 0$, we find $\delta > 0$ such that for every pair $z_1, z_2 \in S^*$, $\nu_\lambda(z_1, z_2) < \delta$ implies $H_\nu(\lambda, \psi(z_1), \psi(z_2)) < \frac{\epsilon}{2}$.

Now let $S_1, S_2 \in K(S^*)$ be such that $H_\nu(\lambda, S_1, S_2) < \delta$. Then,

$$S_2 \subseteq S_1 + \delta \quad \text{and} \quad S_1 \subseteq S_2 + \delta.$$

As, $S_2 \subseteq S_1 + \delta$ and using the uniform continuity of ψ on S^*, we have

$$\psi(S_2) \subseteq \psi(S_1 + \delta) \subseteq \big(\psi(S_1)\big) + \epsilon.$$

By symmetry, $\psi(S_1) \subseteq (\psi(S_2)) + \epsilon$. Hence, $H_\nu(\lambda, \psi(S_1), \psi(S_2)) < \epsilon$. So, F is uniformly continuous on $K(S^*)$, and consequently, $F(S_n) \to F(S)$ as $n \to \infty$. Therefore, F is continuous on $K(\mathcal{X})$. □

Theorem 5.7 *Let $(\mathcal{X},\nu)$ be a (regular) φ-MM space. Let $g_n\colon \mathcal{X}\to\mathcal{X}$ be φ-α-MK contraction for $n\in\mathbb{N}_N$. Then, the function $F\colon K(\mathcal{X})\to K(\mathcal{X})$ defined by $F(A)=\bigcup_{n=1}^N g_n(A)$, where $g_n(A)=\{g_n(a):a\in A\}$ for every $A\in K(\mathcal{X})$ is a φ-MK contraction map with respect to the induced (regular) φ-metric modular H_ν.*

Proof Let $\epsilon>0$ be given. Then, $\exists\delta_n>0$; $n\in\mathbb{N}_N$ such that

$$\epsilon\le\nu_\lambda(x,y)<\varphi(\lambda)(\epsilon+\delta_n)\quad\text{implies}\quad\alpha(x,y)\nu_\lambda\big(g_n(x),g_n(y)\big)<\epsilon.$$

Let $A,B\in K(\mathcal{X})$ be such that $\epsilon\le H_\nu(\lambda,A,B)<\varphi(\lambda)(\epsilon+\delta)$, where $\delta=\min\{\delta_n:n\in\mathbb{N}_N\}$. We shall show that $H_\nu(\lambda,F(A),F(B))<\epsilon$.

Let $z\in F(A)$ be arbitrary. Then, $\exists j\in\mathbb{N}_N$ and $x\in A$ such that $z=g_j(x)$. By Proposition 4.6, $\exists y\in B$ such that

$$\nu_\lambda(x,y)\le H_\nu(\lambda,A,B)<\varphi(\lambda)(\epsilon+\delta).$$

If $\nu_\lambda(x,y)\ge\epsilon$, then $\epsilon\le\nu_\lambda(x,y)<\varphi(\lambda)(\epsilon+\delta)$ and hence $\alpha(x,y)\nu_\lambda(g_j(x),g_j(y))<\epsilon$. Otherwise, $\nu_\lambda(x,y)<\epsilon$ then $\nu_\lambda(g_j(x),g_j(y))<\nu_\lambda(x,y)<\epsilon$.

Therefore, $\nu_\lambda(z,F(B))<\epsilon$. Since $F(A)$ is compact, $\sup_{a\in A}\{\nu_\lambda(F\{a\},F(B))\}<\epsilon$. Similarly, we have, $\sup_{b\in B}\{\nu_\lambda(F(A),F\{b\})\}<\epsilon$.

Consequently, we obtain $H_\nu(\lambda,F(A),F(B))<\epsilon$. Hence, the function F is φ-MK contraction. □

As an application of our main result, we have the following result.

Theorem 5.8 *Let $(\mathcal{X},\nu)$ be a regular φ-MM space such that $(K(\mathcal{X}),H_\nu)$ is a complete regular φ-MM space. Let $g_n\colon\mathcal{X}\to\mathcal{X}$ be a φ-α-MK contraction for $n\in\mathbb{N}_N$. Define $F\colon K(\mathcal{X})\to K(\mathcal{X})$ by $F(A)=\bigcup_{n=1}^N g_n(A)$, where $g_n(A)=\{g_n(a):a\in A\}$ for every $A\in K(\mathcal{X})$. Then, F has a unique fixed point G satisfying the following equation*

$$K=F(G)=\bigcup_{n=1}^N g_n(G).$$

Further, the attractor G can be described as $G=\lim_{n\to\infty}F^n(A)$ for any $A\in K(\mathcal{X})$.

Proof By Theorem 5.7, we conclude that F is a φ-MK contraction map on $K(\mathcal{X})$. By invoking Corollary 3.7, we get that F has a unique fixed point G and $G=\lim_{n\to\infty}F^n(A)$ for any $A\in K(\mathcal{X})$. □

Remark 5.9 Note that in the above theorem, we have assumed $(K(\mathcal{X}),H_\nu)$ to be complete. Without this assumption, using Proposition 4.14, we have the following result:

Corollary 5.10 *Let $(\mathcal{X},\nu)$ be a complete regular φ-MM space such that $\varphi(\lambda)\le\Lambda$ for some $\Lambda\ge1$. Let $g_n\colon\mathcal{X}\to\mathcal{X}$ be φ-α-MK contraction for $n\in\mathbb{N}_N$. Define $F\colon K(\mathcal{X})\to K(\mathcal{X})$ by*

$F(A) = \bigcup_{n=1}^{N} g_n(A)$, where $g_n(A) = \{g_n(a) : a \in A\}$ for every $A \in K(\mathcal{X})$. Then, F has a unique fixed point G satisfying the following equation

$$G = F(G) = \bigcup_{n=1}^{N} g_n(K).$$

Further, the attractor G can be described as $G = \lim_{n\to\infty} F^n(A)$ for any $A \in K(\mathcal{X})$.

Proof It immediately follows from Proposition 4.14 and Theorem 5.8. □

6 Conclusions and future works

In this paper, we studied the notion of φ-MM spaces and φ-α-MK contraction. Based on this contraction, we proved a fixed point result. Also, we provided an example to support our findings. We also explored some topological properties of φ-metric modular space. Moreover, we proved that the space $K(\mathcal{X})$ is complete, which will be required in proving the existence of attractor of an IFS on φ-metric modular space. Further, we defined an IFS structure on the above defined space and proved the existence and uniqueness of the attractor using our main result.

It may be further possible to investigate several other contractive conditions on these spaces to construct fixed point results. Moreover, its related IFS and attractor can also be explored. Common fixed points for single and family of mappings can also be explored. One can also investigate multi-valued mappings for fixed points in these spaces.

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