Fixed Point Theory and Algorithms for Sciences and Engineering

RESEARCH Open Access

On a new generalization of a Perov-type F-contraction with application to a semilinear operator system

Muhammad Sarwar[1,2*], Syed Khayyam Shah[1], Kamaleldin Abodayeh[2], Arshad Khan[1] and Ishak Altun[3]

*Correspondence:
sarwarswati@gmail.com
[1]Department of Mathematics, University of Malakand, Chakdara Dir(L), Khyber Pakhtunkhwa 18000, Pakistan
[2]Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

This manuscript aims to present new results about the generalized F-contraction of Hardy–Rogers-type mappings in a complete vector-valued metric space, and to demonstrate the fixed-point theorems for single and pairs of generalized F-contractions of Hardy–Rogers-type mappings. The established results represent a significant development of numerous previously published findings and results in the existing body of literature. Furthermore, to ensure the practicality and effectiveness of our findings across other fields, we provide an application that demonstrates a unique solution for the semilinear operator system within the Banach space.

Keywords: Fixed point; Vector-valued metric space; Semilinear operators; Generalized F-contraction of a Hardy–Rogers-type mapping

1 Introduction and preliminaries

Fixed-point theory is a beneficial blend of geometry, topology, and analysis. It has numerous applications in diverse scientific disciplines, including physics, mathematical engineering, economics, biology, and chemistry. Poincare [1] introduced the notion of the fixed-point (*FP*) theory for the first time in 1806. Inspired by Poincare's work, another mathematician, Brouwer, demonstrated the *FP* theorem and presented a solution to the equation $\tau(\wp) = \wp$. In the same way, many researchers contributed to developing the *FP* theory. However, Banach had a crucial role in advancing this discipline. In 1922, Banach [2] developed a significant theorem known as the Banach contraction principle (BCP), which holds great importance in mathematics. This principle, which is widely acknowledged, has various applications in the analysis of nonlinear Volterra- and Fredholm-type integral equations [3], nonlinear integrodifferential equations [4], nonlinear differential systems with initial or boundary values, and systems of linear and nonlinear equations of matrices in the context of a Banach space [5].

Moreover, many researchers expanded and generalized this contraction through diverse methods. Hardy and Rogers [6] further developed the contraction presented by Reich and offered a completely novel generalization of the (BCP). Vetro and Cosentino [7] introduced the idea of F-contraction of Hardy–Rogers type and expanded upon the findings of

Wardowski, for more details see [8–12]. Likewise, Naveen [13] analyzed the Wardowski approach and investigated the common fixed-point theorem in complete metric spaces for a pair of multivalued mappings with δ-distance generalized F_δ-multivalued contraction. Piyachat *et al.* [14]) provide an example of generalized (Ψ, α, β)-weakly contractive mappings. They then establish many fixed-point theorems in the context of partially ordered complete metric spaces.

Definition 1.1 [15] Let $F: \mathbb{R}^+ \to \mathbb{R}$ be a mapping in a manner that

- F is increasing strictly, i.e., for every $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathbb{R}^+$ in a sense that $F(\mathfrak{t}_1) < F(\mathfrak{t}_2)$ whenever $\mathfrak{t}_1 < \mathfrak{t}_2$.
- For each sequence $\{t_n\}_{n\in N}$ $\lim_{n\to\infty} t_n = 0$ iff $\lim_{n\to\infty} F(t_n) = -\infty$. The sequence needs to be of positive real numbers.
- There is $k \in (0,1)$ in such a way that $\lim_{t\to 0^+} t^k F(t) = 0$.

We would represent with $\daleth$ the set of all functions F that meet the above requirements.

Definition 1.2 [7] Suppose $(X_s, \mathsf{d_c})$ is a metric space. $\urcorner$ on X_s a is self-map termed a generalized F-contraction of Hardy–Rogers type if there is $\zeta > 0$ and $F \in \daleth$ such that

$$\zeta + F\big(\mathsf{d_c}(\urcorner\epsilon, \urcorner\omega)\big) \leq F\big(\mathfrak{r}.\mathsf{d_c}(\epsilon,\omega) + \mathfrak{s}.\mathsf{d_c}(\epsilon,\urcorner\epsilon) + \mathfrak{t}.\mathsf{d_c}(\omega,\urcorner\omega) + \mathfrak{u}.\mathsf{d_c}(\epsilon,\urcorner\omega) + \mathfrak{v}.\mathsf{d_c}(\omega,\urcorner\epsilon)\big)$$

satisfy $\forall\epsilon,\omega \in X_s$ having $\mathsf{d_c}(\urcorner\epsilon,\urcorner\omega) > 0$, where $\mathfrak{r},\mathfrak{s},\mathfrak{v},\mathfrak{t},\mathfrak{u} \in [0,\infty)$, $\mathfrak{t} \neq 1$, and $\mathfrak{r}+\mathfrak{s}+\mathfrak{t}+2\mathfrak{v} = 1$.

Theorem 1.3 [7] *Considering* $(X_s, \mathsf{d_c})$ *a complete metric space and* $\urcorner: X_s \to X_s$*. Suppose that* $\urcorner$ *is a* F*-contraction of Hardy–Rogers type where* $\mathfrak{t} \neq 1$*. Then,* $\urcorner$ *has a FP. Further, if* $\mathfrak{r}+\mathfrak{u}+\mathfrak{v} \leq 1$*, then* $\urcorner$ *ensures its unique FP.*

Perov [16] modified the classical Banach Contraction Principle (BCP) specifically for contraction mappings in vector-valued metric spaces. The Perov *FP* theorem is highly versatile and applicable in various contexts. For instance, it is employed to illustrate the existence of a solution for the semilinear operator (SLO) system. Before introducing Perov's findings, it is essential to establish clear definitions for the following concepts:

Definition 1.4 [16] Let X_s denotes a nonempty set, $\mathfrak{R}^\eta$ represent a set of $\eta \times 1$ real matrices and $\mathsf{d_c}: X_s \times X_s \to \mathfrak{R}^\eta$, then $(X_s, \mathsf{d_c})$ is termed a vector-valued metric space $(V_v M_s)$, if $\forall\epsilon,\omega,\mathfrak{z} \in X_s$ the following conditions are true:

(i) $\mathsf{d_c}(\epsilon,\omega) \succeq \hat{0}$ and $\mathsf{d_c}(\epsilon,\omega) = \hat{0} \Leftrightarrow \epsilon = \omega$;

(ii) $\mathsf{d_c}(\epsilon,\omega) = \mathsf{d_c}(\omega,\epsilon)$;

(iii) $\mathsf{d_c}(\epsilon,\mathfrak{z}) \preceq \mathsf{d_c}(\epsilon,\omega) + \mathsf{d_c}(\omega,\mathfrak{z})$,

where $\hat{0}$ represents a $\eta \times 1$ zero matrix, $\preccurlyeq$ represents coordinate-wise ordering on $\mathfrak{R}^\eta$, i.e., $\mu = (\mu_j)_{j=1}^\eta$, $\sigma = (\sigma_j)_{j=1}^\eta \in \mathfrak{R}^\eta$, where

$$\mu \preceq \sigma \quad \text{iff} \quad \mu_j \leq \sigma_j, \quad \text{for every } j \in \{1,2,3,\dots,\eta\}$$

and

$$\mu \prec \sigma \quad \text{iff} \quad \mu_j < \sigma_j \quad \text{for every } j \in \{1,2,3,\dots,\eta\}.$$

In the current paper, the symbol $\mathfrak{R}_+^\eta$ is used for the set of $\eta \times 1$ real matrices consisting of nonnegative elements, $\Xi_{(\eta\times\eta)}(\mathfrak{R}_+)$ represent the set of $\eta \times \eta$ matrices consisting of nonnegative elements, Θ is a $\eta \times \eta$ zero matrix, I is a $\eta \times \eta$ identity matrix, and θ is a $\eta \times 1$ zero matrix. It is important to observe that the convergence, Cauchyness, and completeness of a sequence in $(V_v M_s)$ are defined in a sense that is similar to the definitions used in a standard metric space.

Example 1 Consider the set V_s of all 2-dimensional vectors $\mathbf{v} = (v_1, v_2)$ with real components. Define a vector-valued metric $d_c : V_s \times V_s \to \mathbb{R}^2$ such that for any two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ in V_s, the distance $d_c(\mathbf{a}, \mathbf{b})$ is given by $(|a_1 - b_1|, |a_2 - b_2|)$. This metric outputs a vector whose components are the absolute differences of the corresponding components of **a** and **b**.

Definition 1.5 In the context of a vector-valued metric space, an open ball $B_r(\mathbf{a})$ centered at a point **a** with radius r (where r is a vector in this case) is the set of all points **b** in the space such that the vector-valued distance $d(\mathbf{a}, \mathbf{b})$ is less than r in every component.

Example 2 In our vector-valued metric space V_s, an open ball centered at $\mathbf{a} = (1, 2)$ with radius $r = (0.5, 0.5)$ would include all vectors $\mathbf{b} = (b_1, b_2)$ such that $|1 - b_1| < 0.5$ and $|2 - b_2| < 0.5$.

Definition 1.6 A closed ball in a vector-valued metric space, denoted as $\overline{B}_r(\mathbf{a})$, includes all points **b** such that the vector-valued distance $d(\mathbf{a}, \mathbf{b})$ is less than or equal to r in every component.

Example 3 In the same space V_s, a closed ball centered at $\mathbf{a} = (1, 2)$ with radius $r = (0.5, 0.5)$ would include all vectors $\mathbf{b} = (b_1, b_2)$ such that $|1 - b_1| \le 0.5$ and $|2 - b_2| \le 0.5$.

Let $\Lambda \in \Xi_{(\eta\times\eta)}(\mathfrak{R}_+)$ be the matrix, then Λ is convergent to zero iff $\Lambda^\eta \to \Theta$ as $\eta \to \infty$ (see [17]).

Theorem 1.7 [17] *Assume* $\Lambda \in \Xi_{(\eta\times\eta)}(\mathfrak{R}_+)$ *is the matrix, then these statements are equivalent*:

- **(i)** Λ *is convergent to zero*;
- **(ii)** Λ *has eigenvalues that lie in a unit open disc, which means* $|\mu| < 1$ *for every* $\mu \in \mathbb{C}$ *with* $\det(\Lambda - \mu I) = 0$;
- **(iii)** $I - \Lambda$ *is nonsingular and*

$$(I - \Lambda)^{-1} = I + \Lambda + \cdots + \Lambda^\eta + \cdots.$$

Definition 1.8 [17] Let $G = [g_{i,\ell}]$ and $Q = [q_{i,\ell}]$ be two real $n \times \eta$ matrices. Then, $G \ge Q(> Q)$ if $g_{i,\ell} \ge q_{i,\ell}(> q_{i,\ell})$, for all $1 \le i \le n$, $1 \le j \le \eta$. If P is the null matrix and $G \ge P(> P)$, then G is a positive matrix.

In the body of literature, those matrices that are convergent to zero are:

Example 4 Let a matrix $\Lambda \in \Xi_{(2\times 2)}(\mathfrak{R}_+)$ be of the form

$$\Lambda = \begin{pmatrix} p & w \\ p & w \end{pmatrix} \quad \text{or} \quad \Lambda = \begin{pmatrix} p & p \\ w & w \end{pmatrix}$$

with $p + w < 1$ it is convergent to zero.

Example 5 $\Lambda \in \Xi_{(\eta\times\eta)}(\mathfrak{R}_+)$ is of the form

$$\Lambda = \begin{pmatrix} \varpi_1 & 0 & \cdots & 0 \\ 0 & \varpi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varpi_n \end{pmatrix}_{\eta\times\eta}.$$

If $\max\{\varpi_\ell : \ell \in \{1,2,3,\dots,\eta\}\}$ is less than 1, then Λ converges to zero.

Based on the information provided above, Perov [16] modified the concept of (BCP) as:

Theorem 1.9 [16] *Let $(X_s, \mathsf{d_c})$ be a $V_\nu M_s$ and $\daleth : X_s \to X_s$, if there is $\Lambda \in \Xi_{(\eta\times\eta)}(\mathfrak{R}_+)$ convergent to zero as*:

$$\mathsf{d_c}(\daleth\epsilon, \daleth\omega) \preceq \Lambda\big(\mathsf{d_c}(\epsilon,\omega)\big),$$

for every $\epsilon, \omega \in X_s$ then:

(q1) *There must be a unique FP $\mathfrak{z}^* \in X_s$ of $\daleth$*;

(q2) *For every $\epsilon_\circ \in X_s$, the sequence $\{\epsilon_n\}$ defined as $\epsilon_n = \daleth^n \epsilon_\circ$ is convergent to $\mathfrak{z}^*$*;

(q3) *One possesses the estimation,*

$$\mathsf{d_c}\big(\epsilon_n, \mathfrak{z}^*\big) \preceq \Lambda^n (I-\Lambda)^{-1} \mathsf{d_c}(\epsilon_\circ, \daleth\epsilon_\circ).$$

Furthermore, Cvetkovic and Rakocevic [18] extended the work of Perov by introducing Perov-type quasicontractive mapping, where bounded linear operators replaced contractive linear operators. Abbas et al. [19] deduced the common fixed-point result of Perov-type generalized Ciric-type contraction mappings. Safia *et al.* [20] in 2017 demonstrated several common fixed-point (CFP) theorems for a pair of mappings on sets that are associated with vector-valued metrics, either one or two.

Theorem 1.10 [20] *Let $(X_s, \mathsf{d_c})$ be a complete $V_\nu M_s$ with $\daleth, \mathcal{S} : X_s \to X_s$. If there is $\mathcal{T}, \mathcal{P}, \mathcal{Q} \in \Xi_{(\eta\times\eta)}(\mathfrak{R}_+)$ such that*:

(i) *$(I - \mathcal{P} - \mathcal{Q})$ is nonsingular and $(I - \mathcal{P} - \mathcal{Q})^{-1} \in \Xi_{(\eta\times\eta)}(\mathfrak{R}_+)$*;

(ii) *$\mathbb{K} = (I - \mathcal{P} - \mathcal{Q})^{-1}(\mathcal{T} + \mathcal{P} + \mathcal{Q})$ in such a way that $\mathbb{K}$ is convergent to Θ*;

(iii) $\mathsf{d_c}(\daleth\epsilon, \mathcal{S}\omega) \preceq \mathcal{T}\mathsf{d_c}(\epsilon,\omega) + \mathcal{P}[\mathsf{d_c}(\epsilon, \daleth\epsilon) + \mathsf{d_c}(\omega, \daleth\omega)] + \mathcal{Q}[\mathsf{d_c}(\epsilon, \daleth\omega) + \mathsf{d_c}(\omega, \mathcal{S}\epsilon)]$; *for all $\epsilon, \omega \in X_s$. Then, $\daleth$ and $\mathcal{S}$ possesses a CFP $\mathfrak{z}^* \in X_s$*;

(iv) *If $(I - \mathcal{T} - 2\mathcal{Q})$ is nonsingular and $(I - \mathcal{T} - 2\mathcal{Q})^{-1} \in \Xi_{(\eta\times\eta)}(\mathfrak{R}_+)$, then $\mathfrak{z}^*$ is unique.*

Several noteworthy scholars, including Flip and Petrusel [21], Cvetkovic and Rakocevic [22, 23], Minak *et al.* [24], Ilic *et al.* [25], and Vetro and Radenovic [26], have made significant contributions to the advancement of this field and its applications.

Altun and Olgun [27] recently employed the Wardowski technique [15] and Perov *FP* result by defining F-contraction in $V_\nu M_s$.

Definition 1.11 [27] Let $F : \mathfrak{R}^\eta_+ \to \mathfrak{R}^\eta$ such that:

(f1) F is increasing strictly, i.e., for all $\wp = (\wp_\ell)^\eta_{\ell=1}$, $\nu = (\nu_\ell)^\eta_{\ell=1} \in \mathfrak{R}^\eta_+$ and $\wp \prec \nu$, then $F(\wp) \prec F(\nu)$;

(f2) For every $\{\wp_n\} = (\wp^1_n, \wp^2_n, \wp^3_n, \dots, \wp^\eta_n)$ of $\mathfrak{R}^\eta_+$ in such a way that $\lim_{n\to\infty} \wp^j_n = 0$ iff $\lim_{n\to\infty} \nu^j_n = -\infty$ for each $j \in \{1, 2, 3, \dots, \eta\}$, where $(\nu^1_n, \nu^2_n, \nu^3_n, \dots, \nu^\eta_n) = F(\wp^1_n, \wp^2_n, \wp^3_n, \dots, \wp^\eta_n)$;

(f3) There is $\urcorner \in (0, 1)$, such that $\lim_{\wp_\ell \to 0^+} \wp_\ell^{\urcorner} \nu_\ell = 0$ for each $\ell \in \{1, 2, 3, \dots, \eta\}$, where $(\nu^1_n, \nu^2_n, \nu^3_n, \dots, \nu^\eta_n) = F(\wp^1_n, \wp^2_n, \wp^3_n, \dots, \wp^\eta_n)$.

Representing $\beth^\eta$ the set of all mappings F satisfying (f1)–(f3), Altun and Olgun [27] initiated the concept of a Perov-type F-contraction as:

Definition 1.12 [27] Let $(X_s, \mathsf{d_c})$ be a $V_\nu M_s$ with $\urcorner : X_s \to X_s$. If there is $F \in \beth^\eta$ with $\zeta = (\zeta_\ell)^\eta_{\ell=1} \in \mathfrak{R}^\eta_+$ such that:

$$\zeta + F\big(\mathsf{d_c}(\urcorner\epsilon, \urcorner\omega)\big) \preceq F\big(\mathsf{d_c}(\epsilon, \omega)\big),$$

for all $\epsilon, \omega \in X_s$ with $\mathsf{d_c}(\urcorner\epsilon, \urcorner\omega) \succ \hat{0}$, then $\urcorner$ is called a Perov-type F-contraction.

Example 6 Define $F : \mathfrak{R}^\eta_+ \to \mathfrak{R}^\eta$ by:

$$F(\wp_1, \wp_2, \wp_3, \dots \wp_\eta) = (\ln \wp_1, \dots, \ln \wp_\eta).$$

Then, $F \in \beth^\eta$.

Example 7 Define $F : \mathfrak{R}^2_+ \to \mathfrak{R}^2$ by:

$$F(\wp_1, \wp_2) = (\ln \wp_1, \wp_2 + \ln \wp_2).$$

Then, $F \in \beth^2$.

Altun and Olgun [27] generalized the Perov *FP* theorem as:

Theorem 1.13 [27] *Let $(X_s, \mathsf{d_c})$ be a complete $V_\nu M_s$ and $\urcorner : X_s \to X_s$ denotes a Perov-type F-contraction. Then, $\urcorner$ has a unique FP.*

Now, let $X_s \neq \emptyset$ with $\mathsf{d_c} : X^2_s \to \mathfrak{R}^\eta$ is a $V_\nu M_s$, then $\mathsf{d_c}(\epsilon, \omega) = (\mathsf{d}_\ell(\epsilon, \omega))^\eta_{\ell=1}$, such that $\mathsf{d}_\ell : X^2_s \to [0, \infty)$ are pseudometrics for all $\ell = \{1, 2, 3, \dots, \eta\}$ and at least one of the d_ℓ is the ordinary or standard metric (see [28], proposition 2.1). Further, if $F \in \beth^\eta$, so $F = (F_\ell)^\eta_{\ell=1}$, where $F_\ell : [0, \infty) \to (-\infty, +\infty)$ for every $\ell = \{1, 2, \dots, \eta\}$.

Recently, Mirkov *et al.* [29] generalized the findings in [27] by exclusively utilizing the property (f1) of F. Additionally, they extended the results in [7] by introducing the definition of the F-contraction of the Hardy–Rogers type in the following manner.

Definition 1.14 [29] Supposing $(X_s, \mathsf{d_c})$ is a $V_\nu M_s$. A mapping $\urcorner : X_s \to X_s$ is termed an F-contraction of the Hardy–Rogers type if there exists $\zeta \succ \theta$ and strictly increasing mapping

$F:[0,+\infty)^\eta \to (-\infty,+\infty)^\eta$ in such a manner that,

$$\zeta + F\big(\mathsf{d_c}(\urcorner\epsilon,\urcorner\omega)\big) \preceq F\big(\mathcal{A}(\epsilon,\omega)\big),$$

for all $\epsilon,\omega \in X_s$ with $\mathsf{d_c}(\urcorner\epsilon,\urcorner\omega) \succ \theta$, where $\mathcal{A}(\epsilon,\omega) = a\mathsf{d_c}(\epsilon,\omega) + b\mathsf{d_c}(\epsilon,\urcorner\epsilon) + c\mathsf{d_c}(\omega,\urcorner\omega) + \xi\mathsf{d_c}(\epsilon,\urcorner\omega) + e\mathsf{d_c}(\omega,\urcorner\epsilon)$ and a, b, c, ξ, e are nonnegative real numbers such that $\xi < \frac{1}{2}$, $c < 1$, $a + b + c + 2\xi = 1$, and $0 < a + \xi + e \leq 1$.

Theorem 1.15 [29] *Let $(X_s,\mathsf{d_c})$ be a complete $V_\nu M_s$. Then, each F-contraction of Hardy–Rogers type defined in it has a unique fixed point $\epsilon^* \in X_s$ and, for every $\epsilon \in X_s$, the sequence $\{\urcorner^n(\epsilon)\}_{n\in N}$ converges to ϵ^*.*

In the current work, we proposed a very new generalization of the Perov *FP* theorem by defining the generalized F-contraction of Hardy–Rogers type. A CFP theorem for a pair of generalized F-contractive operators of Hardy–Rogers type is also provided. The established results are an extension of recent results found in the literature. To ensure the practicality and effectiveness of our findings across several fields, we provide an application that demonstrates the existence of a singular solution for the semilinear operator system in a Banach space.

2 Fixed-point results

The following notions and definition of a Hardy–Rogers-type generalized F-contraction in $V_\nu M_s$ are very fruitful for proving the main result.

Definition 2.1 If $(X_s,\mathsf{d_c})$ is a $V_\nu M_s$, and $\urcorner : X_s \to X_s$ is a function, then it is said to be a generalized F-contraction of Hardy–Rogers type when it fulfills certain conditions. Specifically, there should exist matrices $\Lambda,\mathcal{B},\mathcal{C},\mathcal{E},\mathcal{G} \in \Xi_{(\eta\times\eta)}(\mathfrak{R}+)$, and a function $F \in \daleth^\eta$ that fulfill several conditions:

$$\begin{aligned}\zeta + F\big(\mathsf{d_c}(\urcorner\epsilon,\urcorner\omega)\big) \preceq F\Big(&\Lambda\big(\mathsf{d_c}(\epsilon,\omega)\big) + \mathcal{B}\big(\mathsf{d_c}(\epsilon,\urcorner\epsilon)\big)\\ &+ \mathcal{C}\big(\mathsf{d_c}(\omega,\urcorner\omega)\big) + \mathcal{E}\big(\mathsf{d_c}(\epsilon,\urcorner\omega)\big) + \mathcal{G}\big(\mathsf{d_c}(\omega,\urcorner\epsilon)\big)\Big),\end{aligned} \tag{1}$$

where $\zeta = (\zeta_\ell)_{\ell=1}^{\eta} \succ \theta$ with for every $\epsilon,\omega \in X_s$ with $\mathsf{d_c}(\urcorner\epsilon,\urcorner\omega) \succ \theta$.

Theorem 2.2 *Let $(X_s,\mathsf{d_c})$ represent a complete $V_\nu M_s$. Then, every F-contraction of Hardy–Rogers type has the following conditions*:

1. *$(\mathcal{I}-\mathcal{C}-\mathcal{E})$ and $(\Lambda+\mathcal{E}+\mathcal{G})$ are nonsingular and $(\mathcal{I}-\mathcal{C}-\mathcal{E})^{-1}$ and $(\Lambda+\mathcal{E}+\mathcal{G})^{-1} \in \Xi_{(\eta\times\eta)}(\mathfrak{R}+)$;*
2. *Q is convergent toward zero, where $Q = (\mathcal{I}-\mathcal{C}-\mathcal{E})^{-1}(\Lambda+\mathcal{B}+\mathcal{E})$.*

Then, $\urcorner$ ensures having a unique fixed point. $\epsilon^* \in X_s$ and, for each $\epsilon \in X_s$, the sequence $\{\urcorner^n(\epsilon)\}_{n\in N}$ converges to ϵ^*.

Proof As every matrix $\Lambda \in \Xi_{(\eta\times\eta)}(\mathfrak{R}_+)$ are the bounded and linear operators in the Banach space $(\mathfrak{R}^\eta,\|\cdot\|)$, where $\|\cdot\|$ is one of the equivalent norms on a finite-dimensional vector space $\mathfrak{R}^\eta$. On $\mathfrak{R}^\eta$ all norms are equivalent. Thus, we can consider any matrix $\Lambda \in \Xi_{(\eta\times\eta)}(\mathfrak{R}_+)$ as a bounded and linear operator in the considered space $(\mathfrak{R}^\eta,\|\cdot\|)$. Using the notation of the coordinate of the form $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_\eta)$, we have,

$$\Lambda\big((\epsilon_1, \epsilon_2, \ldots, \epsilon_\eta)\big) = \big(\Lambda_1(\epsilon_1), \Lambda_2(\epsilon_2), \ldots, \Lambda_\eta(\epsilon_\eta)\big),$$

because $\|\Lambda\| < 1$, $\Lambda_\ell(\epsilon_\ell) = a_\ell \epsilon_\ell$, $\ell = 1, 2, \ldots, \eta$ and $a_\ell \in [0, 1)$. Thus, the contractive condition (1) reduces to the system of η inequalities of the form

$$\begin{aligned}\zeta_\ell + F_\ell\big(\mathsf{d_c}(\urcorner\epsilon, \urcorner\omega)\big) \le F_\ell\big(&\Lambda_\ell\big(\mathsf{d_c}(\epsilon, \omega)\big) + \mathcal{B}_\ell\big(\mathsf{d_c}(\epsilon, \urcorner\epsilon)\big) + \mathcal{C}_\ell\big(\mathsf{d_c}(\omega, \urcorner\omega)\big) \\ &+ \mathcal{E}_\ell\big(\mathsf{d_c}(\epsilon, \urcorner\omega)\big) + \mathcal{G}_\ell\big(\mathsf{d_c}(\omega, \urcorner\epsilon)\big)\big).\end{aligned}$$

This implies that

$$\begin{aligned}\zeta_\ell + F_\ell\big(\mathsf{d_c}(\urcorner\epsilon, \urcorner\omega)\big) \le F_\ell\big(&a_\ell.\mathsf{d_c}(\epsilon, \omega) + b_\ell.\mathsf{d_c}(\epsilon, \urcorner\epsilon) + c_\ell.\mathsf{d_c}(\omega, \urcorner\omega) \\ &+ e_\ell.\mathsf{d_c}(\epsilon, \urcorner\omega) + g_\ell.\mathsf{d_c}(\omega, \urcorner\epsilon)\big).\end{aligned} \tag{2}$$

As stated in proposition 2.1 of [28], there is a $\ell_\circ$ $\mathsf{d_c} : X_s^2 \to [0, +\infty)$ is the ordinary metric. Hence, by (2), we have

$$\begin{aligned}\zeta_{\ell_\circ} + F_{\ell_\circ}\big(\mathsf{d_c}(\urcorner\epsilon, \urcorner\omega)\big) \le F_{\ell_\circ}\big(&a_{\ell_\circ}.\mathsf{d_c}(\epsilon, \omega) + b_{\ell_\circ}.\mathsf{d_c}(\epsilon, \urcorner\epsilon) + c_{\ell_\circ}.\mathsf{d_c}(\omega, \urcorner\omega) \\ &+ e_{\ell_\circ}.\mathsf{d_c}(\epsilon, \urcorner\omega) + g_{\ell_\circ}.\mathsf{d_c}(\omega, \urcorner\epsilon)\big),\end{aligned} \tag{3}$$

where $(X_s, \mathsf{d_c})$ is a complete metric space, $F_{\ell_\circ} : [0, +\infty) \to (-\infty, +\infty)$, $a_{\ell_\circ}, b_{\ell_\circ}, c_{\ell_\circ}, e_{\ell_\circ}, g_{\ell_\circ} \in [0, 1)$. Now, taking $a_{\ell_\circ}, b_{\ell_\circ}, c_{\ell_\circ}, e_{\ell_\circ}, g_{\ell_\circ}$ are such that $e_{\ell_\circ} < \frac{1}{2}$, $c_{\ell_\circ} < 1$, $a_{\ell_\circ} + b_{\ell_\circ} + c_{\ell_\circ} + 2e_{\ell_\circ} = 1$, $0 < a_{\ell_\circ} + e_{\ell_\circ} + g_{\ell_\circ} \le 1$, and $\{\epsilon_n\}_{n \in N} \in X_s$ such that

$$\epsilon_1 = \urcorner\epsilon_\circ, \epsilon_2 = \urcorner\epsilon_1 = \urcorner^2\epsilon_\circ, \ldots, \epsilon_n = \urcorner\epsilon_{n-1} = \urcorner^n\epsilon_\circ$$

for all $n \in N$. If there exists $n \in N \cup \{0\}$ such that $\mathsf{d_c}(\epsilon_n, \urcorner\epsilon_\mathrm{n}) = 0$, then ϵ_n is a fixed point of $\urcorner$ and the proof is complete. Hence, we assume that

$$0 < \mathsf{d_c}(\epsilon_n, \urcorner\epsilon_n) = \mathsf{d_c}(\urcorner\epsilon_{n-1}, \urcorner\epsilon_n)$$

for all $n \in N$. From (3), we have for all $n \in N$

$$\begin{aligned}\zeta_{\ell_\circ} + F_{\ell_\circ}\big(\mathsf{d_c}(\epsilon_n, \epsilon_{n+1})\big) &= \zeta_{\ell_\circ} + F_{\ell_\circ}\big(\mathsf{d_c}(\urcorner\epsilon_{n-1}, \urcorner\epsilon_n)\big) \\ &\le F_{\ell_\circ}\big(a_{\ell_\circ}\mathsf{d_c}(\epsilon_{n-1}, \epsilon_n) + b_{\ell_\circ}\mathsf{d_c}(\epsilon_{n-1}, \urcorner\epsilon_{n-1}) + c_{\ell_\circ}\mathsf{d_c}(\epsilon_n, \urcorner\epsilon_n) \\ &\quad + e_{\ell_\circ}\mathsf{d_c}(\epsilon_{n-1}, \urcorner\epsilon_n) + g_{\ell_\circ}\mathsf{d_c}(\epsilon_n, \urcorner\epsilon_{n-1})\big) \\ &= F_{\ell_\circ}\big(a_{\ell_\circ}\mathsf{d_c}(\epsilon_{n-1}, \epsilon_n) + b_{\ell_\circ}\mathsf{d_c}(\epsilon_{n-1}, \epsilon_n) + c_{\ell_\circ}\mathsf{d_c}(\epsilon_n, \epsilon_{n+1}) \\ &\quad + e_{\ell_\circ}\mathsf{d_c}(\epsilon_{n-1}, \epsilon_{n+1}) + g_{\ell_\circ}\mathsf{d_c}(\epsilon_n, \epsilon_n)\big) \\ &= F_{\ell_\circ}\big(a_{\ell_\circ}\mathsf{d_c}(\epsilon_{n-1}, \epsilon_n) + b_{\ell_\circ}\mathsf{d_c}(\epsilon_{n-1}, \epsilon_n) + c_{\ell_\circ}\mathsf{d_c}(\epsilon_n, \epsilon_{n+1}) \\ &\quad + e_{\ell_\circ}\mathsf{d_c}(\epsilon_{n-1}, \epsilon_{n+1})\big) \\ &\le F_{\ell_\circ}\big(a_{\ell_\circ}\mathsf{d_c}(\epsilon_{n-1}, \epsilon_n) + b_{\ell_\circ}\mathsf{d_c}(\epsilon_{n-1}, \epsilon_n) + c_{\ell_\circ}\mathsf{d_c}(\epsilon_n, \epsilon_{n+1})\end{aligned}$$

$$
\begin{aligned}
&+ e_{\ell_\circ}\big(\mathsf{d_c}(\epsilon_{n-1},\epsilon_n) + \mathsf{d_c}(\epsilon_n,\epsilon_{n+1})\big)\big) \\
&= F_{\ell_\circ}\big((a_{\ell_\circ} + b_{\ell_\circ} + e_{\ell_\circ})\mathsf{d_c}(\epsilon_{n-1},\epsilon_n) + (c_{\ell_\circ} + e_{\ell_\circ})\mathsf{d_c}(\epsilon_n,\epsilon_{n+1})\big).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
F_{\ell_\circ}\big(\mathsf{d_c}(\epsilon_n,\epsilon_{n+1})\big) &\le F_{\ell_\circ}\big((a_{\ell_\circ} + b_{\ell_\circ} + e_{\ell_\circ})\mathsf{d_c}(\epsilon_{n-1},\epsilon_n) + (c_{\ell_\circ} + e_{\ell_\circ})\mathsf{d_c}(\epsilon_n,\epsilon_{n+1})\big) - \zeta_{\ell_\circ} \\
&< F_{\ell_\circ}\big((a_{\ell_\circ} + b_{\ell_\circ} + e_{\ell_\circ})\mathsf{d_c}(\epsilon_{n-1},\epsilon_n) + (c_{\ell_\circ} + e_{\ell_\circ})\mathsf{d_c}(\epsilon_n,\epsilon_{n+1})\big).
\end{aligned}
$$

Using the property $(f1)$ of F, we have

$$
\mathsf{d_c}(\epsilon_n,\epsilon_{n+1}) \le (a_{\ell_\circ} + b_{\ell_\circ} + e_{\ell_\circ})\mathsf{d_c}(\epsilon_{n-1},\epsilon_n) + (c_{\ell_\circ} + e_{\ell_\circ})\mathsf{d_c}(\epsilon_n,\epsilon_{n+1}).
$$

This means that

$$
\mathsf{d_c}(\epsilon_n,\epsilon_{n+1}) \le \frac{(a_{\ell_\circ} + b_{\ell_\circ} + e_{\ell_\circ})}{(1 - c_{\ell_\circ} - e_{\ell_\circ})}\mathsf{d_c}(\epsilon_{n-1},\epsilon_n), \tag{4}
$$

for every $n \in N$. As $a_{\ell_\circ} + b_{\ell_\circ} + c_{\ell_\circ} + 2e_{\ell_\circ} = 1$, thus we obtain that $1 - c_{\ell_\circ} - e_{\ell_\circ} > 0$. Therefore, from (4), we have

$$
\mathsf{d_c}(\epsilon_n,\epsilon_{n+1}) < \mathsf{d_c}(\epsilon_{n-1},\epsilon_n)
$$

for all $n \in N$. Thus, the sequence $\{\mathsf{d_c}(\epsilon_n,\epsilon_{n+1})\}$ is strictly increasing, therefore there exist $\mathsf{d_c}^*$ in a sense that $\lim_{n\to\infty} \mathsf{d_c}(\epsilon_n,\epsilon_{n+1}) = \mathsf{d_c}^*$. Consider that $\mathsf{d_c}^* > 0$, since $F_{\ell_\circ}$ is strictly increasing, so implementing the limit as $n \to \infty$ on inequality (4), we deduce $F_{\ell_\circ}(\mathsf{d_c}^*) < F_{\ell_\circ}(\mathsf{d_c}^*) - \zeta_{\ell_\circ}$, which leads to a contradiction. Thus,

$$
\lim_{n\to\infty} \mathsf{d_c}(\epsilon_n,\epsilon_{n+1}) = 0. \tag{5}
$$

Now, we claim that $\{\epsilon_n\}_{n\in N}$ is a Cauchy sequence by assuming the contrary. If there exist $\epsilon > 0$ and $\{n_k\}$, $\{m_k\}$ are two sequences such that $n_k > m_k > k$,

$$
\mathsf{d_c}(\epsilon_{n_k},\epsilon_{m_k}) > \epsilon, \mathsf{d_c}(\epsilon_{n_k-1},\epsilon_{m_k}) \le \epsilon, \tag{6}
$$

for all $n \in N$, then we have

$$
\epsilon < \mathsf{d_c}(\epsilon_{n_k},\epsilon_{m_k}) \le \mathsf{d_c}(\epsilon_{n_k-1},\epsilon_{n_k}) + \mathsf{d_c}(\epsilon_{n_k-1},\epsilon_{m_k}) \le \mathsf{d_c}(\epsilon_{n_k-1},\epsilon_{n_k}) + \epsilon. \tag{7}
$$

Therefore, from (5) and (7), we have

$$
\lim_{k\to\infty} \mathsf{d_c}(\epsilon_{n_k},\epsilon_{m_k}) = \epsilon. \tag{8}
$$

Since $\mathsf{d_c}(\epsilon_{n_k},\epsilon_{m_k}) > \epsilon > 0$, from (3), we have

$$
\begin{aligned}
\zeta_{\ell_\circ} + F_{\ell_\circ}\big(\mathsf{d_c}(\epsilon_{n_k},\epsilon_{m_k})\big) \le F_{\ell_\circ}\big(&a_{\ell_\circ}.\mathsf{d_c}(\epsilon_{n_k-1},\omega_{m_k-1}) + b_{\ell_\circ}.\mathsf{d_c}(\epsilon_{n_k-1},f\epsilon_{n_k-1}) \\
&+ c_{\ell_\circ}.\mathsf{d_c}(\omega_{m_k-1},\daleth\omega_{m_k-1}) + e_{\ell_\circ}.\mathsf{d_c}(\epsilon_{n_k-1},\daleth\omega_{m_k-1}) \\
&+ g_{\ell_\circ}.\mathsf{d_c}(\omega_{m_k-1},\daleth\epsilon_{n_k-1})\big)
\end{aligned}
$$

$$
\begin{aligned}
&= F_{\ell_\circ}\big(a_{\ell_\circ}.\mathsf{d}_\mathsf{c}(\epsilon_{n_k-1},\omega_{m_k-1}) + b_{\ell_\circ}.\mathsf{d}_\mathsf{c}(\epsilon_{n_k-1},\epsilon_{n_k}) \\
&\quad + c_{\ell_\circ}.\mathsf{d}_\mathsf{c}(\omega_{m_k-1},\omega_{m_k}) + e_{\ell_\circ}.\mathsf{d}_\mathsf{c}(\epsilon_{n_k-1},\omega_{m_k}) + g_{\ell_\circ}.\mathsf{d}_\mathsf{c}(\omega_{m_k-1},\epsilon_{n_k})\big) \\
&\leq F_{\ell_\circ}(a_{\ell_\circ}\big[\mathsf{d}_\mathsf{c}(\epsilon_{n_k},\omega_{m_k}) + \mathsf{d}_\mathsf{c}(\epsilon_{n_k-1},\epsilon_{n_k}) + \mathsf{d}_\mathsf{c}(\omega_{m_k-1},\omega_{m_k})\big] \\
&\quad + b_{\ell_\circ}.\mathsf{d}_\mathsf{c}(\epsilon_{n_k-1},\epsilon_{n_k}) + c_{\ell_\circ}.\mathsf{d}_\mathsf{c}(\omega_{m_k-1},\omega_{m_k}) \\
&\quad + e_{\ell_\circ}\big[\mathsf{d}_\mathsf{c}(\epsilon_{n_k},\omega_{m_k}) + \mathsf{d}_\mathsf{c}(\epsilon_{n_k-1},\epsilon_{n_k}\big] \\
&\quad + g_{\ell_\circ}\big[\mathsf{d}_\mathsf{c}(\epsilon_{n_k},\omega_{m_k}) + \mathsf{d}_\mathsf{c}(\omega_{m_k-1},\omega_{m_k}\big] \\
&= F_{\ell_\circ}\big((a_{\ell_\circ} + e_{\ell_\circ} + g_{\ell_\circ})\mathsf{d}_\mathsf{c}(\epsilon_{n_k},\omega_{m_k}) \\
&\quad + (a_{\ell_\circ} + b_{\ell_\circ} + e_{\ell_\circ})\mathsf{d}_\mathsf{c}(\epsilon_{n_k-1},\epsilon_{n_k}) \\
&\quad + (a_{\ell_\circ} + c_{\ell_\circ} + g_{\ell_\circ})\mathsf{d}_\mathsf{c}(\omega_{m_k-1},\omega_{m_k})\big) \\
&\leq F_{\ell_\circ}\big(\mathsf{d}_\mathsf{c}(\epsilon_{n_k},\omega_{m_k}) + \mathsf{d}_\mathsf{c}(\epsilon_{n_k-1},\epsilon_{n_k}) \\
&\quad + (a_{\ell_\circ} + c_{\ell_\circ} + g_{\ell_\circ})\mathsf{d}_\mathsf{c}(\omega_{m_k-1},\omega_{m_k})\big).
\end{aligned}
$$

Now, taking the limit as $k \to \infty$ on the above inequality, we obtain

$$\zeta_{\ell_\circ} + F_{\ell_\circ}(\epsilon) \leq F_{\ell_\circ}(\epsilon),$$

which is a contradiction, hence $\{\epsilon_n\}_{n\in N}$ is a Cauchy sequence. Since $(X_s, \mathsf{d}_\mathsf{c})$ is a complete metric space, there exist $\epsilon^* \in X_s$ such that $\lim_{n\to\infty} \epsilon_n = \epsilon^*$. Suppose that $\urcorner\epsilon^* \neq \epsilon^*$, then by hypothesis, we have

$$
\begin{aligned}
\zeta_{\ell_\circ} + F_{\ell_\circ}\big(\mathsf{d}_\mathsf{c}\big(\urcorner\epsilon_n, \urcorner\epsilon^*\big)\big) \leq F_{\ell_\circ}\big(&a_{\ell_\circ}\mathsf{d}_\mathsf{c}\big(\epsilon_n, \epsilon^*\big) + b_{\ell_\circ}\mathsf{d}_\mathsf{c}(\epsilon_n, \urcorner\epsilon_n) + c_{\ell_\circ}\mathsf{d}_\mathsf{c}\big(\epsilon^*, \urcorner\epsilon^*\big) \\
&+ e_{\ell_\circ}\mathsf{d}_\mathsf{c}\big(\epsilon_n, \urcorner\epsilon^*\big) + g_{\ell_\circ}\mathsf{d}_\mathsf{c}\big(\epsilon^*, \urcorner\epsilon_n\big)\big).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
F_{\ell_\circ}\big(\mathsf{d}_\mathsf{c}\big(\epsilon_{n+1}, \urcorner\epsilon^*\big)\big) < F_{\ell_\circ}\big(&a_{\ell_\circ}\mathsf{d}_\mathsf{c}\big(\epsilon_n, \epsilon^*\big) + b_{\ell_\circ}\mathsf{d}_\mathsf{c}(\epsilon_n, \epsilon_{n+1}) + c_{\ell_\circ}\mathsf{d}_\mathsf{c}\big(\epsilon^*, \urcorner\epsilon^*\big) \\
&+ e_{\ell_\circ}\mathsf{d}_\mathsf{c}\big(\epsilon_n, \urcorner\epsilon^*\big) + g_{\ell_\circ}\mathsf{d}_\mathsf{c}\big(\epsilon^*, \epsilon_{n+1}\big)\big).
\end{aligned}
$$

Using property $(f1)$ of F, we obtain

$$
\begin{aligned}
\mathsf{d}_\mathsf{c}\big(\epsilon_{n+1}, \urcorner\epsilon^*\big) < &a_{\ell_\circ}\mathsf{d}_\mathsf{c}\big(\epsilon_n, \epsilon^*\big) + b_{\ell_\circ}\mathsf{d}_\mathsf{c}(\epsilon_n, \epsilon_{n+1}) + c_{\ell_\circ}\mathsf{d}_\mathsf{c}\big(\epsilon^*, \urcorner\epsilon^*\big) \\
&+ e_{\ell_\circ}\mathsf{d}_\mathsf{c}\big(\epsilon_n, \urcorner\epsilon^*\big) + g_{\ell_\circ}\mathsf{d}_\mathsf{c}\big(\epsilon^*, \epsilon_{n+1}\big).
\end{aligned}
$$

Taking the limit as $n \to \infty$, we obtain

$$\mathsf{d}_\mathsf{c}\big(\epsilon^*, \urcorner\epsilon^*\big) < c_{\ell_\circ}\mathsf{d}_\mathsf{c}\big(\epsilon^*, \urcorner\epsilon^*\big) + e_{\ell_\circ}\mathsf{d}_\mathsf{c}\big(\epsilon^*, \urcorner\epsilon^*\big).$$

Since $a_{\ell_\circ} + b_{\ell_\circ} + c_{\ell_\circ} + 2e_{\ell_\circ} = 1$, therefore, we conclude that $c_{\ell_\circ} + e_{\ell_\circ} < 1$. Thus, we have

$$\mathsf{d}_\mathsf{c}\big(\epsilon^*, \urcorner\epsilon^*\big) < \mathsf{d}_\mathsf{c}\big(\epsilon^*, \urcorner\epsilon^*\big).$$

This produces a contradiction. Therefore, $\urcorner\epsilon^* = \epsilon^*$.

Now, we want to justify that $\urcorner$ possesses a unique fixed point. Let $\partial, \flat$ be two distinct fixed points of $\urcorner$ such that $\partial \neq \flat$. Since $0 < a_{\ell_\circ} + e_{\ell_\circ} + g_{\ell_\circ} \leq 1$ and $\mathsf{d_c}(\urcorner\partial, \urcorner\flat) = \mathsf{d_c}(\partial, \flat) > 0$, then by hypothesis, we have

$$\begin{aligned}\zeta_{\ell_\circ} + F_{\ell_\circ}\big(\mathsf{d_c}(\partial,\flat)\big) &= \zeta_{\ell_\circ} + F_{\ell_\circ}\big(\mathsf{d_c}(\urcorner\partial,\urcorner\flat)\big)\\ &\leq F_{\ell_\circ}\big(a_{\ell_\circ}\mathsf{d_c}(\partial,\flat) + b_{\ell_\circ}\mathsf{d_c}(\partial,\partial) + c_{\ell_\circ}\mathsf{d_c}(\flat,\flat) + e_{\ell_\circ}\mathsf{d_c}(\partial,\flat) + g_{\ell_\circ}\mathsf{d_c}(\flat,\partial)\big)\\ &= F_{\ell_\circ}\big((a_{\ell_\circ} + e_{\ell_\circ} + g_{\ell_\circ})\mathsf{d_c}(\partial,\flat)\big)\\ &\leq F_{\ell_\circ}\big(\mathsf{d_c}(\partial,\flat)\big).\end{aligned}$$

This leads to a contradiction, hence $\partial = \flat$. This justifies a unique FP of $\urcorner$. □

Next, for two self-mappings we extend the definition 2.1.

Definition 2.3 Let $(X_s, \mathsf{d_c})$ denote $V_\nu M_s$, then $\urcorner, \mathcal{S} : X_s \to X_s$ is termed a generalized F-contraction of Hardy–Rogers type, if there exists $\Lambda, \mathcal{B}, \mathcal{C}, \mathcal{E}, \mathcal{G} \in \Xi_{(\eta\times\eta)}(\mathfrak{R}_+)$, $F \in \daleth^\eta$ and $\zeta = (\zeta_\ell)_{\ell=1}^{\eta} \succ \theta$ in such a sense that

$$\begin{aligned}\zeta + F\big(\mathsf{d_c}(\urcorner\epsilon, \mathcal{S}\omega)\big) \preceq F\Big(&\Lambda\big(\mathsf{d_c}(\epsilon,\omega)\big) + \mathcal{B}\big(\mathsf{d_c}(\epsilon,\urcorner\epsilon)\big) + \mathcal{C}\big(\mathsf{d_c}(\omega,\mathcal{S}\omega)\big)\\ &+ \mathcal{E}\big(\mathsf{d_c}(\epsilon,\mathcal{S}\omega)\big) + \mathcal{G}\big(\mathsf{d_c}(\omega,\urcorner\epsilon)\big)\Big).\end{aligned} \tag{9}$$

For all $\epsilon, \omega \in X_s$ with $\mathsf{d_c}(\urcorner\epsilon, \mathcal{S}\omega) > \theta$.

Theorem 2.4 *Let $(X_s, \mathsf{d_c})$ be a complete $V_\nu M_s$. Then, each F-contraction of Hardy–Rogers type with the following holds true:*

1. *$(\mathcal{I} - \mathcal{B} - \mathcal{G})$ and $(\Lambda + \mathcal{E} + \mathcal{G})$ are nonsingular, and $(\mathcal{I} - \mathcal{B} - \mathcal{G})^{-1}$ and $(\Lambda + \mathcal{E} + \mathcal{G})^{-1} \in \Xi_{(\eta\times\eta)}(\mathfrak{R}+)$;*
2. *Q is convergent toward zero, where $Q^* = (\mathcal{I} - \mathcal{B} - \mathcal{G})^{-1}(\Lambda + \mathcal{C} + \mathcal{G})$.*

Then, the operators $\urcorner$ and $\mathcal{S}$ have a unique fixed point $\epsilon^* \in X_s$.

Proof Now, according to the previous results, the contractive condition (9) become

$$\begin{aligned}\zeta_{\ell_\circ} + F_{\ell_\circ}\big(\mathsf{d_c}(\urcorner\epsilon, \mathcal{S}\omega)\big) \leq F_{\ell_\circ}\big(&a_{\ell_\circ}.\mathsf{d_c}(\epsilon,\omega) + b_{\ell_\circ}.\mathsf{d_c}(\epsilon,\urcorner\epsilon) + c_{\ell_\circ}.\mathsf{d_c}(\omega,\mathcal{S}\omega)\\ &+ e_{\ell_\circ}.\mathsf{d_c}(\epsilon,\mathcal{S}\omega) + g_{\ell_\circ}.\mathsf{d_c}(\omega,\urcorner\epsilon)\big),\end{aligned} \tag{10}$$

where $(X_s, \mathsf{d_c})$ is a complete metric space, $F_{\ell_\circ} : [0,+\infty) \to (-\infty,+\infty)$, $a_{\ell_\circ}, b_{\ell_\circ}, c_{\ell_\circ}, e_{\ell_\circ}, g_{\ell_\circ} \in [0,1)$. Now, taking $a_{\ell_\circ}$, $b_{\ell_\circ}$, $c_{\ell_\circ}$, $e_{\ell_\circ}$, $g_{\ell_\circ}$ such that $e_{\ell_\circ}, c_{\ell_\circ} < \frac{1}{2}$, $a_{\ell_\circ} + b_{\ell_\circ} + c_{\ell_\circ} + 2g_{\ell_\circ} = 1$, $0 < a_{\ell_\circ} + e_{\ell_\circ} + g_{\ell_\circ} \leq 1$, and $\{\epsilon_n\}_{n\in N} \in X_s$ such that

$$\begin{aligned}\epsilon_{2n+1} &= \urcorner(\epsilon_{2n})\\ \epsilon_{2n+2} &= \mathcal{S}(\epsilon_{2n+1}),\end{aligned}$$

where $n \in N \cup \{0\}$. If $\epsilon_{2n_0} = \epsilon_{2n_0+1}$, then ϵ_{2n_0} is a CFP of $\mathcal{S}$, $\urcorner$. Assume that $\epsilon_{2n} \neq \epsilon_{2n+1}$ for all $n \in N \cup \{0\}$. Hence, we assume that

$$0 < \mathsf{d_c}(\epsilon_{2n+1}, \epsilon_{2n}) = \mathsf{d_c}(\urcorner\epsilon_{2n}, \mathcal{S}\epsilon_{2n-1})$$

for all $n \in N$. Then, from (10), we have for all $n \in N$

$$\begin{aligned}
\zeta_{\ell_\circ} + F_{\ell_\circ}\big(\mathsf{d_c}(\epsilon_{2n+1}, \epsilon_{2n})\big) &= \zeta_{\ell_\circ} + F_{\ell_\circ}\big(\mathsf{d_c}(\urcorner\epsilon_{2n}, \mathcal{S}\epsilon_{2n-1})\big) \\
&\le F_{\ell_\circ}\big(a_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n-1}) + b_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n}, \urcorner\epsilon_{2n}) + c_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n-1}, \mathcal{S}\epsilon_{2n-1}) \\
&\quad + e_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n}, \mathcal{S}\epsilon_{2n-1}) + g_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n-1}, \urcorner\epsilon_{2n})\big) \\
&= F_{\ell_\circ}\big(a_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n-1}) + b_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n+1}) + c_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n-1}, \epsilon_{2n}) \\
&\quad + e_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n}) + g_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n-1}, \epsilon_{2n+1})\big) \\
&= F_{\ell_\circ}\big(a_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n-1}) + b_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n+1}) + c_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n-1}, \epsilon_{2n}) \\
&\quad + g_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n-1}, \epsilon_{2n+1})\big) \\
&\le F_{\ell_\circ}\big(a_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n-1}) + b_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n+1}) + c_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n-1}, \epsilon_{2n}) \\
&\quad + g_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n-1}, \epsilon_{2n}) + g_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n+1})\big) \\
&= F_{\ell_\circ}\big((a_{\ell_\circ} + c_{\ell_\circ} + g_{\ell_\circ})\mathsf{d_c}(\epsilon_{2n-1}, \epsilon_{2n}) + (b_{\ell_\circ} + g_{\ell_\circ})\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n+1})\big).
\end{aligned}$$

It follows that

$$\begin{aligned}
F_{\ell_\circ}\big(\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n+1})\big) &\le F_{\ell_\circ}\big((a_{\ell_\circ} + c_{\ell_\circ} + g_{\ell_\circ})\mathsf{d_c}(\epsilon_{2n-1}, \epsilon_{2n}) + (b_{\ell_\circ} + g_{\ell_\circ})\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n+1})\big) - \zeta_{\ell_\circ} \\
&< F_{\ell_\circ}\big((a_{\ell_\circ} + c_{\ell_\circ} + g_{\ell_\circ})\mathsf{d_c}(\epsilon_{2n-1}, \epsilon_{2n}) + (b_{\ell_\circ} + g_{\ell_\circ})\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n+1})\big).
\end{aligned}$$

Using the property $(f1)$ of F, we have

$$\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n+1}) < a_{\ell_\circ} + c_{\ell_\circ} + g_{\ell_\circ}\mathsf{d_c}(\epsilon_{2n-1}, \epsilon_{2n}) + (b_{\ell_\circ} + g_{\ell_\circ})\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n+1}).$$

This implies that

$$\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n+1}) < \frac{a_{\ell_\circ} + c_{\ell_\circ} + g_{\ell_\circ}}{1 - b_{\ell_\circ} - g_{\ell_\circ}}\mathsf{d_c}(\epsilon_{2n-1}, \epsilon_{2n}) \tag{11}$$

for all $n \in N$. Since $a_{\ell_\circ} + b_{\ell_\circ} + c_{\ell_\circ} + 2g_{\ell_\circ} = 1$, thus we obtain that $1 - b_{\ell_\circ} - g_{\ell_\circ} > 0$. Therefore, from (11), we have

$$\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n+1}) < \mathsf{d_c}(\epsilon_{2n-1}, \epsilon_{2n}).$$

Similarly, it can be shown that

$$\mathsf{d_c}(\epsilon_{2n+1}, \epsilon_{2n+2}) < \mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n+1}).$$

In general,

$$\mathsf{d_c}(\epsilon_n, \epsilon_{n+1}) < \mathsf{d_c}(\epsilon_{n-1}, \epsilon_n)$$

for all $n \in N$. Thus, the sequence $\{\mathsf{d_c}(\epsilon_n, \epsilon_{n+1})\}$ is strictly increasing, therefore there exist z^* such that $\lim_{n\to\infty} \mathsf{d_c}(\epsilon_n, \epsilon_{n+1}) = \mathsf{z}^*$. Suppose that $\mathsf{z}^* > 0$, since $F_{\ell_\circ}$ is strictly increasing, so by the limit as $n \to \infty$ on inequality (11), we deduce $F_{\ell_\circ}(\mathsf{z}^*) < F_{\ell_\circ}(\mathsf{z}^*) - \zeta_{\ell_\circ}$, which is a contradiction. Thus,

$$\lim_{n\to\infty} \mathsf{d_c}(\epsilon_n, \epsilon_{n+1}) = 0. \tag{12}$$

By using the previous results, we can easily verify that the $\{\epsilon_n\}_{n\in N}$ is a Cauchy sequence. Since $(X_s, \mathsf{d_c})$ is a complete metric space, there exist $\omega^* \in X_s$, such that $\lim_{n\to\infty} \epsilon_n = \omega^*$.

Next, we show that ω^* is a common fixed point of $\urcorner, \mathcal{S}$. Let $\mathcal{S}\omega^* \neq \omega^*$. Then, by hypothesis, we have

$$\begin{aligned}\zeta_{\ell_\circ} + F_{\ell_\circ}\big(\mathsf{d_c}\big(\urcorner\epsilon_{2n}, \mathcal{S}\omega^*\big)\big) \le F_{\ell_\circ}\big(&a_{\ell_\circ}.\mathsf{d_c}\big(\epsilon_{2n}, \omega^*\big) + b_{\ell_\circ}.\mathsf{d_c}(\epsilon_{2n}, \urcorner\epsilon_{2n}) + c_{\ell_\circ}.\mathsf{d_c}\big(\omega^*, \mathcal{S}\omega^*\big)\\ &+ e_{\ell_\circ}.\mathsf{d_c}\big(\epsilon_{2n}, \mathcal{S}\omega^*\big) + g_{\ell_\circ}.\mathsf{d_c}\big(\omega^*, \urcorner\epsilon_{2n}\big)\big).\end{aligned} \tag{13}$$

This implies that

$$\begin{aligned}\zeta_{\ell_\circ} + F_{\ell_\circ}\big(\mathsf{d_c}\big(\epsilon_{2n+1}, \mathcal{S}\omega^*\big)\big) \le F_{\ell_\circ}\big(&a_{\ell_\circ}.\mathsf{d_c}\big(\epsilon_{2n}, \omega^*\big) + b_{\ell_\circ}.\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n+1}) + c_{\ell_\circ}.\mathsf{d_c}\big(\omega^*, \mathcal{S}\omega^*\big)\\ &+ e_{\ell_\circ}.\mathsf{d_c}\big(\epsilon_{2n}, \mathcal{S}\omega^*\big) + g_{\ell_\circ}.\mathsf{d_c}\big(\omega^*, \epsilon_{2n+1}\big)\big).\end{aligned}$$

Using the property $(f1)$ of F, we obtain

$$\begin{aligned}\mathsf{d_c}\big(\epsilon_{2n+1}, \mathcal{S}\omega^*\big) < &a_{\ell_\circ}.\mathsf{d_c}\big(\epsilon_{2n}, \omega^*\big) + b_{\ell_\circ}.\mathsf{d_c}(\epsilon_{2n}, \epsilon_{2n+1}) + c_{\ell_\circ}.\mathsf{d_c}\big(\omega^*, \mathcal{S}\omega^*\big)\\ &+ e_{\ell_\circ}.\mathsf{d_c}\big(\epsilon_{2n}, \mathcal{S}\omega^*\big) + g_{\ell_\circ}.\mathsf{d_c}\big(\omega^*, \epsilon_{2n+1}\big).\end{aligned}$$

Hence, letting $n \to \infty$, we obtain

$$\mathsf{d_c}\big(\omega^*, \mathcal{S}\omega^*\big) < c_{\ell_\circ}.\mathsf{d_c}\big(\omega^*, \mathcal{S}\omega^*\big) + e_{\ell_\circ}.\mathsf{d_c}\big(\omega^*, \mathcal{S}\omega^*\big).$$

This implies that

$$\mathsf{d_c}\big(\omega^*, \mathcal{S}\omega^*\big) < (c_{\ell_\circ} + e_{\ell_\circ})\mathsf{d_c}\big(\omega^*, \mathcal{S}\omega^*\big),$$

since $c_{\ell_\circ}, e_{\ell_\circ} < \frac{1}{2}$. Then, $c_{\ell_\circ} + e_{\ell_\circ}) < 1$. Therefore,

$$\mathsf{d_c}\big(\omega^*, \mathcal{S}\omega^*\big) < \mathsf{d_c}\big(\omega^*, \mathcal{S}\omega^*\big),$$

which is a contradiction, therefore $\mathcal{S}\omega^* = \omega^*$. Similarly, we can prove that $\urcorner\omega^* = \omega^*$. Next, we want to prove ω^* is unique, let us consider $\partial, \flat$ to be two distinct common fixed points of $\urcorner, \mathcal{S}$. Then, by hypothesis, we have

$$\begin{aligned}\zeta_{\ell_\circ} + F_{\ell_\circ}\big(\mathsf{d_c}(\urcorner\partial, \mathcal{S}\flat)\big) \le F_{\ell_\circ}\big(&a_{\ell_\circ}.\urcorner d_{c\ell_\circ}(\partial, \flat) + b_{\ell_\circ}.\mathsf{d_c}(\partial, \urcorner\partial) + c_{\ell_\circ}.\mathsf{d_c}(\flat, \mathcal{S}\flat)\\ &+ e_{\ell_\circ}.\mathsf{d_c}(\partial, \mathcal{S}\flat) + g_{\ell_\circ}.\mathsf{d_c}(\flat, \urcorner\partial)\big).\end{aligned} \tag{14}$$

This implies that

$$\begin{aligned}\zeta_{\ell_\circ} + F_{\ell_\circ}\big(\mathsf{d_c}(\partial, \flat)\big) &\le F_{\ell_\circ}\big(a_{\ell_\circ}.\mathsf{d_c}(\partial, \flat) + e_{\ell_\circ}.\mathsf{d_c}(\partial, \flat) + g_{\ell_\circ}.\mathsf{d_c}(\flat, \partial)\big).\\ &= F_{\ell_\circ}\big((a_{\ell_\circ} + e_{\ell_\circ} + g_{\ell_\circ})\mathsf{d_c}(\partial, \flat)\big).\end{aligned} \tag{15}$$

By property $(f1)$ of F, we have

$$\mathsf{d_c}(\partial,\flat) < (a_{\ell_\circ} + e_{\ell_\circ} + g_{\ell_\circ})\mathsf{d_c}(\partial,\flat),$$

since $0 < (a_{\ell_\circ} + e_{\ell_\circ} + g_{\ell_\circ}) \leq 1$, we obtain

$$\mathsf{d_c}(\partial,\flat) < \mathsf{d_c}(\partial,\flat),$$

which is a contradiction, therefore $\partial = \flat$. Hence, $\urcorner$, $\mathcal{S}$ has a unique common fixed point. □

If $\mathcal{B} = \mathcal{C}$ and $\mathcal{E} = \mathcal{G}$, since $\Lambda, \mathcal{B}, \mathcal{C}, \mathcal{E}, \mathcal{G}$ are linear in the Banach space $(\mathfrak{R}^\eta, \|\cdot\|)$, then we have the following corollary.

Corollary 1 *Let $(X_s, \mathsf{d_c})$ be a complete V_vM_s, $\urcorner : X_s \to X_s$. If there exist $\Lambda, \mathcal{C}, \mathcal{G} \in \Xi_{(\eta\times\eta)}(\mathfrak{R}_+)$ such that*:

$$\zeta + F\big(\mathsf{d_c}(\urcorner\epsilon, \urcorner\omega)\big) \preceq F\big(\Lambda\big(\mathsf{d_c}(\epsilon,\omega)\big) + \mathcal{C}\big(\mathsf{d_c}(\epsilon,\urcorner\epsilon) + \mathsf{d_c}(\omega,\urcorner\omega)\big) + \mathcal{G}\big(\mathsf{d_c}(\epsilon,\urcorner\omega) + \mathsf{d_c}(\omega,\urcorner\epsilon)\big)\big)$$

with the following conditions holding true:
1. *$(\mathcal{I} - \mathcal{C} - \mathcal{G})$ and $(\Lambda + 2\mathcal{G})$ are nonsingular and $(\mathcal{I} - \mathcal{C} - \mathcal{G})^{-1}$ and $(\Lambda + 2\mathcal{G})^{-1} \in \Xi_{(\eta\times\eta)}(\mathfrak{R}+)$;*
2. *Q is convergent toward zero, where $Q = (\mathcal{I} - \mathcal{C} - \mathcal{G})^{-1}(\Lambda + \mathcal{C} + \mathcal{G})$,*

then there is a unique fixed point of $\urcorner$.

If $\Lambda = I$ and $\mathcal{B} = \mathcal{C} = \mathcal{E} = \mathcal{G} = \Theta$, we obtain the following corollary.

Corollary 2 *Let $(X_s, \mathsf{d_c})$ be a complete V_vM_s. $\urcorner : X_s \to X_s$ and $F \in \exists^\eta$. If there exist $\zeta = (\zeta_\ell)_{\ell=1}^{\eta} \succ \theta$ such that*

$$\zeta + F\big(\mathsf{d_c}(\urcorner\epsilon, \urcorner\omega)\big) \preceq F\big(\mathsf{d_c}(\epsilon,\omega)\big),$$

for all $\epsilon, \omega \in X_s$ with $\mathsf{d_c}(\urcorner\epsilon, \urcorner\omega) \succ \theta$, then $\urcorner$ has a unique FP.

If $\mathcal{B} = \mathcal{C}$ and $\mathcal{E} = \mathcal{G}$, then we have the following corollary.

Corollary 3 *Let $(X_s, \mathsf{d_c})$ be a complete V_vM_s, $\mathcal{S}, \urcorner : X_s \to X_s$. If there exist $\Lambda, \mathcal{C}, \mathcal{G} \in \Xi_{(\eta\times\eta)}(\mathfrak{R}_+)$ such that*:

$$\zeta + F\big(\mathsf{d_c}(\urcorner\epsilon, \mathcal{S}\omega)\big) \preceq F\big(\Lambda\big(\mathsf{d_c}(\epsilon,\omega)\big) + \mathcal{C}\big(\mathsf{d_c}(\epsilon,\urcorner\epsilon) + \mathsf{d_c}(\omega,\mathcal{S}\omega)\big) + \mathcal{G}\big(\mathsf{d_c}(\epsilon,\mathcal{S}\omega) + \mathsf{d_c}(\omega,\urcorner\epsilon)\big)\big),$$

with the following holding:
1. *$(\mathcal{I} - \mathcal{C} - \mathcal{G})$ and $(\Lambda + 2\mathcal{G})$ are nonsingular, and $(\mathcal{I} - \mathcal{C} - \mathcal{G})^{-1}$ and $(\Lambda + 2\mathcal{G})^{-1} \in \Xi_{(\eta\times\eta)}(\mathfrak{R}+)$;*
2. *Q is convergent toward zero, where $Q^* = (\mathcal{I} - \mathcal{C} - \mathcal{G})^{-1}(\Lambda + \mathcal{C} + \mathcal{G})$.*

Then, there is a unique CFP of $\urcorner$ and $\mathcal{S}$.

Example 8 Let $(X_s, \mathsf{d_c})$ be a complete vector-valued metric space, where $X_s = \{\epsilon_n = \frac{1}{n^2}\}$: $n \in 1, 2, 3 \ldots \cup \{0\}$ and $\mathsf{d_c} : \mathsf{X_s} \times \mathsf{X_s} \to \mathsf{R}^2$ is given by:

$$\mathsf{d}_c(\epsilon, \omega) = \big(|\epsilon - \omega|, |\epsilon - \omega|\big).$$

Define $\urcorner : X_s \to X_s$ by:

$$\urcorner(\epsilon) = \begin{cases} 0, & \epsilon = 0 \\ \epsilon_{n+1}, & \epsilon = \epsilon_n \end{cases}$$

and $F : R^2 \to R^2$by:

$$F(\omega_1, \omega_2) = \begin{cases} (\frac{\ln(\omega_1)}{\sqrt{\omega_1}}, -\frac{1}{\sqrt{\omega_2}}), & \text{if } \omega_1 \le e \\ (\frac{\omega_1}{e\sqrt{e}}, -\frac{1}{\sqrt{\omega_2}}) & \text{if } \omega_1 \ge e. \end{cases}$$

Now, taking the contractive factor $\zeta = (\ln 2, 1)$, $\epsilon = 0$, $\omega = \epsilon_n$ and

$$\Lambda = \begin{pmatrix} -4 & -5 \\ -3 & -6 \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}, \qquad \mathcal{G} = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

Now, utilizing the contraction of 1

$$\begin{aligned} (\ln 2, 1) + F\big(d_c(0, \epsilon_{n+1})\big) \le F \Bigg\{ & \begin{pmatrix} -4 & -5 \\ -3 & -6 \end{pmatrix} d_c(0, \epsilon_n) + \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \big\{ d_c(0, 0) + d_c(\epsilon_n, \epsilon_{n+1}) \big\} \\ & + \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \big\{ d_c(0, \epsilon_{n+1}) + d_c(\epsilon_n, 0) \big\} \Bigg\} \\ (\ln 2, 1) + F\big(d_c(\urcorner\epsilon, \urcorner\omega)\big) \le F \Bigg\{ & \begin{pmatrix} -4 & -5 \\ -3 & -6 \end{pmatrix} \begin{pmatrix} \epsilon_n \\ \epsilon_n \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} |\epsilon_n - \epsilon_{n+1}| \\ |\epsilon_n - \epsilon_{n+1}| \end{pmatrix} \\ & + \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \Bigg\{ \begin{pmatrix} \epsilon_{n+1} \\ \epsilon_{n+1} \end{pmatrix} + \begin{pmatrix} \epsilon_n \\ \epsilon_n \end{pmatrix} \Bigg\} \Bigg\}. \end{aligned}$$

This implies that

$$(\ln 2, 1) + F\big(d_c(\urcorner\epsilon, \urcorner\omega)\big) \le F \begin{pmatrix} \epsilon_n \\ \epsilon_n. \end{pmatrix}.$$

Thus,

$$\begin{aligned} & (\ln 2, 1) + F\big(d_c(\urcorner\epsilon, \urcorner\omega)\big) \le F\big(d_c(0, \epsilon_n)\big). \\ & (\ln 2, 1) + F\big(d_c(\urcorner\epsilon, \urcorner\omega)\big) \le F\big(d_c(\epsilon, \omega)\big). \\ & (\ln 2, 1) + \left(\frac{\ln|\urcorner\epsilon - \urcorner\omega|}{\sqrt{|\urcorner\epsilon - \urcorner\omega|}}, -\frac{1}{\sqrt{|\urcorner\epsilon - \urcorner\omega|}} \right) \le \left(\frac{\ln|\epsilon - \omega|}{\sqrt{|\epsilon - \omega|}}, -\frac{1}{\sqrt{|\epsilon - \omega|}} \right) \\ & \left(\ln 2 + \frac{\ln|\urcorner\epsilon - \urcorner\omega|}{\sqrt{|\urcorner\epsilon - \urcorner\omega|}}, 1 - \frac{1}{\sqrt{|\urcorner\epsilon - \urcorner\omega|}} \right) \le \left(\frac{\ln|\epsilon - \omega|}{\sqrt{|\epsilon - \omega|}}, -\frac{1}{\sqrt{|\epsilon - \omega|}} \right) \end{aligned}$$

$$\ln 2 + \frac{\ln|\daleth\epsilon - \daleth\omega|}{\sqrt{|\daleth\epsilon - \daleth\omega|}} \leq \frac{\ln|\epsilon - \omega|}{\sqrt{|\epsilon - \omega|}} \quad \text{and} \quad 1 - \frac{1}{\sqrt{|\daleth\epsilon - \daleth\omega|}} \leq -\frac{1}{\sqrt{|\epsilon - \omega|}}.$$

This implies that:

$$|\daleth\epsilon - \daleth\omega|^{\frac{1}{\sqrt{|\daleth\epsilon - \daleth\omega|}}} |\epsilon - \omega|^{\frac{-1}{\sqrt{|\epsilon - \omega|}}} \leq \frac{1}{2}$$

and

$$\frac{1}{\sqrt{|\daleth\epsilon - \daleth\omega|}} - \frac{1}{\sqrt{|\epsilon - \omega|}} \geq 1.$$

Thus, all the conditions of Corollary 1 hold true, so $\daleth$ must has a unique fixed point.

Application In this section, we have employed our main result to demonstrate a unique solution for a semilinear operator system within the context of a Banach space. This exploration is crucial, as it provides a concrete instance where our theoretical findings can be applied to solve practical problems in mathematical analysis. Specifically, by considering a semilinear operator defined on a Banach space, we have illustrated how our result contributes to establishing not only the existence but also the uniqueness of solutions in such systems.

Let us consider a Banach space $(\triangle, \|\cdot\|)$ and let $P, Q : \triangle \times \triangle \to \triangle$ be nonlinear operators. In this section, we will utilize the previous findings to offer the theorem that establishes the existence of the semilinear operator (SLO) system in the given form:

$$\begin{cases} P(\epsilon, \omega) = \epsilon \\ Q(\epsilon, \omega) = \omega. \end{cases} \tag{16}$$

The nonlinear differential system, which includes initial or boundary values, is expressed in the form of the operators (16). The theorems of Schauder, Leray-Schauder, Krasnoselskii, and Parov are employed to establish the existence and uniqueness of a solution to such a system.

Let $X_s = \triangle^2$ and $\mathsf{d_c} : X_s \times X_s \to \Re^2$ by $\mathsf{d_c}(\epsilon, \omega) = (\|\epsilon_1 - \epsilon_2\|, \|\omega_1 - \omega_2\|)$ $\epsilon = (\epsilon_1, \omega_1)$, $\omega = (\epsilon_2, \omega_2) \in X_s$, then $(X_s, \mathsf{d_c})$ is a generalized complete metric space. If , $\daleth : X_s \to X_s$ define by $\daleth u = (Pu, Qu)$, then (16) can be represented in the FP problem as

$$u = \daleth(u). \tag{17}$$

The following theorem guarantees the existence of the solution of the FP problem (17).

Theorem 2.5 *Let* $\zeta_\ell > 0$ *for* $\ell \in \{1, 2\}$ *and* $g_{ij}, h_{ij}, k_{i\ell}, l_{i\ell}, w_{i\ell} \in [0, 1)$ *such that*

$$\Lambda = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix},$$

$$\mathcal{E} = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}, \qquad \mathcal{G} = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix},$$

for each $\epsilon = (\epsilon_1, \omega_1)$, $\omega = (\epsilon_2, \omega_2) \in X_s$ *such that the following conditions hold*:

(i) $\|P(\epsilon_1,\omega_1)-P(\epsilon_2,\omega_2)\| \le \exp(\zeta_1)(g_{11}\|\epsilon_1-\epsilon_2\|+g_{12}\|\omega_1-\omega_2\|+h_{11}\|\epsilon_1-P\epsilon\|+h_{12}\|\omega_1-Q\epsilon\|+l_{11}\|\epsilon_2-P\omega\|+l_{12}\|\omega_2-Q\omega\|)+k_{11}\|\epsilon_1-P\omega\|+k_{12}\|\omega_1-Q\omega\|+w_{11}\|\epsilon_2-P\epsilon\|+w_{12}\|\omega_2-Q\epsilon\|)$;

(ii) $\|Q(\epsilon_1,\omega_1)-Q(\epsilon_2,\omega_2)\| \le \exp(\zeta_2)(g_{21}\|\epsilon_1-\epsilon_2\|+g_{22}\|\omega_1-\omega_2\|+h_{21}\|\epsilon_1-P\epsilon\|+h_{22}\|\omega_1-Q\epsilon\|+l_{21}\|\epsilon_2-P\omega\|+l_{22}\|\omega_2-Q\omega\|)+k_{21}\|\epsilon_1-P\omega\|+k_{22}\|\omega_1-Q\omega\|+w_{21}\|\epsilon_2-P\epsilon\|+w_{22}\|\omega_2-Q\epsilon\|)$.

Then, the system (16) *has a unique solution in* $\triangle^2$.

Proof Due to **(i)** and **(ii)**, we can write

$$\begin{aligned}
&\zeta_1+\ln\big\|P(\epsilon_1,\omega_1)-P(\epsilon_2,\omega_2)\big\| \\
&\quad\le \ln\big(g_{11}\|\epsilon_1-\epsilon_2\|+g_{12}\|\omega_1-\omega_2\|+h_{11}\|\epsilon_1-P\epsilon\|+h_{12}\|\omega_1-Q\epsilon\| \\
&\qquad+l_{11}\|\epsilon_2-P\omega\|+l_{12}\|\omega_2-Q\omega\|+k_{11}\|\epsilon_1-P\omega\|+k_{12}\|\omega_1-Q\omega\| \\
&\qquad+w_{11}\|\epsilon_2-P\epsilon\|+w_{12}\|\omega_2-Q\epsilon\|\big)
\end{aligned}\tag{18}$$

and

$$\begin{aligned}
&\zeta_2+\ln\big\|Q(\epsilon_1,\omega_1)-Q(\epsilon_2,\omega_2)\big\| \\
&\quad\le \ln\big(g_{21}\|\epsilon_1-\epsilon_2\|+g_{22}\|\omega_1-\omega_2\|+h_{21}\|\epsilon_1-P\epsilon\|+h_{22}\|\omega_1-Q\epsilon\| \\
&\qquad+l_{21}\|\epsilon_2-P\omega\|+l_{22}\|\omega_2-Q\omega\|+k_{21}\|\epsilon_1-P\omega\|+k_{22}\|\omega_1-Q\omega\| \\
&\qquad+w_{21}\|\epsilon_2-P\epsilon\|+w_{22}\|\omega_2-Q\epsilon\|\big),
\end{aligned}\tag{19}$$

respectively. Now, combining (18) and (19), we have

$$\begin{aligned}
&(\zeta_1+\ln\|P(\epsilon_1,\omega_1)-P(\epsilon_2,\omega_2)\|,\zeta_2+\ln\|Q(\epsilon_1,\omega_1)-Q(\epsilon_2,\omega_2)\|) \\
&\quad\preceq \big(\ln\big(g_{11}\|\epsilon_1-\epsilon_2\|+g_{12}\|\omega_1-\omega_2\|+h_{11}\|\epsilon_1-P\epsilon\|+h_{12}\|\omega_1-Q\epsilon\| \\
&\qquad+l_{11}\|\epsilon_2-P\omega\|+l_{12}\|\omega_2-Q\omega\|+k_{11}\|\epsilon_1-P\omega\|+k_{12}\|\omega_1-Q\omega\| \\
&\qquad+w_{11}\|\epsilon_2-P\epsilon\|+w_{12}\|\omega_2-Q\epsilon\|\big), \\
&\qquad\ln\big(g_{21}\|\epsilon_1-\epsilon_2\|+g_{22}\|\omega_1-\omega_2\|+h_{21}\|\epsilon_1-P\epsilon\|+h_{22}\|\omega_1-Q\epsilon\| \\
&\qquad+l_{21}\|\epsilon_2-P\omega\|+l_{22}\|\omega_2-Q\omega\|+k_{21}\|\epsilon_1-P\omega\|+k_{22}\|\omega_1-Q\omega\| \\
&\qquad+w_{21}\|\epsilon_2-P\epsilon\|+w_{22}\|\omega_2-Q\epsilon\|\big)\big).
\end{aligned}\tag{20}$$

Considering $F(a_1,a_2)=(F_1(a_1),F(a_2))=(\ln a_1,\ln a_2)$, then $F\in \lrcorner^2$ and therefore from (20), we have

$$\begin{aligned}
&(\zeta_1,\zeta_2)+F(\|P(\epsilon_1,\omega_1)-P(\epsilon_2,\omega_2)\|,\|Q(\epsilon_1,\omega_1)-Q(\epsilon_2,\omega_2)\|) \\
&\quad\preceq F\big(g_{11}\|\epsilon_1-\epsilon_2\|+g_{12}\|\omega_1-\omega_2\|+h_{11}\|\epsilon_1-P\epsilon\|+h_{12}\|\omega_1-Q\epsilon\| \\
&\qquad+l_{11}\|\epsilon_2-P\omega\|+l_{12}\|\omega_2-Q\omega\| \\
&\qquad+k_{11}\|\epsilon_1-P\omega\|+k_{12}\|\omega_1-Q\omega\|+w_{11}\|\epsilon_2-P\epsilon\|+w_{12}\|\omega_2-Q\epsilon\|, \\
&\qquad g_{21}\|\epsilon_1-\epsilon_2\|+g_{22}\|\omega_1-\omega_2\|+h_{21}\|\epsilon_1-P\epsilon\|+h_{22}\|\omega_1-Q\epsilon\|
\end{aligned}$$

$$+ l_{21}\|\epsilon_2 - P\omega\| + l_{22}\|\omega_2 - Q\omega\|$$
$$+ k_{21}\|\epsilon_1 - P\omega\| + k_{22}\big[\|\omega_1 - Q\omega\| + w_{21}\|\epsilon_2 - P\epsilon\| + w_{22}\|\omega_2 - Q\epsilon\|\big]\big). \tag{21}$$

We also write (21) as

$$\begin{aligned}
&\zeta + F\big(\|P\epsilon - P\omega\|, \|Q\epsilon - Q\omega\|\big)\\
&\preceq F\left[\left(\begin{bmatrix} g_{11}\\ g_{12}\end{bmatrix}^T \begin{bmatrix}\|\epsilon_1-\epsilon_2\|\\ \|\omega_1-\omega_2\|\end{bmatrix}, \begin{bmatrix} g_{21}\\ g_{22}\end{bmatrix}^T \begin{bmatrix}\|\epsilon_1-\epsilon_2\|\\ \|\omega_1-\omega_2\|\end{bmatrix}\right)\right.\\
&\quad + \left(\begin{bmatrix} h_{11}\\ h_{12}\end{bmatrix}^T \begin{bmatrix}\|\epsilon_1-P\epsilon\|\\ \|\omega_1-Q\epsilon\|\end{bmatrix}, \begin{bmatrix} h_{21}\\ h_{22}\end{bmatrix}^T \begin{bmatrix}\|\epsilon_1-P\epsilon\|\\ \|\omega_1-Q\epsilon\|\end{bmatrix}\right)\\
&\quad + \left(\begin{bmatrix} l_{11}\\ l_{12}\end{bmatrix}^T \begin{bmatrix}\|\epsilon_2-P\omega\|\\ \|\omega_2-Q\omega\|\end{bmatrix}, \begin{bmatrix} l_{21}\\ l_{22}\end{bmatrix}^T \begin{bmatrix}\|\epsilon_2-P\omega\|\\ \|\omega_2-Q\omega\|\end{bmatrix}\right)\\
&\quad + \left(\begin{bmatrix} k_{11}\\ k_{12}\end{bmatrix}^T \begin{bmatrix}\|\epsilon_1-P\omega\|\\ \|\omega_1-Q\omega\|\end{bmatrix}, \begin{bmatrix} k_{21}\\ k_{22}\end{bmatrix}^T \begin{bmatrix}\|\epsilon_1-P\omega\|\\ \|\omega_1-Q\omega\|\end{bmatrix}\right)\\
&\quad \left.+ \left(\begin{bmatrix} w_{11}\\ w_{12}\end{bmatrix}^T \begin{bmatrix}\|\epsilon_2-P\epsilon\|\\ \|\omega_2-Q\epsilon\|\end{bmatrix}, \begin{bmatrix} w_{21}\\ w_{22}\end{bmatrix}^T \begin{bmatrix}\|\epsilon_2-P\epsilon\|\\ \|\omega_2-Q\epsilon\|\end{bmatrix}\right)\right].
\end{aligned}$$

We can also write

$$\begin{aligned}
&\zeta + F\big(\|P\epsilon - P\omega\|, \|Q\epsilon - Q\omega\|\big)\\
&\preceq F\left[\begin{bmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\begin{pmatrix}\|\epsilon_1-\epsilon_2\|\\ \|\omega_1-\omega_2\|\end{pmatrix}\right.\\
&\quad + \begin{bmatrix} h_{11} & h_{12}\\ h_{21} & h_{22}\end{bmatrix}\begin{pmatrix}\|\epsilon_1-P\epsilon\|\\ \|\omega_1-Q\epsilon\|\end{pmatrix} + \begin{bmatrix} l_{11} & l_{12}\\ l_{21} & l_{22}\end{bmatrix}\begin{pmatrix}\|\epsilon_2-P\omega\|\\ \|\omega_2-Q\omega\|\end{pmatrix}\\
&\quad \left.+ \begin{bmatrix} k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}\begin{pmatrix}\|\epsilon_1-P\omega\|\\ \|\omega_1-Q\omega\|\end{pmatrix} + \begin{bmatrix} w_{11} & w_{12}\\ w_{21} & w_{22}\end{bmatrix}\begin{pmatrix}\|\epsilon_2-P\epsilon\|\\ \|\omega_2-Q\epsilon\|\end{pmatrix}\right].
\end{aligned}$$

Consequently,

$$\begin{aligned}
\zeta + F\big(\mathsf{d_c}(\urcorner\epsilon, \urcorner\omega)\big) &\preceq F\big(\Lambda\big(\mathsf{d_c}(\epsilon,\omega)\big) + \mathcal{B}\big(\mathsf{d_c}(\epsilon,\urcorner\epsilon)\big) + \mathcal{C}\big(\mathsf{d_c}(\omega,\urcorner\omega)\big)\\
&\quad + \mathcal{E}\big(\mathsf{d_c}(\epsilon,\urcorner\omega)\big) + \mathcal{G}\big(\mathsf{d_c}(\omega,\urcorner\epsilon)\big)\big),
\end{aligned}$$

where $\zeta = (\zeta_1, \zeta_2)$, thus $\urcorner$ ensures a unique FP in $X_s = \triangle^2$ or we can say that the (SLO) system (16) has a unique solution in $\triangle^2$. □

3 Conclusion and future work

This manuscript contributes significantly to the field of mathematical analysis by presenting new results on a generalized F-contraction of Hardy–Rogers-type mappings in a complete vector-valued metric space. The work extends and enhances numerous findings already established in the literature, offering a more comprehensive understanding of fixed-point theorems for single and pairs of these mappings. Applying these theorems to demonstrate the existence of a unique solution for a semilinear operator system in a Banach space not only validates the theoretical results but also showcases their practical relevance.

Looking ahead, several avenues for future research present themselves. First, extending these results to more generalized metric spaces, such as modular metric spaces or spaces with a more complex structure, could provide further insights. Additionally, exploring the applicability of these theorems in solving more complex differential and integral equations could prove beneficial. Another potential area of exploration is applying these theorems in computational mathematics, particularly in machine-learning and data-analysis algorithms.

Acknowledgements
The authors would like to thank the Prince Sultan University for the support of this work through TAS LAB.

Funding
This paper has not received any external funding.

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
All authors contribute equally to the writing of this manuscript.

Author details
[1]Department of Mathematics, University of Malakand, Chakdara Dir(L), Khyber Pakhtunkhwa 18000, Pakistan. [2]Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia. [3]Department of Mathematics, Faculty of Science and Arts, Kirikkale University, Yahsihan, Kirikkale 71450, Turkey.

Received: 24 December 2023 Accepted: 30 January 2024 Published online: 18 March 2024

References

1. Poincare, H.: Surless courbes define barles equations differentiate less. J. Math. **2**, 54–65 (1886)
2. Banach, S.: Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales. Fundam. Math. **3**, 133–181 (1922)
3. Asif, A., Khan, S., Abdeljawad, T., Muhammad, A., Savas, E.: 3D analysis of modified F-contractions in convex b-metric spaces with application to Fredholm integral equations. AIMS Math. **5**(6), 6929–6948 (2020)
4. Amin, R., Shah, K., Gao, L., Abdeljawad, T.: On existence and numerical solution of higher order nonlinear integro-differential equations involving variable coefficients. Results Appl. Math. **20**(3), 100399 (2023)
5. Hammad, H.A., Abdeljawad, T.: Quadruple fixed-point techniques for solving integral equations involved with matrices and the Markov process in generalized metric spaces. J. Inequal. Appl. **2022**, 44 (2022)
6. Hardy, G.E., Rogers, A.D.: A generalization of fixed point theorem of Reich. Can. Math. Bull. **16**, 201–208 (1973)
7. Cosentino, V., Vetro, P.: Fixed point result for *F*-contractive mappings of Hardy-Rogers-type. Filomat **28**, 715–722 (2014)
8. Rasham, T., Shoaib, A., Hussain, N., Arshad, M., Khan, S.U.: Common fixed point results for new Ciric-type rational multivalued F-contraction with an application. J. Fixed Point Theory Appl. **20**, 1–16 (2018)
9. Mahmood, Q., Shoaib, A., Rasham, T., Arshad, M.: Fixed point results for the family of multivalued F-contractive mappings on closed ball in complete dislocated b-metric spaces. Mathematics **7**(1), 56 (2019). https://doi.org/10.3390/math7010056

10. Rasham, T., Asif, A., Aydi, H., De La Sen, M.: On pairs of fuzzy dominated mappings and applications. Adv. Differ. Equ. **2021**(1), 417 (2021)
11. Tahair, R., Shoaib, A., Park, C., Arshad, M.: Fixed point results for a pair of multi dominated mappings on a smallest subset in K-sequentially dislocated quasi metric space with application. J. Comput. Anal. Appl. **25**(5), 975–986 (2018)
12. Rasham, T., Shoaib, A., Park, C., de la Sen, M., Aydi, H., Rye Lee, J.: Multivalued fixed point results for two families of mappings in modular-like metric spaces with applications. Complexity **2020**, 1–10 (2020)
13. Mani, N., Gupta, V., Kanwar, A., Bhardwaj, R.: Generalized F_δ-contractions and multivalued common fixed point theorem. Proc. Jangjeon Math. Soc. **21**(4), 703–708 (2018)
14. Borisut, P., Kumam, P., Gupta, V., Mani, N.: Generalized (Ψ, α, β)-weak contractions for initial value problems. Mathematics **7**(3), 266 (2019)
15. Wardowski, D.: Fixed points of a new type of contractive mappings in complete metric spaces. Fixed Point Theory Appl. **2012**, 94 (2012)
16. Perov, A.I.: On the Cauchy problem for a system of ordinary differential equations. Priblizhen. Met. Reshen. Differ. Uravn. **2**, 115–134 (1964)
17. Varga, R.: Matrix Itrative Analysis. Springer Series in Computational Mathematics, vol. 27. Springer, Berlin (2000)
18. Cvetkovic, M., Rakvcevic, V.: Quasi-contraction of Perov-type contraction. Appl. Math. Comput. **235**, 712–722 (2014)
19. Abbas, M., Rakocevic, V., Iqbal, A.: Coincidence and common fixed point of Perov type generalized Ciric-contraction mappings. Mediterr. J. Math. **13**, 3537–3555 (2016)
20. Safia, B., fateh, E., Abdelkrim, A.: Fixed point theory on spaces with vector-valued metrics and applications. Hacet. J. Math. Stat. **46**(3), 419–426 (2017)
21. Filip, A.D., Petrusel, A.: Fixed point theorems on space endowed with vector valued metrics. Fixed Point Theory Appl. (2010)
22. Petrusel, A., Mlesnite, O., Urs, C.: Vector-valued metrics in fixed point theory. In: Reich, S., Zaslavski, A.J. (eds.) Infinite Products of Operators and Their Application. American Mathematical Society, Israel Mathematical Confrence Proceedings, Contemporary Mathematics, vol. 636, pp. 149–165 (2015)
23. Petrusel, A., Petrusel, G.: A study of a general system of operator equations in b-metric space via the vector approach in the fixed point theory. J. Fixed Point Theory Appl. **19**(3), 1793–1814 (2017)
24. Minak, G., Helvaci, A., Altun, I.: Ciric type generalization F-contraction on comlete metric spaces and fixed point results. Filomat **28**(6), 1143–1151 (2014)
25. Ilic, D., Cvetkovic, M., Gajic, L., Rakocevic, V.: Fixed points of sequence of Ciric generalized contractions of Perov type. Mediterr. J. Math. **13**, 3921–3937 (2016)
26. Vetro, F., Radenovic, S.: Some results of Perov type in rectangular cone metric spaces. J. Fixed Point Theory Appl. **20**, 41 (2018)
27. Altun, I., Olgun, M.: Fixed point results for perove type F-contractions and an applications. J. Fixed Point Theory Appl. **22**, 1–11 (2020)
28. Jachymski, J., Klima, J.: Around Perov's fixed point theorem for mappings on generalized metric spaces. Fixed Point Theory **17**, 367–380 (2016)
29. Nikola, M., Stojan, R., Slobodan, R.: Some new observations for F-contractions in vector-valued metric spaces of Perov's type. Axioms **10**, 127 (2021)

Publisher's Note