Fixed Point Theory and Algorithms for Sciences and Engineering

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Fixed point results for Geraghty–Ćirić-type contraction mappings in b-metric space with applications

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Abstract

In this study, we manifest a new class of mappings that satisfy Geraghty–Ćirić-type contractive conditions in the context of b-metric spaces and prove a theorem on the existence and uniqueness of fixed points. Our results unify and generalize the results of Geraghty; Ćirić; Dukic, Kadelburg, and Radenović; and Shu-fang Li, Fei Hi, and Ning Lu in the setting of b-metric spaces. Furthermore, we provide examples to verify the correctness and applicability of our results. We also utilize our findings to show the existence of a unique solution for a nonlinear integral equation.

Keywords: Fixed point; b-metric spaces; Geraghty–Ćirić-type contraction; Nonlinear integral equations

1 Introduction and preliminaries

Fixed point theory is one of the fundamental and most significant areas in the advancement of mathematics and nonlinear analysis. This theory has been used extensively in numerous scientific disciplines, including computer science, engineering, chemistry, biology, economics, medical sciences, and telecommunication. One of the key findings of traditional functional analysis, known as the Banach contraction principle, was initially presented by Banach [9] in 1922. This principle is a well-known and widely recognized result of fixed point theory. Since its initial proposal and successful demonstration, numerous mathematicians have extended and generalized the Banach contraction mapping concept in several intriguing ways.

Geraghty [18] demonstrated a significant extension of the Banach contraction principle in 1973 by using an auxiliary function instead of a constant and establishing a fixed point result for these mappings in the setting of complete metric spaces. Geraghty [18] stated and proved the following result:

Theorem 1 ([18]) *Let $(\mathcal{M}, d)$ be a complete metric space and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a mapping satisfying*

$$d(\mathcal{T}x, \mathcal{T}y) \leq \beta\big(d(x,y)\big)d(x,y),$$

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for all $x, y \in \mathcal{M}$, *where* $\beta \in \mathcal{S} = \{\beta : \Re^+ \to [0,1) | \lim_{n\to\infty} \beta(\zeta_n) = 1 \implies \lim_{n\to\infty} \zeta_n = 0\}$. *Then,* $\mathcal{T}$ *has a unique fixed point* x^* *in* $\mathcal{M}$.

Later on, many researchers have generalized and extended the result obtained by Geraghty in diverse ways, see [4, 5, 7, 14, 15, 17, 19] and the references therein.

Ćirić [10, 11] established the famous Ćirić-type fixed point theorem in the context of metric spaces, which is regarded as one of the most well-known findings that generalizes the Banach contraction principle. An important generalization of the Banach contraction principle obtained by Ćirić is as follows:

Theorem 2 ([11]) *Let* $(\mathcal{M}, d)$ *be a complete metric space and* $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ *be a mapping. If there exists a* $\lambda \in [0,1)$ *satisfying*

$$d(\mathcal{T}x, \mathcal{T}y) \le \lambda \max\bigl\{d(x,y), d(x,\mathcal{T}x), d(y,\mathcal{T}y), d(x,\mathcal{T}y), d(y,\mathcal{T}x)\bigr\},$$

for all $x, y \in \mathcal{M}$, *then* $\mathcal{T}$ *has a unique fixed point* x^* *in* $\mathcal{M}$.

Over the past few decades, fixed point theorems of the Ćirić kind have been generalized and extended in various ways by several authors. In 2013, Kumam et al. [23] reported one of the most interesting results and established a new fixed point theorem, which is a generalization of the Ćirić fixed point theorem. Karapınar [21] explored a Ćirić-type nonunique fixed point result in the framework of Branciari metric spaces and generalized the Ćirić-type fixed point theorem.

Recently, some authors studied fixed point theorems that incorporate Geraghty and Ćirić type contraction conditions in the framework of complete metric spaces. In 2019, Alqahtani et al. [2] introduced the notion of Ćirić-type φ-Geraghty contraction mappings and investigated under which conditions such mappings possess a unique fixed point in complete metric spaces.

In 2022, Shu-fang Li et al. [25] unified Geraghty and Ćirić type contractive mappings and obtained a fixed point for such mappings in a complete metric space. Shu-fang Li et al. [25] defined Geraghty–Ćirić-type contraction as follows:

A self-map $\mathcal{T}$ on a metric space $(\mathcal{M}, d)$ is said to be a Geraghty–Ćirić-type contraction mapping if there exists $\beta \in \mathcal{S}$ such that, for all $x, y \in \mathcal{M}$,

$$d(\mathcal{T}x, \mathcal{T}y) \le M(x,y),$$

where

$$\begin{aligned} M(x,y) = \max\bigl\{&\beta\bigl(d(x,y)\bigr)d(x,y), \beta\bigl(d(x,\mathcal{T}x)\bigr)d(x,\mathcal{T}x), \beta\bigl(d(y,\mathcal{T}y)\bigr)d(y,\mathcal{T}y), \\ &\beta\bigl(d(x,\mathcal{T}y)\bigr)d(x,\mathcal{T}y), \beta\bigl(d(y,\mathcal{T}x)\bigr)d(y,\mathcal{T}x)\bigr\}. \end{aligned}$$

Theorem 3 ([25]) *Let* $(\mathcal{M}, d)$ *be a complete metric space and* $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ *be a Geraghty–Ćirić-type contraction with some* $\beta \in \mathcal{S}$. *Then* $\mathcal{T}$ *has a unique fixed point* $x^* \in \mathcal{M}$.

Notation 1 Throughout the paper, we use the following notations:

(i) $\Re$ stands for the set of all real numbers;

(ii) $\Re^+$ stands for the set of all nonnegative real numbers;
(iii) $\mathbb{N}$ stands for the set of all positive integers;
(iv) $\mathbb{N}_0$ stands for the set of all nonnegative integers;
(v) $\mathcal{M}$ and $\mathcal{T}$ denote a nonempty set and a self-mapping on $\mathcal{M}$, respectively.

The notion of a b-metric space was introduced by Bakhtin [8] and Czerwik [12] as a generalization of metric space, and they demonstrated fixed point results for contractive mappings in such spaces. Subsequently, several papers on the fixed point theory for different classes of mappings satisfying various contractive conditions have been published in b-metric spaces. Karapinar et al. [22] obtained a fixed point theorem of Ćirić type in b-metric spaces. In 2016, Pant and Panicker [27] introduced Geraghty and Ćirić type fixed point theorems and obtained fixed point results for admissible mappings in the setting of b-metric spaces. In 2019, Mlaiki et al. [26] discussed the fixed point results given by Pant and Panicker [27] and improved some related fixed point theorems in b-metric spaces. For some recent significant developments in the area of b-metric spaces and their extensions with various contractive conditions, we refer to the work of Karapinar [20], Afshari et al. [1], Ding et al. [13], Latif et al. [24], Faraji et al. [17], Eshraghi et al. [16], Amirbostagi and Asad [3], Asadi and Afshar [6], Erhan [15], and the references therein.

The concept of a b-metric space was defined independently by Bakhtin [8] and Czerwik [12] as follows:

Definition 1 ([8, 12]) A mapping $d : \mathcal{M} \times \mathcal{M} \to \Re^+$ is said to be a b-metric if it satisfies the following three conditions:
(i) $d(x,y) = 0$ if and only if $x = y$ for any $x,y \in \mathcal{M}$;
(ii) $d(x,y) = d(y,x)$ for any $x,y \in \mathcal{M}$;
(iii) there exists a real number $\nu \geq 1$ such that $d(x,y) \leq \nu[d(x,z) + d(z,y)]$ for any $x,y,z \in \mathcal{M}$.
In this case, the triplet $(\mathcal{M}, d, \nu)$ is called a b-metric space.

Remark 1 ([1]) Every metric is a b-metric with $\nu = 1$ but not conversely. Thus, the class of b-metrics is effectively larger than that of metrics.

We illustrate the above remark by using the following two examples.

Example 1 ([1]) Let $\mathcal{M} = [0,1]$ be a set and $d : \mathcal{M} \times \mathcal{M} \to \Re^+$ be a function given by $d(x,y) = |x - y|^2$ for all $x,y \in \mathcal{M}$. For any $x,y,z \in \mathcal{M}$, we have

$$d(x,y) \leq 2\bigl[d(x,z) + d(z,y)\bigr].$$

So, d is a b-metric with $\nu = 2$. But d is not a metric. In fact, if we take $x = 0$ and $y = 1$, we obtain

$$d(0,1) = 1 > 0.5 = 0.25 + 0.25 = d(1,0.5) + d(0.5,0).$$

Example 2 ([20]) Given a set $\mathcal{M} = \{0,1,2\}$. Define the function $d : \mathcal{M} \times \mathcal{M} \to \Re^+$ by

$$\begin{cases} d(x,x) = 0, \quad \text{for all } x \in \mathcal{M}, \qquad d(x,y) = d(y,x) \quad \text{for all } x,y \in \mathcal{M}, \\ d(1,2) = d(1,0) = 1, \qquad d(0,2) = 3. \end{cases}$$

Letting $x = 2$ and $y = 0$, we get

$$d(2,0) = 3 > 2 = d(2,1) + d(1,0).$$

However, for all $x, y, z \in \mathcal{M}$, we have

$$d(x,y) \le \frac{3}{2}\big[d(x,z) + d(z,y)\big].$$

Therefore, d is a b-metric with $\nu = \frac{3}{2}$ but not a metric.

Definition 2 ([12]) Let $\{x_n\}$ be a sequence on a b-metric space $(\mathcal{M}, d, \nu)$ with $\nu \ge 1$. Then

(i) $\{x_n\}$ is called b-convergent if there exists $x \in \mathcal{M}$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) $\{x_n\}$ is called b-Cauchy if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$, that is, if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \ge n_0$.

Definition 3 ([12]) A b-metric space $(\mathcal{M}, d, \nu)$ with $\nu \ge 1$ is said to be complete if every Cauchy sequence in $\mathcal{M}$ is b-convergent in $\mathcal{M}$.

Remark 2 ([12]) Let $(\mathcal{M}, d, \nu)$ be a b-metric space with $\nu \ge 1$ and $\{x_n\}$ is a b-convergent sequence in $\mathcal{M}$. Then, the sequence $\{x_n\}$ has a unique limit and it is b-Cauchy.

In 2011, Dukic et al. [14] obtained fixed points for Geraghty-type mappings in b-metric spaces by considering the class of functions

$$\mathcal{S}_\nu = \left\{\beta : \Re^+ \to \left[0, \frac{1}{\nu}\right) \Big| \lim_{n\to\infty} \beta(\zeta_n) = \frac{1}{\nu} \implies \lim_{n\to\infty} \zeta_n = 0\right\},$$

where $\nu \ge 1$. For instance, the function $\beta : \Re^+ \to [0, \frac{1}{\nu})$ defined by $\beta(\zeta) = \frac{1}{\nu} e^{-\zeta}$ for $\zeta > 0$ and $\beta(0) \in [0, \frac{1}{\nu})$ is in $\mathcal{S}_\nu$.

Theorem 4 ([14]) *Let $(\mathcal{M}, d, \nu)$ be a complete b-metric space with $\nu > 1$. Suppose that a mapping $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ satisfies*

$$d(\mathcal{T}x, \mathcal{T}y) \le \beta\big(d(x,y)\big) d(x,y),$$

for all $x, y \in \mathcal{M}$ and for some $\beta \in \mathcal{S}_\nu$. Then $\mathcal{T}$ has a unique fixed point $x^* \in \mathcal{M}$.

In 2019, Faraji et al. [17] established a fixed point theorem satisfying Geraghty-type contractive conditions in b-metric spaces by defining a class of function $\mathcal{S}_\nu^*$, for $\nu \ge 1$, as

$$\mathcal{S}_\nu^* = \left\{\beta : \Re^+ \to \left[0, \frac{1}{\nu}\right) \Big| \limsup_{n\to\infty} \beta(\zeta_n) = \frac{1}{\nu} \implies \lim_{n\to\infty} \zeta_n = 0\right\}.$$

Theorem 5 ([17]) *Let $(\mathcal{M}, d, \nu)$ be a b-complete b-metric space with $\nu \ge 1$ and let $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a self-mapping. If there exists $\beta \in \mathcal{S}_\nu^*$ such that*

$$d(\mathcal{T}x, \mathcal{T}y) \le \beta\big(M(x,y)\big) M(x,y), \quad \textit{for all } x, y \in \mathcal{M},$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,\mathcal{T}x), d(y,\mathcal{T}y), \frac{1}{2\nu}\big[d(x,\mathcal{T}y) + d(y,\mathcal{T}x)\big]\right\},$$

then $\mathcal{T}$ has a unique fixed point.

The aim of this paper is to introduce a new class of mappings satisfying Geraghty–Ćirić-type contraction condition in the context of b-metric spaces and prove a theorem on the existence and uniqueness of fixed points for the mappings introduced. Our result generalize, include, and unify the results defined by Geraghty [18], Ćirić [11], Dukic et al. [14] and Shu-fang Li et al. [25], and also various existing results on the topic in the corresponding literature. Furthermore, we provide examples to illustrate the validity of our main result and apply our findings in establishing the existence and uniqueness of a solution of a nonlinear integral equation.

The following lemma is useful in proving our main result.

Lemma 1 ([24]) *Let $(\mathcal{M}, d, \nu)$ be a b-metric space with $\nu \geq 1$ and let $\{x_n\}$ and $\{y_n\}$ be b-convergent to $x, y \in \mathcal{M}$, respectively. Then, the following inequality holds*:

$$\frac{1}{\nu^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq \nu^2 d(x,y).$$

In particular, if $x = y$, we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in \mathcal{M}$, we have

$$\frac{1}{\nu} d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq \nu d(x,z).$$

2 Main results

In this section, we introduce Geraghty–Ćirić-type contraction mappings and study fixed point results for such mappings in the setting of b-metric spaces.

Here, we use a comparison function $\beta : \Re^+ \to [0, \frac{1}{\nu})$, where $\nu \geq 1$, satisfying the condition

$$\limsup_{n\to\infty} \beta(\zeta_n) = \frac{1}{\nu} \quad \Longrightarrow \quad \lim_{n\to\infty} \zeta_n = 0.$$

We denote the set of all functions β satisfying the above condition by $\mathcal{F}$.

The main result in this paper is based on the following contractive condition.

Definition 4 A mapping $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ on a b-metric space $(\mathcal{M}, d, \nu)$ with $\nu \geq 1$ is called a Geraghty–Ćirić-type contraction mapping if there exists $\beta \in \mathcal{F}$ such that

$$d(\mathcal{T}x, \mathcal{T}y) \leq \mathcal{L}(x,y), \quad \text{for all } x, y \in \mathcal{M}, \tag{1}$$

where

$$\begin{aligned}\mathcal{L}(x,y) = \max\big\{&\beta\big(d(x,y)\big)d(x,y), \beta\big(d(x,\mathcal{T}x)\big)d(x,\mathcal{T}x), \beta\big(d(y,\mathcal{T}y)\big)d(y,\mathcal{T}y),\\ &\beta\big(d(x,\mathcal{T}y)\big)d(x,\mathcal{T}y), \beta\big(d(y,\mathcal{T}x)\big)d(y,\mathcal{T}x)\big\}.\end{aligned}$$

Lemma 2 *Let $(\mathcal{M}, d, \nu)$ be a b-metric space with $\nu \geq 1$ and let $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a self-mapping. Let $x_0 \in \mathcal{M}$ be given and $\{x_n\}$ be a sequence in $\mathcal{M}$ such that $x_n = \mathcal{T}x_{n-1}$ for all $n \in \mathbb{N}$. Consider the sequence defined by*

$$A_n = \max\{d(x_p, x_q) | 0 \leq p, q \leq n \text{ and } p, q \in \mathbb{N}_0\}, \tag{2}$$

for $n \in \mathbb{N}_0$. If $\mathcal{T}$ satisfies the contractivity condition in (1), then $\{A_n\}$ is bounded.

Proof Let $n \in \mathbb{N}$ be arbitrary and fixed. Then, for any $p, q \in \mathbb{N}$ with $1 \leq p, q \leq n$, using (1), we have

$$\begin{aligned}
d(x_p, x_q) &= d(\mathcal{T}x_{p-1}, \mathcal{T}x_{q-1}) \\
&\leq \mathcal{L}(x_{p-1}, x_{q-1}) \\
&= \max\{\beta(d(x_{p-1}, x_{q-1}))d(x_{p-1}, x_{q-1}), \beta(d(x_{p-1}, x_p))d(x_{p-1}, x_p), \\
&\quad \beta(d(x_{q-1}, x_q))d(x_{q-1}, x_q), \beta(d(x_{p-1}, x_q))d(x_{p-1}, x_q), \\
&\quad \beta(d(x_p, x_{q-1}))d(x_p, x_{q-1})\} \\
&< \frac{1}{\nu} \max\{d(x_{p-1}, x_{q-1}), d(x_{p-1}, x_p), d(x_{q-1}, x_q), d(x_{p-1}, x_q), d(x_p, x_{q-1})\} \\
&\leq A_n.
\end{aligned}$$

As a result, $\max\{d(x_p, x_q) | 1 \leq p, q \leq n; p, q \in \mathbb{N}_0\} < A_n$. From this, we conclude that there is $\omega_n \in \mathbb{N}$ with $1 \leq \omega_n \leq n$ such that

$$A_n = d(x_0, x_{\omega_n}).$$

Observe that $0 \leq A_n \leq A_{n+1}$ for all $n \in \mathbb{N}_0$. We aim to show that the sequence $\{A_n\}$ is bounded. Assume, to the contrary, that $\{A_n\}$ is unbounded. Then, since $\{A_n\}$ is an increasing sequence of nonnegative real numbers, we have $\lim_{n\to\infty} A_n = +\infty$.

Now, applying b-triangle inequality on the term $d(x_0, x_{\omega_n})$ and using (1), we get

$$\begin{aligned}
A_n = d(x_0, x_{\omega_n}) &\leq \nu[d(x_0, x_1) + d(x_1, x_{\omega_n})] \\
&= \nu d(x_0, x_1) + \nu \mathcal{L}(x_0, x_{\omega_{n-1}}),
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
\mathcal{L}(x_0, x_{\omega_{n-1}}) = \max\{&\beta(d(x_0, x_{\omega_{n-1}}))d(x_0, x_{\omega_{n-1}}), \beta(d(x_0, x_1))d(x_0, x_1), \\
&\beta(d(x_{\omega_{n-1}}, x_{\omega_n}))d(x_{\omega_{n-1}}, x_{\omega_n}), \beta(d(x_0, x_{\omega_n}))d(x_0, x_{\omega_n}), \\
&\beta(d(x_1, x_{\omega_{n-1}}))d(x_1, x_{\omega_{n-1}})\}.
\end{aligned}$$

Since the sequences

$$\begin{aligned}
&\{\beta(d(x_0, x_{\omega_{n-1}}))d(x_0, x_{\omega_{n-1}})\}, \qquad \{\beta(d(x_0, x_1))d(x_0, x_1)\}, \\
&\{\beta(d(x_{\omega_{n-1}}, x_{\omega_n}))d(x_{\omega_{n-1}}, x_{\omega_n})\}, \qquad \{\beta(d(x_0, x_{\omega_n}))d(x_0, x_{\omega_n})\}, \quad \text{and}
\end{aligned}$$

$$\{\beta(d(x_1, x_{\omega_{n-1}}))d(x_1, x_{\omega_{n-1}})\}$$

are real-number sequences, there is a subsequence $\{\mathcal{L}(x_0, x_{\omega_{n_k-1}})\}$ of $\{\mathcal{L}(x_0, x_{\omega_{n-1}})\}$ that is equal to one of the following five terms:

$$\begin{aligned}
&\beta(d(x_0, x_{\omega_{n_k-1}}))d(x_0, x_{\omega_{n_k-1}}), \qquad \beta(d(x_0, x_1))d(x_0, x_1), \\
&\beta(d(x_{\omega_{n_k-1}}, x_{\omega_{n_k}}))d(x_{\omega_{n_k-1}}, x_{\omega_{n_k}}), \qquad \beta(d(x_0, x_{\omega_{n_k}}))d(x_0, x_{\omega_{n_k}}), \quad \text{or} \\
&\beta(d(x_1, x_{\omega_{n_k-1}}))d(x_1, x_{\omega_{n_k-1}}).
\end{aligned}$$

So, we have five cases to consider:

Case 1. Suppose that

$$\mathcal{L}(x_0, x_{\omega_{n_k-1}}) = \beta(d(x_0, x_{\omega_{n_k-1}}))d(x_0, x_{\omega_{n_k-1}}). \tag{4}$$

Using (3) and (4), we get

$$\begin{aligned}
A_{n_k} &\le \nu d(x_0, x_1) + \nu\beta(d(x_0, x_{\omega_{n_k-1}}))d(x_0, x_{\omega_{n_k-1}}) \\
&\le \nu d(x_0, x_1) + \nu\beta(d(x_0, x_{\omega_{n_k-1}}))A_{n_k}.
\end{aligned} \tag{5}$$

Rearranging the terms in the second inequality of (5) yields

$$\frac{1}{\nu} - \frac{d(x_0, x_1)}{A_{n_k}} \le \beta(d(x_0, x_{\omega_{n_k-1}})) < \frac{1}{\nu}.$$

Since $A_{n_k} \to +\infty$ as $k \to \infty$, we can see that $\lim_{k\to\infty}(\frac{1}{\nu} - \frac{d(x_0,x_1)}{A_{n_k}}) = \frac{1}{\nu}$. Hence $\limsup_{k\to\infty} \beta(d(x_0, x_{\omega_{n_k-1}})) = \frac{1}{\nu}$. This implies that $\lim_{k\to\infty} d(x_0, x_{\omega_{n_k-1}}) = 0$ since $\beta \in \mathcal{F}$. Taking the limit on both sides of (5), we obtain

$$\begin{aligned}
\lim_{k\to\infty} A_{n_k} &\le \lim_{k\to\infty}[\nu d(x_0, x_1) + \nu\beta(d(x_0, x_{\omega_{n_k-1}}))d(x_0, x_{\omega_{n_k-1}})] \\
&= \nu d(x_0, x_1).
\end{aligned}$$

This contradicts our assumption $\lim_{k\to\infty} A_{n_k} = +\infty$.

Case 2. Suppose that

$$\mathcal{L}(x_0, x_{\omega_{n_{k-1}}}) = \beta(d(x_0, x_1))d(x_0, x_1). \tag{6}$$

By means of (3) and (6), we have

$$\begin{aligned}
A_{n_k} &\le \nu[d(x_0, x_1) + \beta(d(x_0, x_1))d(x_0, x_1)] \\
&< (\nu + 1)d(x_0, x_1).
\end{aligned}$$

This contradicts $A_{n_k} \to +\infty$ as $k \to \infty$.

Case 3. Suppose that

$$\mathcal{L}(x_0, x_{\omega_{n_{k-1}}}) = \beta(d(x_{\omega_{n_k-1}}, x_{\omega_{n_k}}))d(x_{\omega_{n_k-1}}, x_{\omega_{n_k}}). \tag{7}$$

Combining (3) and (7), we obtain

$$\begin{aligned} A_{n_k} &\le \nu\bigl[d(x_0,x_1)+\beta\bigl(d(x_{\omega_{n_k-1}},x_{\omega_{n_k}})\bigr)d(x_{\omega_{n_k-1}},x_{\omega_{n_k}})\bigr] \\ &\le \nu d(x_0,x_1)+\nu\beta\bigl(d(x_{\omega_{n_k-1}},x_{\omega_{n_k}})\bigr)A_{n_k}. \end{aligned} \tag{8}$$

It follows from the second inequality in (8) that

$$\frac{1}{\nu}-\frac{d(x_0,x_1)}{A_{n_k}} \le \beta\bigl(d(x_{\omega_{n_k-1}},x_{\omega_{n_k}})\bigr) < \frac{1}{\nu}.$$

As $k\to+\infty$, we have $(\frac{1}{\nu}-\frac{d(x_0,x_1)}{A_{n_k}})\to\frac{1}{\nu}$ and hence $\beta(d(x_{\omega_{n_k-1}},x_{\omega_{n_k}}))\to\frac{1}{\nu}$. Since $\beta\in\mathcal{F}$, we obtain $\lim_{k\to\infty} d(x_{\omega_{n_k-1}},x_{\omega_{n_k}})=0$. Letting $k\to\infty$ on both sides of the inequality in (8), we get

$$\begin{aligned} \lim_{k\to\infty} A_{n_k} &\le \lim_{k\to\infty}\bigl[\nu d(x_0,x_1)+\nu\beta\bigl(d(x_{\omega_{n_k-1}},x_{\omega_{n_k}})\bigr)d(x_{\omega_{n_k-1}},x_{\omega_{n_k}})\bigr] \\ &= \nu d(x_0,x_1). \end{aligned}$$

This contradicts our assumption $A_{n_k}\to+\infty$ as $k\to\infty$.

Case 4. Suppose that

$$\mathcal{L}(x_0,x_{\omega_{n_k-1}})=\beta\bigl(d(x_0,x_{\omega_{n_k}})\bigr)d(x_0,x_{\omega_{n_k}}). \tag{9}$$

Substituting (9) into (3), we get

$$\begin{aligned} A_{n_k} &\le \nu\bigl[d(x_0,x_1)+\beta\bigl(d(x_0,x_{\omega_{n_k}})\bigr)d(x_0,x_{\omega_{n_k}})\bigr] \\ &\le \nu d(x_0,x_1)+\nu\beta\bigl(d(x_0,x_{\omega_{n_k}})\bigr)A_{n_k}. \end{aligned} \tag{10}$$

From the second inequality in (10) it follows that

$$\frac{1}{\nu}-\frac{d(x_0,x_1)}{A_{n_k}} \le \beta\bigl(d(x_0,x_{\omega_{n_k}})\bigr) < \frac{1}{\nu}.$$

Since $A_{n_k}\to+\infty$ as $k\to\infty$, then $\lim_{k\to\infty}(\frac{1}{\nu}-\frac{d(x_0,x_1)}{A_{n_k}})=\frac{1}{\nu}$ and hence $\limsup_{k\to\infty}\beta(d(x_0, x_{\omega_{n_k}}))=\frac{1}{\nu}$. Since $\beta\in\mathcal{F}$, we have $\lim_{k\to\infty} d(x_0,x_{\omega_{n_k}})=0$. Letting $k\to\infty$ in (10), we obtain

$$\begin{aligned} \lim_{k\to\infty} A_{n_k} &\le \lim_{k\to\infty}\bigl[\nu d(x_0,x_1)+\nu\beta\bigl(d(x_0,x_{\omega_{n_k}})\bigr)d(x_0,x_{\omega_{n_k}})\bigr] \\ &= \nu d(x_0,x_1). \end{aligned}$$

This contradicts $A_{n_k}\to+\infty$ as $k\to\infty$.

Case 5. Suppose that

$$\mathcal{L}(x_0,x_{\omega_{n_k-1}})=\beta\bigl(d(x_1,x_{\omega_{n_k-1}})\bigr)d(x_1,x_{\omega_{n_k-1}}). \tag{11}$$

Putting (11) into (3), we have

$$\begin{aligned} A_{n_k} &\leq \nu\big[d(x_0,x_1)+\beta\big(d(x_1,x_{\omega_{n_k-1}})\big)d(x_1,x_{\omega_{n_k-1}})\big] \\ &\leq \nu d(x_0,x_1)+\nu\beta\big(d(x_1,x_{\omega_{n_k-1}})\big)A_{n_k}. \end{aligned} \tag{12}$$

The second inequality in (12) yields that

$$\frac{1}{\nu}-\frac{d(x_0,x_1)}{A_{n_k}} \leq \beta\big(d(x_1,x_{\omega_{n_k-1}})\big) < \frac{1}{\nu}.$$

Following similar argument as in case 4, we have that $d(x_1,x_{\omega_{n_{k-1}}}) \to 0$ as $k\to\infty$ and using (12), we obtain

$$\begin{aligned} \lim_{k\to\infty} A_{n_k} &\leq \lim_{k\to\infty}\big[\nu d(x_0,x_1)+\nu\beta\big(d(x_1,x_{\omega_{n_k-1}})\big)d(x_1,x_{\omega_{n_k-1}})\big] \\ &= \nu d(x_0,x_1). \end{aligned}$$

This contradicts our assumption $A_{n_k}\to+\infty$ as $k\to\infty$.

The contradictions in all the cases considered above guarantee that $\{A_n\}$ is a bounded sequence. □

Theorem 6 *Let $(\mathcal{M},d,\nu)$ be a complete b-metric space with $\nu\geq 1$ and let $\mathcal{T}:\mathcal{M}\to\mathcal{M}$ be a mapping satisfying Geraghty–Ćirić-type contraction condition in (1). Then, $\mathcal{T}$ has a unique fixed point $x^*\in\mathcal{M}$.*

Proof Let $x_0\in\mathcal{M}$ be arbitrary. Construct a sequence $\{x_n\}$ in $\mathcal{M}$ by

$$x_n=\mathcal{T}x_{n-1}=\mathcal{T}^n x_0, \quad \text{for all } n\in\mathbb{N}.$$

Step 1. We show that $\{x_n\}_{n\in\mathbb{N}_0}$ is a b-Cauchy sequence in $\mathcal{M}$. Consider a sequence defined by

$$A_n=\max\big\{d(x_p,x_q)|0\leq p,q\leq n \text{ and } p,q\in\mathbb{N}_0\big\}.$$

Using Lemma 2, there exists $K>0$ such that $A_n\leq K$ for all $n\in\mathbb{N}$. Since $\{A_n\}$ is an increasing sequence, we have $\lim_{n\to\infty}A_n\leq K$.

Define a sequence $\{\Upsilon_n\}$ on a b-metric space $(\mathcal{M},d,\nu)$ by

$$\Upsilon_n=\sup\big\{d(x_p,x_q)|p,q\geq n \text{ and } p,q\in\mathbb{N}_0\big\}.$$

From this, we can see that

$$0\leq\Upsilon_n\leq\Upsilon_{n-1}\leq\cdots\leq\Upsilon_0=\lim_{n\to\infty}A_n\leq K \quad \text{for all } n\in\mathbb{N}_0.$$

Thus, $\{\Upsilon_n\}$ is a decreasing and bounded sequence of nonnegative real numbers, so that it converges to some $r\geq 0$, that is, $\lim_{n\to\infty}\Upsilon_n=r$. Then there exist two subsequences $\{x_{p_k}\}$ and $\{x_{q_k}\}$ of $\{x_n\}$ with $q_k>p_k\geq k$ for $k\in\mathbb{N}$ such that

$$d(x_{p_k},x_{q_k})\to r \quad \text{as } k\to\infty. \tag{13}$$

We claim that $r = 0$. Assume that $r > 0$. Putting $x = x_{p_k-1}$ and $y = x_{q_k-1}$ in (1), we have

$$d(x_{p_k}, x_{q_k}) = d(\mathcal{T} x_{p_k-1}, \mathcal{T} x_{q_k-1}) \leq \mathcal{L}(x_{p_k-1}, x_{q_k-1}), \tag{14}$$

where

$$\begin{aligned}\mathcal{L}(x_{p_k-1}, x_{q_k-1}) = \max\bigl\{&\beta\bigl(d(x_{p_k-1}, x_{q_k-1})\bigr)d(x_{p_k-1}, x_{q_k-1}), \beta\bigl(d(x_{p_k-1}, x_{p_k})\bigr)d(x_{p_k-1}, x_{p_k}),\\ &\beta\bigl(d(x_{q_k-1}, x_{q_k})\bigr)d(x_{q_k-1}, x_{q_k}), \beta\bigl(d(x_{p_k-1}, x_{q_k})\bigr)d(x_{p_k-1}, x_{q_k}),\\ &\beta\bigl(d(x_{p_k}, x_{q_k-1})\bigr)d(x_{p_k}, x_{q_k-1})\bigr\}.\end{aligned}$$

Thus, $\mathcal{L}(x_{p_k-1}, x_{q_k-1})$ equals to one of the five terms on the right-hand side, so that we have five cases to consider.

First, consider the case $\mathcal{L}(x_{p_k-1}, x_{q_k-1}) = \beta(d(x_{p_k-1}, x_{q_k-1}))d(x_{p_k-1}, x_{q_k-1})$ for all $k \in \mathbb{N}$. Then, condition (14) becomes

$$\begin{aligned}d(x_{p_k}, x_{q_k}) &\leq \beta\bigl(d(x_{p_k-1}, x_{q_k-1})\bigr)d(x_{p_k-1}, x_{q_k-1})\\ &\leq \beta\bigl(d(x_{p_k-1}, x_{q_k-1})\bigr)\Upsilon_{k-1}.\end{aligned} \tag{15}$$

Taking the upper limit as $k \to \infty$ on both sides of (15), it follows that

$$\limsup_{k\to\infty} d(x_{p_k}, x_{q_k}) \leq \limsup_{k\to\infty} \beta\bigl(d(x_{p_k-1}, x_{q_k-1})\bigr) \limsup_{k\to\infty} \Upsilon_{k-1},$$

and, using (13), we have

$$\begin{aligned}&r \leq \limsup_{k\to\infty} \beta\bigl(d(x_{p_k-1}, x_{q_k-1})\bigr)r,\\ &\frac{1}{\nu} \leq 1 \leq \limsup_{k\to\infty} \beta\bigl(d(x_{p_k-1}, x_{q_k-1})\bigr) < \frac{1}{\nu}.\end{aligned}$$

Since $\beta \in \mathcal{F}$, we have $\lim_{k\to\infty} d(x_{p_k-1}, x_{q_k-1}) = 0$. Using (13) and the first inequality of (15), we get

$$r = \lim_{k\to\infty} d(x_{p_k}, x_{q_k}) = 0.$$

This contradicts our assumption $r > 0$. Hence, $\lim_{n\to\infty} \Upsilon_n = r = 0$.

The other four cases can be handled similarly. Therefore, we have $r = \lim_{n\to\infty} \Upsilon_n = 0$.

Now, let $m, n \in \mathbb{N}_0$ with $m > n$. Then we get

$$\lim_{n\to\infty} d(x_n, x_m) \leq \lim_{n\to\infty} \Upsilon_n = 0.$$

Hence, $\{x_n\}$ is a b-Cauchy sequence in $\mathcal{M}$. By completeness of $\mathcal{M}$, the sequence $\{x_n\}$ converges to some $x^* \in \mathcal{M}$.

Step 2. We show that x^* is a fixed point of $\mathcal{T}$.

Assume that $\mathcal{T}x^* \neq x^*$, that is, $d(\mathcal{T}x^*, x^*) > 0$. Letting $x = x_n$ and $y = x^*$ in (1), we have

$$d\bigl(x_{n+1}, \mathcal{T}x^*\bigr) = d\bigl(\mathcal{T}x_n, \mathcal{T}x^*\bigr) \leq \mathcal{L}\bigl(x_n, x^*\bigr), \tag{16}$$

where

$$\begin{aligned}\mathcal{L}(x_n,x^*) &= \max\{\beta(d(x_n,x^*))d(x_n,x^*),\beta(d(x_n,\mathcal{T}x_n))d(x_n,\mathcal{T}x_n),\\ &\quad \beta(d(x^*,\mathcal{T}x^*))d(x^*,\mathcal{T}x^*),\beta(d(x_n,\mathcal{T}x^*))d(x_n,\mathcal{T}x^*),\\ &\quad \beta(d(x^*,\mathcal{T}x_n))d(x^*,\mathcal{T}x_n)\}\\ &= \max\{\beta(d(x_n,x^*))d(x_n,x^*),\beta(d(x_n,x_{n+1}))d(x_n,x_{n+1}),\\ &\quad \beta(d(x^*,\mathcal{T}x^*))d(x^*,\mathcal{T}x^*),\beta(d(x_n,\mathcal{T}x^*))d(x_n,\mathcal{T}x^*),\\ &\quad \beta(d(x^*,x_{n+1}))d(x^*,x_{n+1})\}.\end{aligned}$$

We consider the following five cases:

Case 1. Suppose that $\mathcal{L}(x_n,x^*) = \beta(d(x_n,x^*))d(x_n,x^*)$. Then, from (16), we have

$$d(x_{n+1},\mathcal{T}x^*) \le \beta(d(x_n,x^*))d(x_n,x^*).$$

Taking the upper limit as $n \to \infty$ on both sides of the above inequality together with Lemma 1 and noting that $\lim_{n\to\infty} d(x_n,x^*) = 0$, we have

$$\begin{aligned}\frac{1}{\nu}d(x^*,\mathcal{T}x^*) &\le \limsup_{n\to\infty} d(x_{n+1},\mathcal{T}x^*)\\ &\le \limsup_{n\to\infty} \beta(d(x_n,x^*)) \limsup_{n\to\infty} d(x_n,x^*) = 0.\end{aligned}$$

Consequently, we get $d(x^*,\mathcal{T}x^*) = 0$. This contradicts our assumption $d(\mathcal{T}x^*,x^*) > 0$.

Case 2. Suppose that $\mathcal{L}(x_n,x^*) = \beta(d(x_n,x_{n+1}))d(x_n,x_{n+1})$. In a similar way as above, using (16) and Lemma 1 together with $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$, we have

$$\begin{aligned}\frac{1}{\nu}d(x^*,\mathcal{T}x^*) &\le \limsup_{n\to\infty} d(x_{n+1},\mathcal{T}x^*)\\ &\le \limsup_{n\to\infty} \beta(d(x_n,x_{n+1})) \limsup_{n\to\infty} d(x_n,x_{n+1}) = 0.\end{aligned}$$

This implies $d(x^*,\mathcal{T}x^*) = 0$, contradicting our assumption $d(\mathcal{T}x^*,x^*) > 0$.

Case 3. Suppose that $\mathcal{L}(x_n,x^*) = \beta(d(x^*,\mathcal{T}x^*))d(x^*,\mathcal{T}x^*)$. Then, using (16) and Lemma 1, we have

$$\begin{aligned}\frac{1}{\nu}d(x^*,\mathcal{T}x^*) &\le \limsup_{n\to\infty} d(x_{n+1},\mathcal{T}x^*)\\ &\le \limsup_{n\to\infty} \beta(d(x^*,\mathcal{T}x^*))d(x^*,\mathcal{T}x^*)\\ &= \beta(d(x^*,\mathcal{T}x^*))d(x^*,\mathcal{T}x^*) < \frac{1}{\nu}d(x^*,\mathcal{T}x^*),\end{aligned}$$

which is a contradiction.

Case 4. Suppose that $\mathcal{L}(x_n,x^*) = \beta(d(x_n,\mathcal{T}x^*))d(x_n,\mathcal{T}x^*)$. Then, using (16), we have

$$d(x_{n+1},\mathcal{T}x^*) \le \beta(d(x_n,\mathcal{T}x^*))d(x_n,\mathcal{T}x^*).$$

Applying b-triangle inequality on $d(x_n, Tx^*)$, we have

$$d\big(x_{n+1}, \mathcal{T}x^*\big) \leq \nu\beta\big(d\big(x_n, \mathcal{T}x^*\big)\big)\big[d(x_n, x_{n+1}) + d\big(x_{n+1}, \mathcal{T}x^*\big)\big],$$

which is equivalent to

$$\big[1 - \nu\beta\big(d\big(x_n, \mathcal{T}x^*\big)\big)\big]d\big(x_{n+1}, \mathcal{T}x^*\big) \leq \nu\beta\big(d\big(x_n, \mathcal{T}x^*\big)\big)d(x_n, x_{n+1}).$$

Taking the upper limit as $n \to \infty$ on both sides of the above inequality, we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\big\{\big[1 - \nu\beta\big(d\big(x_n, \mathcal{T}x^*\big)\big)\big]d\big(x_{n+1}, \mathcal{T}x^*\big)\big\}\\&\quad\leq \nu \limsup_{n\to\infty} \beta\big(d\big(x_n, \mathcal{T}x^*\big)\big) \limsup_{n\to\infty} d(x_n, x_{n+1}) = 0.\end{aligned}$$

That is,

$$\limsup_{n\to\infty}\big[1 - \nu\beta\big(d\big(x_n, \mathcal{T}x^*\big)\big)\big] \limsup_{n\to\infty} d\big(x_{n+1}, \mathcal{T}x^*\big) \leq 0.$$

Using Lemma 1, we have $0 < \frac{1}{\nu}d(x^*, \mathcal{T}x^*) \leq \limsup_{n\to\infty} d(x_{n+1}, \mathcal{T}x^*) \leq \nu d(x^*, \mathcal{T}x^*)$, so that

$$\limsup_{n\to\infty}\big[1 - \nu\beta\big(d\big(x_n, \mathcal{T}x^*\big)\big)\big] \leq 0.$$

From this, it follows that

$$\frac{1}{\nu} \leq \limsup_{n\to\infty} \beta\big(d\big(x_n, \mathcal{T}x^*\big)\big) < \frac{1}{\nu}.$$

Since $\beta \in \mathcal{F}$, we get $\lim_{n\to\infty} d(x_n, \mathcal{T}x^*) = 0$. That is, $\lim_{n\to\infty} x_n = \mathcal{T}x^*$. By the uniqueness of limit of a b-convergent sequence, we have $x^* = \mathcal{T}x^*$. This contradicts our assumption $x^* \neq \mathcal{T}x^*$.

Case 5. Suppose that $\mathcal{L}(x_n, x^*) = \beta(d(x^*, x_{n+1}))d(x^*, x_{n+1})$. Similarly, using (16) and Lemma 1 together with $\lim_{n\to\infty} d(x^*, x_{n+1}) = 0$, we have

$$\begin{aligned}\frac{1}{\nu}d\big(x^*, \mathcal{T}x^*\big) &\leq \limsup_{n\to\infty} d\big(x_{n+1}, \mathcal{T}x^*\big)\\&\leq \limsup_{n\to\infty} \beta\big(d\big(x^*, x_{n+1}\big)\big) \limsup_{n\to\infty} d\big(x^*, x_{n+1}\big) = 0.\end{aligned}$$

This implies $d(x^*, \mathcal{T}x^*) = 0$ and it contradicts the assumption $d(\mathcal{T}x^*, x^*) > 0$.

Therefore, from cases 1 to 5, we deduce that $x^* = \mathcal{T}x^*$ and hence x^* is a fixed point of $\mathcal{T}$.

Step 3. Now, we show the uniqueness of the fixed point.

Assume $u \in \mathcal{M}$ is another fixed point of $\mathcal{T}$ such that $x^* \neq u$ (or $d(x^*, u) > 0$). Then, by means of (1), we get

$$\begin{aligned}
d(x^*, u) = d(\mathcal{T}x^*, \mathcal{T}u) &\leq \mathcal{L}(x^*, u) \\
&= \max\bigl\{\beta\bigl(d(x^*, u)\bigr)d(x^*, u), \beta\bigl(d(x^*, \mathcal{T}x^*)\bigr)d(x^*, \mathcal{T}x^*), \beta\bigl(d(u, \mathcal{T}u)\bigr)d(u, \mathcal{T}u), \\
&\quad \beta\bigl(d(x^*, \mathcal{T}u)\bigr)d(x^*, \mathcal{T}u), \beta\bigl(d(\mathcal{T}x^*, u)\bigr)d(\mathcal{T}x^*, u)\bigr\} \\
&= \max\bigl\{\beta\bigl(d(x^*, u)\bigr)d(x^*, u), \beta\bigl(d(x^*, x^*)\bigr)d(x^*, x^*), \beta\bigl(d(u, u)\bigr)d(u, u)\bigr\} \\
&= \beta\bigl(d(x^*, u)\bigr)d(x^*, u) < \frac{1}{\nu}d(x^*, u),
\end{aligned}$$

which is a contradiction. Therefore, $x^* = u$ and hence x^* is the only fixed point of $\mathcal{T}$ in $\mathcal{M}$. □

The following corollary is an immediate consequence of Theorem 6.

Corollary 1 *Let $(\mathcal{M}, d, \nu)$ be a complete b-metric space with $\nu \geq 1$. Suppose $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is a self-mapping and $\beta \in \mathcal{F}$ satisfies the following condition for any $x, y \in \mathcal{M}$:*

$$d(\mathcal{T}x, \mathcal{T}y) \leq \beta\bigl(\mathcal{N}(x, y)\bigr)\mathcal{N}(x, y), \tag{17}$$

where $\mathcal{N}(x, y) = \max\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), d(x, \mathcal{T}y), d(y, \mathcal{T}x)\}$. Then $\mathcal{T}$ has a unique fixed point $x^* \in \mathcal{M}$.

Proof For any $x, y \in \mathcal{M}$, the value of $\mathcal{N}(x, y)$ is always equal to at least one of the terms $d(x, y)$, $d(x, \mathcal{T}x)$, $d(y, \mathcal{T}y)$, $d(x, \mathcal{T}y)$, or $d(y, \mathcal{T}x)$. It follows that

$$\begin{aligned}
d(\mathcal{T}x, \mathcal{T}y) &\leq \beta\bigl(\mathcal{N}(x, y)\bigr)\mathcal{N}(x, y) \\
&\leq \max\bigl\{\beta\bigl(d(x, y)\bigr)d(x, y), \beta\bigl(d(x, \mathcal{T}x)\bigr)d(x, \mathcal{T}x), \beta\bigl(d(y, \mathcal{T}y)\bigr)d(y, \mathcal{T}y), \\
&\quad \beta\bigl(d(x, \mathcal{T}y)\bigr)d(x, \mathcal{T}y), \beta\bigl(d(y, \mathcal{T}x)\bigr)d(y, \mathcal{T}x)\bigr\} \\
&= \mathcal{L}(x, y).
\end{aligned}$$

All the assumptions of Theorem 6 are satisfied, and we deduce that $\mathcal{T}$ has a unique fixed point in $\mathcal{M}$. □

Remark 3 Let $(\mathcal{M}, d, \nu)$ be a complete b-metric space with $\nu \geq 1$ and let $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a self-mapping. Since $\beta(d(x, y))d(x, y) \leq \mathcal{L}(x, y)$ for all $x, y \in \mathcal{M}$, then we conclude that Theorem 4 is a special case of Theorem 6.

The following example illustrates the validity of our result in Theorem 6.

Example 3 Let $\mathcal{M} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{0\}$ and let the function $d : \mathcal{M} \times \mathcal{M} \to \Re^+$ be defined by

$$d(x, y) = |x - y|^2.$$

Define the mappings $\mathcal{T}:\mathcal{M}\to\mathcal{M}$ and $\beta:\Re^+\to[0,\frac{1}{\nu})$, respectively, by

$$\mathcal{T}x=\begin{cases}0, & \text{if } x=0,\\ \frac{1}{a^2+1}, & \text{if } x=\frac{1}{a}, a\in\mathbb{N},\end{cases}\quad\text{and}\quad \beta(\zeta)=\begin{cases}\frac{1}{\zeta+2}, & \text{if } \zeta\in\Gamma,\\ 0, & \text{otherwise},\end{cases}$$

where $\Gamma=\{\frac{\kappa}{\omega}|\text{ for some }\kappa,\omega\in\mathbb{N},\kappa\text{ is odd},\gcd(\kappa,\omega)=1\}$.

We show that all the conditions of Theorem 6 are fulfilled to conclude the existence of a unique fixed point of $\mathcal{T}$, whereas condition (17) in Corollary 1 is not with the $\beta(\zeta)$.

Proof Clearly, d is a b-metric with $\nu=2$ and hence $(\mathcal{M},d,\nu)$ is a complete b-metric space. Also, $\beta\in\mathcal{F}$.

To show that the conditions in Theorem 6 are satisfied, it is sufficient to prove that (1) holds with β.

In case of $x=y$ for any $x,y\in\mathcal{M}$, from (1) we have $d(\mathcal{T}x,\mathcal{T}y)=0\le L(x,y)$, so that all the conditions in Theorem 6 are satisfied.

Thus, we suppose that $x\neq y$. Without loss of generality, assume that $x>y$. Now, we consider the following two cases:

Case 1. Suppose $y=0$. Let $x=\frac{1}{a}$, for some $a\in\mathbb{N}$. Then,

$$\begin{aligned}d(\mathcal{T}x,\mathcal{T}y)=|\mathcal{T}x-\mathcal{T}y|^2&=\frac{1}{(a^2+1)^2}=\frac{1}{a^4+2a^2+1}\\ &\le\frac{1}{2a^2+1}=\frac{1}{(2+\frac{1}{a^2})}\frac{1}{a^2}=\beta\left(\frac{1}{a^2}\right)\frac{1}{a^2}\\ &=\beta\big(d(x,y)\big)d(x,y)\le\mathcal{L}(x,y).\end{aligned}$$

Case 2. Suppose $x=\frac{1}{a}$ and $y=\frac{1}{b}$ for some $a,b\in\mathbb{N}$. Since $x>y$, then $b>a$.

Here we consider the following two subcases:

Case 2.1. Suppose a is odd and b is even, or vice versa. In this case, $b-a$ is odd. Then,

$$d(x,y)=\left|\frac{1}{a}-\frac{1}{b}\right|^2=\frac{(b-a)^2}{a^2b^2},$$

and since $(b-a)^2$ is odd, we have

$$\beta\big(d(x,y)\big)=\beta\left(\frac{(b-a)^2}{b^2a^2}\right)=\frac{1}{\frac{(b-a)^2}{b^2a^2}+2}=\frac{b^2a^2}{2b^2a^2+b^2-2ab+a^2}.$$

Now, using the values of $d(x,y)$ and $\beta(d(x,y))$ obtained above, we have

$$\begin{aligned}d(\mathcal{T}x,\mathcal{T}y)=\left|\frac{1}{a^2+1}-\frac{1}{b^2+1}\right|^2&=\frac{(b^2-a^2)^2}{((a^2+1)(b^2+1))^2}\\ &\le\frac{(b-a)^2(b+a)^2}{(b+a)^2(2b^2a^2+b^2+a^2-2ab)}\\ &=\frac{(b-a)^2}{2b^2a^2+b^2+a^2-2ab}\\ &=\frac{b^2a^2}{2b^2a^2+b^2+a^2-2ab}\left(\frac{b-a}{ab}\right)^2\end{aligned}$$

$$
\begin{aligned}
&= \beta\left(\frac{(b-a)^2}{a^2b^2}\right)\frac{(b-a)^2}{a^2b^2} \\
&= \beta\big(d(x,y)\big)d(x,y) \le \mathcal{L}(x,y).
\end{aligned}
$$

Case 2.2. Suppose both a and b are even (or odd). In both cases, b^2+1-a is odd. Then,

$$
d(x,\mathcal{T}y) = \left|\frac{1}{a} - \frac{1}{b^2+1}\right|^2 = \left(\frac{b^2+1-a}{a(b^2+1)}\right)^2,
$$

and

$$
\begin{aligned}
\beta\big(d(x,\mathcal{T}y)\big) &= \beta\left(\left(\frac{b^2+1-a}{a(b^2+1)}\right)^2\right) = \frac{1}{(\frac{b^2+1-a}{a(b^2+1)})^2+2} \\
&= \frac{1}{2+\frac{1}{a^2}-\frac{2}{a(b^2+1)}+\frac{1}{(b^2+1)^2}} = \frac{1}{2+\frac{1}{a^2}-\frac{1}{b^2+1}(\frac{2}{a}-\frac{1}{b^2+1})} \\
&> \frac{1}{2+\frac{1}{a^2}} = \beta\left(\frac{1}{a^2}\right).
\end{aligned}
$$

Observe that, since $b>a$, we have $(b^2-a^2)^2 \le (b^2+1-a)^2$ and $(a^2+1)^2 \ge 2a^2+1$.

Thus,

$$
\begin{aligned}
d(\mathcal{T}x,\mathcal{T}y) &= \left|\frac{1}{a^2+1} - \frac{1}{b^2+1}\right|^2 = \frac{(b^2-a^2)^2}{(a^2+1)^2(b^2+1)^2} \\
&\le \frac{(b^2+1-a)^2}{(a^2+1)^2(b^2+1)^2} \le \left(\frac{1}{2a^2+1}\right)\left(\frac{b^2+1-a}{b^2+1}\right)^2 \\
&= \left(\frac{1}{2+\frac{1}{a^2}}\right)\left(\frac{1}{a^2}\right)\left(\frac{b^2+1-a}{b^2+1}\right)^2 \\
&= \left(\frac{1}{2+\frac{1}{a^2}}\right)\left(\frac{b^2+1-a}{a(b^2+1)}\right)^2 = \beta\left(\frac{1}{a^2}\right)\left(\frac{b^2+1-a}{a(b^2+1)}\right)^2 \\
&< \beta\big(d(x,\mathcal{T}y)\big)d(x,\mathcal{T}y) \le \mathcal{L}(x,y).
\end{aligned}
$$

Therefore, from cases 1 and 2 all the conditions of Theorem 6 are satisfied and $\mathcal{T}$ has a unique fixed point in $\mathcal{M}$. Thus, 0 is the only fixed point of $\mathcal{T}$ in $\mathcal{M}$.

To show that condition (17) of Corollary 1 fails with this β, let us take $x = \frac{1}{j+1}$ and $y = \frac{1}{j+2}$, for $j \in \mathbb{N}_0$. Then,

$$
\begin{aligned}
\mathcal{N}(x,y) &= \max\big\{d(x,y), d(x,\mathcal{T}x), d(y,\mathcal{T}y), d(x,\mathcal{T}y), d(y,\mathcal{T}x)\big\} \\
&= \max\left\{\frac{1}{(j+1)^2(j+2)^2}, \frac{(j^2+j+1)^2}{(j+1)^2[((j+1)^2+1)]^2}, \frac{(j^2+3j+3)^2}{(j+2)^2[(j+2)^2+1]^2},\right. \\
&\qquad \left.\frac{(j^2+3j+4)^2}{(j+1)^2[(j+2)^2+1]^2}, \frac{(j^2+j)^2}{(j+2)^2[(j+1)^2+1]^2}\right\} \\
&= \left(\frac{j^2+3j+4}{(j+1)[(j+2)^2+1]}\right)^2 = d(x,\mathcal{T}y).
\end{aligned}
$$

Since $j^2 + 3j + 4$ is even for all $j \in \mathbb{N}_0$, from the definition of β, we obtain

$$\beta\big(\mathcal{N}(x,y)\big) = \beta\big(d(x,\mathcal{T}y)\big) = \beta\left(\left(\frac{j^2+3j+4}{(j+1)[(j+2)^2+1]}\right)^2\right) = 0.$$

Thus,

$$\begin{aligned} d(\mathcal{T}x,\mathcal{T}y) &= \left(\frac{1}{j^2+2j+2} - \frac{1}{j^2+4j+5}\right)^2 = \frac{(2j+3)^2}{(j^2+2j+2)^2(j^2+4j+5)^2} > 0 \\ &= \beta\big(\mathcal{N}(x,y)\big)\mathcal{N}(x,y). \end{aligned}$$

In particular, if $x = \frac{1}{2}$ and $y = \frac{1}{3}$, then

$$\begin{aligned} \mathcal{N}(x,y) &= \max\big\{d(x,y), d(x,\mathcal{T}x), d(y,\mathcal{T}y), d(x,\mathcal{T}y), d(y,\mathcal{T}x)\big\} \\ &= \max\left\{\frac{1}{36}, \frac{9}{100}, \frac{49}{900}, \frac{4}{25}, \frac{4}{225}\right\} = \frac{4}{25} = d(x,\mathcal{T}y). \end{aligned}$$

From the definition of β, we obtain

$$\beta\big(\mathcal{N}(x,y)\big) = \beta\big(d(x,Ty)\big) = \beta\left(\frac{4}{25}\right) = 0.$$

Thus,

$$d(\mathcal{T}x,\mathcal{T}y) = \left(\frac{1}{5} - \frac{1}{10}\right)^2 = \frac{1}{100} > 0 = \beta\big(\mathcal{N}(x,y)\big)\mathcal{N}(x,y).$$

Therefore, from this example we can see that condition (17) of Corollary 1 is not satisfied with the $\beta(\zeta)$ and hence Corollary 1 is not applicable to show the existence and uniqueness of the fixed point. □

The following example indicate that our result is an actual generalization of Theorem 4.

Example 4 Let $\mathcal{M} = \{x_1, x_2, x_3, x_4\}$. Define $d : \mathcal{M} \times \mathcal{M} \to \Re^+$ by

$$\begin{cases} d(x,x) = 0 \quad \text{and} \quad d(x,y) = d(y,x) \quad \text{for all } x,y \in \mathcal{M}, \\ d(x_1,x_2) = d(x_1,x_3) = \frac{1}{4}, \qquad d(x_2,x_4) = 1, \\ d(x_1,x_4) = d(x_2,x_3) = d(x_3,x_4) = \frac{1}{2}. \end{cases}$$

Clearly, $(\mathcal{M}, d, \nu)$ is a complete b-metric space with $\nu = \frac{4}{3} > 1$. Define a mapping $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ by

$$\mathcal{T}x_1 = \mathcal{T}x_2 = x_1, \qquad \mathcal{T}x_3 = \mathcal{T}x_4 = x_2,$$

and define a function $\beta : \Re^+ \to [0, \frac{1}{\nu})$ by $\beta(\zeta) = \frac{1}{\nu+\zeta}$ for $\zeta > 0$ and $\beta(0) = 0$.

We show that all the conditions of Theorem 6 are satisfied and conclude that $x = x_1$ is the unique fixed point of $\mathcal{T}$, but the conditions in Theorem 4 are not satisfied.

Proof As $d(x_2,x_4) = 1 > \frac{3}{4} = \frac{1}{4} + \frac{1}{2} = d(x_2,x_1) + d(x_1,x_4)$, the function d is not a metric on $\mathcal{M}$.

The condition in Theorem 4 is fulfilled if, with the $\beta \in \mathcal{F}$, the inequality

$$d(\mathcal{T}x,\mathcal{T}y) \le \beta\bigl(d(x,y)\bigr)d(x,y) \quad \text{for all } x,y \in \mathcal{M}, \tag{18}$$

holds. Since $\beta(\zeta) < \frac{1}{\nu}$ for $\zeta > 0$, we have $d(\mathcal{T}x,\mathcal{T}y) < d(x,y)$ for all $x,y \in \mathcal{M}$. Let us take $x = x_1$ and $y = x_3$. Then, we have

$$d(\mathcal{T}x_1,\mathcal{T}x_3) = \frac{1}{4} > \frac{3}{19} = \beta\bigl(d(x_1,x_3)\bigr)d(x_1,x_3),$$

and this contradicts $d(\mathcal{T}x_1,\mathcal{T}x_3) \le \beta(d(x_1,x_3))d(x_1,x_3)$. So, the inequality in (18) does not hold for all $x_i,x_j \in \mathcal{M}$, and hence Theorem 4 is not applicable to conclude the existence of fixed point of $\mathcal{T}$ with the $\beta \in \mathcal{F}$.

On the other hand, for all $x_i,x_j \in X$, where $i,j = 1,2,3,4$,

$$d(\mathcal{T}x_i,\mathcal{T}x_j) \le \beta\bigl(\mathcal{N}(x_i,x_j)\bigr)\mathcal{N}(x_i,x_j) \le \mathcal{L}(x_i,x_j).$$

To see this, observe that

$$\begin{aligned}
&d(\mathcal{T}x_1,\mathcal{T}x_2) = d(\mathcal{T}x_3,\mathcal{T}x_4) = 0 \quad \text{and}\\
&d(\mathcal{T}x_1,\mathcal{T}x_3) = d(\mathcal{T}x_1,\mathcal{T}x_4) = d(\mathcal{T}x_2,\mathcal{T}x_3) = d(\mathcal{T}x_2,\mathcal{T}x_4) = \frac{1}{4}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&d(\mathcal{T}x_1,\mathcal{T}x_2) = 0 \le \frac{3}{19} = \beta\bigl(\mathcal{N}(x_1,x_2)\bigr)\mathcal{N}(x_1,x_2) \le \frac{3}{11} = \mathcal{L}(x_1,x_2),\\
&d(\mathcal{T}x_1,\mathcal{T}x_3) = \frac{1}{4} \le \frac{3}{11} = \beta\bigl(\mathcal{N}(x_1,x_3)\bigr)\mathcal{N}(x_1,x_3) = \mathcal{L}(x_1,x_3),\\
&d(\mathcal{T}x_1,\mathcal{T}x_4) = \frac{1}{4} \le \frac{3}{7} = \beta\bigl(\mathcal{N}(x_1,x_4)\bigr)\mathcal{N}(x_1,x_4) = \mathcal{L}(x_1,x_4),\\
&d(\mathcal{T}x_2,\mathcal{T}x_3) = \frac{1}{4} \le \frac{3}{11} = \beta\bigl(\mathcal{N}(x_2,x_3)\bigr)\mathcal{N}(x_2,x_3) = \mathcal{L}(x_2,x_3),\\
&d(\mathcal{T}x_2,\mathcal{T}x_4) = \frac{1}{4} \le \frac{3}{7} = \beta\bigl(\mathcal{N}(x_2,x_4)\bigr)\mathcal{N}(x_2,x_4) = \mathcal{L}(x_2,x_4),\\
&d(\mathcal{T}x_3,\mathcal{T}x_4) = 0 \le \frac{3}{7} = \beta\bigl(\mathcal{N}(x_3,x_4)\bigr)\mathcal{N}(x_3,x_4) = \mathcal{L}(x_3,x_4).
\end{aligned}$$

Therefore, all the conditions of Theorem 6 and Corollary 1 are satisfied with the β defined above and x_1 is the only fixed point of $\mathcal{T}$. □

3 Applications to nonlinear integral equations

In this section, we discuss an existence result for the solution to a nonlinear integral equation using Theorem 6. We developed this application inspired by [17].

Let $\mathcal{M} = C[\alpha, \gamma]$ be the set of all continuous real-valued functions defined on $[\alpha, \gamma]$, where $0 \leq \alpha < \gamma$. Let $d : \mathcal{M} \times \mathcal{M} \to \Re^+$ be defined by

$$d(x,y) = \max_{\alpha \leq t \leq \gamma} \left| x(t) - y(t) \right|^2, \quad \text{for all } x, y \in \mathcal{M}. \tag{19}$$

Clearly, $(\mathcal{M}, d, \nu)$ is a complete b-metric space with $\nu = 2$.

Our aim is to find a function $x(t) \in \mathcal{M}$, $t \in [\alpha, \gamma]$, such that for $f : [\alpha, \gamma] \to \Re$, $g : [\alpha, \gamma] \times [\alpha, \gamma] \to \Re$ and $\mathcal{A} : [\alpha, \gamma] \times [\alpha, \gamma] \times \Re \to \Re$, it satisfies the nonlinear integral equation

$$x(t) = f(t) + \int_\alpha^\gamma g(t,\tau)\mathcal{A}\big(t,\tau,x(\tau)\big)\,d\tau. \tag{20}$$

Theorem 7 *The nonlinear integral equation* (20) *has a unique solution in* $\mathcal{M}$ *provided that the following hypotheses hold*:

(i) *the functions* $f : [\alpha, \gamma] \to \Re$, $g : [\alpha, \gamma] \times [\alpha, \gamma] \to \Re$ *and* $\mathcal{A} : [\alpha, \gamma] \times [\alpha, \gamma] \times \Re \to \Re$ *are continuous on* $[\alpha, \gamma]$, $[\alpha, \gamma]^2$, *and* $[\alpha, \gamma]^2 \times \Re$, *respectively.*

(ii) *for all* $t, \tau \in [\alpha, \gamma]$ *and for all* $x, y \in \mathcal{M}$, *there exists* $\phi : \mathcal{M} \times \mathcal{M} \to \Re^+$ *such that*

$$\left|\mathcal{A}\big(t,\tau,x(\tau)\big) - \mathcal{A}\big(t,\tau,y(\tau)\big)\right| \leq \phi(x,y)\sqrt{\ln\Big(1 + \max_{\alpha \leq \tau \leq \gamma}\left|x(\tau) - y(\tau)\right|^2\Big)}.$$

(iii) *for all* $t, \tau \in [\alpha, \gamma]$,

$$\max_{\alpha \leq t \leq \gamma} \int_\alpha^\gamma \left|g(t,\tau)\phi(x,y)\right|^2 d\tau \leq \frac{1}{\nu(\gamma - \alpha)}.$$

Proof Define a mapping $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ by

$$\mathcal{T}x(t) = f(t) + \int_\alpha^\gamma g(t,\tau)\mathcal{A}\big(t,\tau,x(\tau)\big)\,d\tau. \tag{21}$$

The existence of a unique solution of the nonlinear integral equation (20) is equivalent to the existence of a fixed point of $\mathcal{T}$ in (21).

Now, we prove that $\mathcal{T}$ is a Geraghty–Ćirić-type contraction mapping. From conditions (ii) and (iii), we have

$$\begin{aligned}
&\left|\mathcal{T}x(t) - \mathcal{T}y(t)\right|^2 \\
&\quad = \left(f(t) + \int_\alpha^\gamma g(t,\tau)\mathcal{A}\big(t,\tau,x(\tau)\big)\,d\tau - f(t) - \int_\alpha^\gamma g(t,\tau)\mathcal{A}\big(t,\tau,y(\tau)\big)\,d\tau\right)^2 \\
&\quad = \left(\int_\alpha^\gamma g(t,\tau)\Big(\mathcal{A}\big(t,\tau,x(\tau)\big) - \mathcal{A}\big(t,\tau,y(\tau)\big)\Big)\,d\tau\right)^2 \\
&\quad \leq \left(\int_\alpha^\gamma \big[g(t,\tau)\big]^2 d\tau\right)\left(\int_\alpha^\gamma \left|\mathcal{A}\big(t,\tau,x(\tau)\big) - \mathcal{A}\big(t,\tau,y(\tau)\big)\right|^2 d\tau\right) \\
&\quad \leq \left(\int_\alpha^\gamma \big[g(t,\tau)\phi(x,y)\big]^2 d\tau\right)\left(\int_\alpha^\gamma \ln\Big(1 + \max_{\alpha \leq \tau \leq \gamma}\left|x(\tau) - y(\tau)\right|^2\Big)\,d\tau\right) \\
&\quad \leq \frac{1}{\nu(\gamma - \alpha)}\left(\ln\Big(1 + \max_{\alpha \leq t \leq \gamma}\left|x(t) - y(t)\right|^2\Big)\right)\int_\alpha^\gamma d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\ln(1 + \max_{\alpha \le t \le \gamma} |x(t) - y(t)|^2)}{2} \\
&= \frac{\ln(1 + \max_{\alpha \le t \le \gamma} |x(t) - y(t)|^2)}{2 \max_{\alpha \le t \le \gamma} |x(t) - y(t)|^2} \max_{\alpha \le t \le \gamma} \bigl|x(t) - y(t)\bigr|^2 \\
&= \frac{\ln(1 + d(x(t), y(t)))}{2d(x(t), y(t))} d\bigl(x(t), y(t)\bigr).
\end{aligned}
$$

Then, by (19),

$$
d(\mathcal{T}x, \mathcal{T}y) = \max_{\alpha \le t \le \gamma} \bigl|\mathcal{T}x(t) - \mathcal{T}y(t)\bigr|^2 \le \max_{\alpha \le t \le \gamma} \left\{ \frac{\ln(1 + d(x(t), y(t)))}{2d(x(t), y(t))} d\bigl(x(t), y(t)\bigr) \right\}.
$$

Define $\beta : \Re^+ \to [0, \frac{1}{2})$ by $\beta(\zeta) = \frac{\ln(1+\zeta)}{2\zeta}$ for $\zeta > 0$ and $\beta(0) \in [0, \frac{1}{2})$. Then, $\beta \in \mathcal{F}$ and

$$
\begin{aligned}
d(\mathcal{T}x, \mathcal{T}y) &\le \max_{\alpha \le t \le \gamma} \left\{ \frac{\ln(1 + d(x(t), y(t)))}{2d(x(t), y(t))} d\bigl(x(t), y(t)\bigr) \right\} \\
&= \max_{\alpha \le t \le \gamma} \bigl\{ \beta\bigl(d\bigl(x(t), y(t)\bigr)\bigr) d\bigl(x(t), y(t)\bigr) \bigr\} \\
&\le \max_{\alpha \le t \le \gamma} \bigl\{ \beta\bigl(d\bigl(x(t), y(t)\bigr)\bigr) d\bigl(x(t), y(t)\bigr), \beta\bigl(d\bigl(x(t), \mathcal{T}x(t)\bigr)\bigr) d\bigl(x(t), \mathcal{T}x(t)\bigr), \\
&\quad \beta\bigl(d\bigl(y(t), \mathcal{T}y(t)\bigr)\bigr) d\bigl(y(t), \mathcal{T}y(t)\bigr), \beta\bigl(d\bigl(x(t), \mathcal{T}y(t)\bigr)\bigr) d\bigl(x(t), \mathcal{T}y(t)\bigr), \\
&\quad \beta\bigl(d\bigl(y(t), \mathcal{T}x(t)\bigr)\bigr) d\bigl(y(t), \mathcal{T}x(t)\bigr) \bigr\} \\
&= \mathcal{L}\bigl(x(t), y(t)\bigr).
\end{aligned}
$$

Thus, for all $t \in [\alpha, \gamma]$ and for all $x, y \in C[\alpha, \gamma]$, we have $d(\mathcal{T}x, \mathcal{T}y) \le \mathcal{L}(x, y)$. Hence, $\mathcal{T}$ is a Geraghty–Ćirić-type contraction mapping with $\beta(\zeta) = \frac{\ln(1+\zeta)}{2\zeta}$ for $\zeta > 0$ and $\beta(0) \in [0, \frac{1}{2})$.

Therefore, by Theorem 6, $\mathcal{T}$ has a unique fixed point in $\mathcal{M} = C[\alpha, \gamma]$. Hence, the nonlinear integral equation (20) has a unique solution in $\mathcal{M} = C[\alpha, \gamma]$. □

Example 5 Consider $\mathcal{M} = C[0, 1]$ the space of real-valued continuous functions on $[0, 1]$. Assume that for any $x \in \mathcal{M}$, we have $x(t) > 0$ for all $t \in [0, 1]$. Let $d : C[0, 1] \times C[0, 1] \to \Re^+$ be a b-metric given by

$$
d(x, y) = \max_{0 \le t \le 1} \bigl|x(t) - y(t)\bigr|^2,
$$

for all $x, y \in C[0, 1]$. Consider a nonlinear integral equation

$$
x(t) = \frac{1}{2}t - \int_0^1 t\tau \frac{e^{-t} x(\tau)}{1 + x(\tau)} \, d\tau, \tag{22}
$$

for $x \in C[0, 1]$. We show that, by using Theorem 6, the nonlinear integral equation (22) has a unique solution.

Proof We can easily verify that $(\mathcal{M}, d, \nu)$ is a b-metric space with $\nu = 2$. Define $\mathcal{T} : C[0, 1] \to C[0, 1]$ by

$$
\mathcal{T}x(t) = \frac{1}{2}t - \int_0^1 t\tau \frac{e^{-t} x(\tau)}{1 + x(\tau)} \, d\tau, \tag{23}
$$

for $x \in C[0,1]$. The existence of a unique fixed point of $\mathcal{T}$ in (23) is equivalent to the existence of a unique solution of the nonlinear integral equation (22).

Observe that (22) is a particular case of (20) with $f(t) = \frac{1}{2}t$, $g(t,\tau) = t\tau$ and $\mathcal{A}(t,\tau,x(\tau)) = \frac{e^{-t}x(\tau)}{1+x(\tau)}$. Also, the functions $f(t)$ and $g(t,\tau)$ are continuous on $[0,1]$, and $\mathcal{A}(t,\tau,x(\tau))$ is integrable with respect to τ on $[0,1]$.

For every sequence $\{t_n\} \subset [0,1]$, we have $t \in [0,1]$ such that $\lim_{n\to\infty} t_n = t$. Then, for any $x \in C[0,1]$, we get

$$\begin{aligned}
&\left|\mathcal{T}x(t_n) - \mathcal{T}x(t)\right| \\
&= \left| f(t_n) - f(t) - \int_0^1 \bigl(g(t_n,\tau) - g(t,\tau)\bigr)\bigl(\mathcal{A}\bigl(t_n,\tau,x(\tau)\bigr) - \mathcal{A}\bigl(t,\tau,x(\tau)\bigr)\bigr)\,d\tau \right| \\
&\leq \left|\frac{1}{2}t_n - \frac{1}{2}t\right| + \left|\int_0^1 (t_n - t)\tau\left(e^{-t_n} - e^{-t}\right)\frac{x(\tau)}{1+x(\tau)}\,d\tau\right| \\
&\leq \frac{1}{2}|t_n - t| + |t_n - t|\left|e^{-t_n} - e^{-t}\right| \int_0^1 \tau\left|\frac{x(\tau)}{1+x(\tau)}\right| d\tau \\
&\leq \frac{1}{2}|t_n - t| + \frac{1}{2}|t_n - t|\left|e^{-t_n} - e^{-t}\right|.
\end{aligned}$$

Letting $n \to \infty$, we have $|\mathcal{T}x(t_n) - \mathcal{T}x(t)| \to 0$. That is, $\lim_{n\to\infty} \mathcal{T}x(t_n) = \mathcal{T}x(t)$. Hence, $\mathcal{T}x \in C[0,1]$ for all $x \in C[0,1]$. Then, for all $t,\tau \in [0,1]$ and for all $x,y \in C[0,1]$, we have

$$\begin{aligned}
\left|\mathcal{A}\bigl(t,\tau,x(\tau)\bigr) - \mathcal{A}\bigl(t,\tau,y(\tau)\bigr)\right|^2 &= \left|\frac{e^{-t}x(\tau)}{1+x(\tau)} - \frac{e^{-t}y(\tau)}{1+y(\tau)}\right|^2 = \frac{e^{-2t}(x(\tau)-y(\tau))^2}{(1+x(\tau))^2(1+y(\tau))^2} \\
&\leq \frac{e^{-2t}(x(\tau)-y(\tau))^2}{1+(x(\tau)-y(\tau))^2} \leq \ln\bigl(1 + \left|x(\tau)-y(\tau)\right|^2\bigr) \\
&\leq \ln\Bigl(1 + \max_{0\leq\tau\leq1}\left|x(\tau)-y(\tau)\right|^2\Bigr),
\end{aligned}$$

and

$$\max_{t\in[0,1]} \int_0^1 (\tau t)^2\,d\tau = \max_{t\in[0,1]} t^2 \int_0^1 \tau^2\,d\tau = \frac{1}{3}\max_{t\in[0,1]} t^2 = \frac{1}{3} \leq \frac{1}{2}.$$

Let $\phi(x,y) = 1$ for all $(x,y) \in C[0,1] \times C[0,1]$. Then

$$\begin{aligned}
\left|\mathcal{T}x(t) - \mathcal{T}y(t)\right|^2 &= \left(f(t) - f(t) - t\int_0^1 \tau\left(\frac{e^{-t}x(\tau)}{1+x(\tau)} - \frac{e^{-t}y(\tau)}{1+y(\tau)}\right)d\tau\right)^2 \\
&= \left(\int_0^1 t\tau\frac{e^{-t}(x(\tau)-y(\tau))}{(1+x(\tau))(1+y(\tau))}\,d\tau\right)^2 \\
&\leq \left(\int_0^1 (t\tau)^2\,d\tau\right)\left(\int_0^1 \frac{e^{-2t}(x(\tau)-y(\tau))^2}{1+(x(\tau)-y(\tau))^2}\,d\tau\right) \\
&\leq \frac{1}{3}\ln\Bigl(1 + \max_{0\leq t\leq1}\left|x(t)-y(t)\right|^2\Bigr)\int_0^1 d\tau \\
&\leq \frac{1}{2}\ln\Bigl(1 + \max_{0\leq t\leq1}\left|x(t)-y(t)\right|^2\Bigr)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\ln(1 + \max_{0\le t\le 1} |x(t) - y(t)|^2)}{2\max_{0\le t\le 1} |x(t) - y(t)|^2} \max_{0\le t\le 1} \left|x(t) - y(t)\right|^2 \\
&= \frac{\ln(1 + d(x(t), y(t))}{2d(x(t), y(t))} d\big(x(t), y(t)\big).
\end{aligned}
$$

Define $\beta : \Re^+ \to [0, \frac{1}{2})$ by $\beta(\zeta) = \frac{\ln(1+\zeta)}{2\zeta}$ for $\zeta > 0$ and $\beta(0) \in [0, \frac{1}{2})$. Then $\beta \in \mathcal{F}$. Thus,

$$
\begin{aligned}
d\big(\mathcal{T}x(t), \mathcal{T}y(t)\big) &= \max_{0\le t\le 1} \left|\mathcal{T}x(t) - \mathcal{T}y(t)\right|^2 \le \max_{0\le t\le 1} \left\{ \frac{\ln(1 + d(x(t), y(t))}{2d(x(t), y(t))} d\big(x(t), y(t)\big) \right\} \\
&= \max_{0\le t\le 1} \left\{\beta\big(d\big(x(t), y(t)\big)\big) d\big(x(t), y(t)\big)\right\} \\
&\le \max_{0\le t\le 1} \left\{\beta\big(d\big(x(t), y(t)\big)\big) d\big(x(t), y(t)\big), \beta\big(d\big(x(t), \mathcal{T}x(t)\big)\big) d\big(x(t), \mathcal{T}x(t)\big),\right. \\
&\quad \beta\big(d\big(y(t), \mathcal{T}y(t)\big)\big) d\big(y(t), \mathcal{T}y(t)\big), \beta\big(d\big(x(t), \mathcal{T}y(t)\big)\big) d\big(x(t), \mathcal{T}y(t)\big), \\
&\quad \left.\beta\big(d\big(y(t), \mathcal{T}x(t)\big)\big) d\big(y(t), \mathcal{T}x(t)\big)\right\} \\
&\le \mathcal{L}\big(x(t), y(t)\big).
\end{aligned}
$$

Thus, for all $t \in [0, 1]$, we have $d(\mathcal{T}x, \mathcal{T}y) \le \mathcal{L}(x, y)$. Therefore, by Theorem 6, we see that $\mathcal{T}$ has a unique fixed point in $\mathcal{M} = C[0, 1]$. Hence, the nonlinear integral equation (22) has unique solution in $\mathcal{M} = C[0, 1]$. □

4 Conclusion

In this manuscript, we introduced a new fixed point theorem for a self-mapping satisfying Geraghty–Ćirić-type contraction conditions in the framework of b-metric spaces. We established the existence and uniqueness of fixed points for such mappings. The presented main theorem unifies and generalizes some fixed point results in the related literature. We have provided an example to demonstrate the superiority of our results compared to some corresponding fixed point results. In addition, we presented the applicability of our primary finding to show the existence of a unique solution to a nonlinear integral equation.

Author contributions
A.G. and K.K. involved in conceptualization, methodology. K.K. contributed in supervision and editing the manuscript. A.G. and H.E. contributed in formal analysis. H.E. contributed in revising and proof reading of the manuscript. All authors contributed equally. All authors read and approved the final manuscript.

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