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RESEARCH

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Demiclosed principle and some fixed-point theorems for generalized nonexpansive mappings in Banach spaces

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Abstract
The aim of this paper is to discuss some results concerning the demiclosedness principle of generalized, nonexpansive mappings in uniformly convex spaces. Further, we present some new fixed-point theorems for generalized nonexpansive mappings in different settings of Banach spaces.

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1 Introduction

Let $(\mathcal{M}, \|.\|)$ be a Banach space and $\mathcal{C}$ a nonempty subset of $\mathcal{M}$. A mapping $S : \mathcal{C} \to \mathcal{C}$ is said to be nonexpansive if

$$\left\|S(u) - S(v)\right\| \leq \|u - v\|, \quad \forall u, v \in \mathcal{C}.$$

A point $u^\dagger \in \mathcal{C}$ is said to be a fixed point of S if $S(u^\dagger) = u^\dagger$. In the context of Banach spaces, a nonexpansive mapping may not necessarily possess a fixed point. However, it is possible to obtain fixed points for such mappings by enriching the space with certain geometric properties. In 1965, Browder [5] and Göhde [15] separately established that nonexpansive mappings have fixed points in every uniformly convex Banach space. Kirk [20], on the other hand, extended the fixed-point theorem for nonexpansive mappings to the broader category of reflexive Banach spaces with normal structure. Recall that a Banach space $(\mathcal{M}, \|.\|)$ is said to have normal structure, if for each bounded, closed, and convex subset $\mathcal{C}$ of $\mathcal{M}$ consisting of more than one point there is a point $u \in \mathcal{C}$ such that

$$\sup\left\{\|v - u\| : v \in \mathcal{C}\right\} < \operatorname{diam}_{\|.\|}(\mathcal{C}) = \sup\left\{\|v - u\| : u, v \in \mathcal{C}\right\}.$$

In [3], Baillon and Schöneberg weakened the concept of normal structure and introduced the asymptotic normal structure as follows: A Banach space $(\mathcal{M}, \|.\|)$ is said to have asymptotic normal structure, if for each bounded, closed, and convex subset $\mathcal{C}$ of $\mathcal{M}$ consisting of more than one point and each sequence $\{u_n\}$ in $\mathcal{C}$ satisfying $u_n - u_{n+1} \to 0$ as $n \to \infty$, there is a point $u \in \mathcal{C}$ such that

$$\liminf_{n\to\infty} \|u_n - u\| < \operatorname{diam}_{\|.\|}(\mathcal{C}).$$

The relationship between fixed-point theory and the geometry of Banach spaces has been highly productive and significant. In the context of metric fixed-point problems, geometric properties are particularly influential. Nonexpansive mappings are a prominent area of study in metric fixed-point theory. Many authors have since derived generalizations and extensions of nonexpansive mappings and their associated results. The literature contains a considerable body of research on classes of mappings that are more general than the nonexpansive ones. Some of the notable extensions and generalizations of nonexpansive mappings can be found in [1, 2, 4, 6, 7, 11, 13, 21, 22, 24–26]. Some classes of mappings are not necessarily continuous on their domains, unlike nonexpansive mappings. In 2008, Suzuki [27] introduced a new class of nonexpansive-type mappings, referred to as mappings satisfying condition (C), and derived some significant fixed-point results for them. Suzuki [27] also demonstrated that this class of mappings does not necessarily exhibit continuity, unlike nonexpansive mappings. García-Falset *et al.* [11] explored a broader version of condition (C), called mappings satisfying condition (E). In 2011, Llorens-Fuster and Moreno-Galvez [21] introduced a general class of mappings called (L)-type mappings (or condition (L)), which is contingent on two conditions. First, the existence of an approximate fixed-point sequence (a.f.p.s.) for S in all nonempty, closed, convex, and S-invariant subsets of C. Secondly, the distances between points and their images in the limiting case from the a.f.p.s. For this class of mappings, the nonexpansiveness condition need not hold for all points but only for certain points in the domain. They obtained several fixed-point results for their new class of nonexpansive-type mappings.

It is noted herein that the normal structure condition depends on the distance between all points of set $\mathcal{C}$ and point u, while the asymptotic normal structure condition depends on the limiting distance between sequence $\{u_n\}$ and point u. This condition seems similar to the second condition of (L)-type mappings. It looks natural to investigate the fixed-point theorem for (L)-type mappings in the setting of Banach spaces having asymptotic normal structure. In this paper, we present some results concerning the demiclosedness principle of a mapping satisfying condition (L-1) in uniform convex spaces. Further, we obtain some fixed-point theorems for (L-1)-type mappings in the setting of Banach spaces having asymptotic normal structure. Moreover, we show that in ℓ_1 and J_0 (James space), (L-1)-type self-mapping of a bounded weak* closed convex subset has a fixed point. In this way, results in [11, 18, 21, 27] have been extended, generalized, and complemented.

2 Preliminaries

Definition 1 [12]. Let $\mathcal{C}$ be a nonempty subset of a Banach space $\mathcal{M}$. A sequence $\{u_n\}$ in $\mathcal{C}$ is said to be an approximate fixed-point sequence (in short, a.f.p.s.) for a mapping $S : \mathcal{C} \to \mathcal{C}$ if $\lim_{n\to\infty} \|u_n - S(u_n)\| = 0$.

Definition 2 [12]. Let $\mathcal{C}$ be a subset of a Banach space $\mathcal{M}$. A mapping $G : \mathcal{C} \to \mathcal{C}$ is said to be demiclosed if for any sequence $\{u_n\}$ in $\mathcal{C}$ the following implication holds:

$$\{u_n\} \text{ converges weakly to } u \text{ and } \lim_{n\to\infty} \left\| G(u_n) - w \right\| = 0$$

that implies

$$u \in \mathcal{C} \quad \text{and} \quad G(u) = w.$$

Definition 3 [16]. Let $\mathcal{M}$ be a Banach space and $u, v \in \mathcal{M}$. A vector u is orthogonal to v if $\|u\| \leq \|u + \mu v\|$ for all scalars μ. We use to denote $u \perp v$ if u is orthogonal to v.

In general, the relation $\perp$ is not symmetric cf. [18].

Definition 4 [18]. Let $\mathcal{M}$ be a Banach space. The relation $\perp$ is said to be approximately symmetric if for each $u \in \mathcal{M}$ and $\varepsilon > 0$, there exists a closed, linear subspace $\mathcal{Y} = \mathcal{Y}(u, \varepsilon)$ such that the following two conditions hold:

(i) $\mathcal{Y}$ has finite codimension;

(ii) $\|z\| \leq \|z + \mu u\|$ for all $z \in \mathcal{Y}$, $\|z\| = 1$, and each μ with $\mu \geq \varepsilon$.

Definition 5 [18]. Let $\mathcal{M}$ be a conjugate space, that is, there exists a normed space $\mathcal{Z}$ such that $\mathcal{M} = \mathcal{Z}^*$. The relation $\perp$ is said to be weak* approximately symmetric if conditions (i) and (ii) in Definition 4 hold along with $\mathcal{Y}$ is weak* closed.

Definition 6 [18]. Let $\mathcal{M}$ be a Banach space. The relation $\perp$ is said to be uniformly approximately symmetric (uniformly weak* approximately symmetric) if it is approximately symmetric (weak* approximately symmetric) and condition (ii) in Definition 4 is replaced by the following stronger condition:

(iii) there exists $\delta = \delta(u, \varepsilon) > 0$ such that $\|z\| \leq \|z + \mu u\| - \delta$, for all $z \in Y$, $\|z\| = 1$, and each μ with $\mu \geq \varepsilon$.

In the spaces ℓ_p, $p \in (1, \infty)$, the relation $\perp$ is uniformly approximately symmetric. In spaces ℓ_1 and J_0 (James space [17]) the relation $\perp$ is uniformly weak* approximately symmetric. However, in both spaces L_p, $p \neq 2$ and c_0, the relation $\perp$ fails to be uniformly approximately symmetric.

Lemma 1 (Goebel–Karlovitz) [14]. *Let $\mathcal{C}$ be a subset of a reflexive Banach space $\mathcal{M}$, and suppose $\mathcal{C}$ is minimally invariant with respect to being nonempty, bounded, closed, convex, and S-invariant for some nonexpansive mapping S. Let $\{x_n\}$ be a sequence in $\mathcal{C}$ that satisfies $\lim_{n\to\infty} \|u_n - S(u_n)\| = 0$. Then, for each $u \in \mathcal{C}$, $\lim_{n\to\infty} \|u_n - u\| = \operatorname{diam}(\mathcal{C})$.*

Theorem 1 [14] *Let $\mathcal{M}$ be a uniformly convex Banach space. Then, for any $d > 0$, $\varepsilon > 0$ and $u, v \in X$ with $\|u\| \leq d$, $\|v\| \leq d$, $\|u - v\| \geq \varepsilon$, there exists a $\delta > 0$ such that*

$$\left\| \frac{1}{2}(u + v) \right\| \leq \left[1 - \delta\left(\frac{\varepsilon}{d}\right) \right] d.$$

Theorem 2 [23]. *Let $\mathcal{M}$ be a Banach space. The following conditions are equivalent*:

(i) *$\mathcal{M}$ is strictly convex;*

(ii) *If $u, v \in \mathcal{M}$ and $\|u + v\| = \|u\| + \|v\|$, then $u = 0$ or $v = 0$ or $v = cu$ for some $c > 0$.*

Theorem 3 [3]. *Let $\beta \geq 1$ and let $\mathcal{M}_\beta$ be the real space ℓ_2 renormed according to*

$$|u|_\beta = \max\left\{ \|u\|_2, \beta \|u\|_\infty \right\},$$

where $\|u\|_\infty$ denotes the ℓ_∞-norm and $\|u\|_2$ the ℓ_2 norm. Then,

(1) *$\mathcal{M}_\beta$ has normal structure if and only if $\beta < \sqrt{2}$; and*
(2) *$\mathcal{M}_\beta$ has asymptotic normal structure if and only if $\beta < 2$.*

Lemma 2 [3]. *Let $\beta \geq 1$, $x, y, z \in \mathcal{M}_\beta$ and $\alpha \in [0,1]$. Then,*

$$\left\|x - \big((1-\alpha)y + \alpha z\big)\right\|_2^2 + \alpha(1-\alpha)\|y - z\|_2^2 = (1-\alpha)\|x - y\|_2^2 + \alpha\|x - z\|_2^2.$$

Lemma 3 [3, 9, 10]. *Let $\beta \geq 1$ and $\mathcal{C}$ be a bounded, closed, and convex subset of $\mathcal{M}_\beta$. Let $\{u_n\}$ be a sequence in $\mathcal{C}$. Then, there exists a unique point $w \in \mathcal{C}$ that satisfies the following conditions*:

(i) $\limsup_{n\to\infty} \|u_n - w\|_2^2 + \|w - u\|_2^2 \leq \limsup_{n\to\infty} \|u_n - u\|_2^2$ *for all $u \in \mathcal{C}$; and*
(ii) $2\limsup_{n\to\infty} \|u_n - w\|_2^2 \leq \limsup_{p\to\infty}\{\limsup_{n\to\infty} \|u_n - u_p\|_2^2\}$.

Lemma 4 [3]. *Let $1 \leq \beta \leq 2$ and $\mathcal{C}$ be a bounded, closed, and convex subset of $\mathcal{M}_\beta$ with $d = \operatorname{diam}_{|.|_\beta}(\mathcal{C})$. Let $\{u_n\}$ be a sequence in $\mathcal{C}$ such that $u_n - u_{n+1} \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} |u_n - u|_\beta = d$ for all $u \in \mathcal{C}$, let $w \in \mathcal{C}$ be the $\|.\|_2$-asymptotic-center of $\{u_n\}$ in $\mathcal{C}$. Then,* $\limsup_{n\to\infty} \|u_n - w\|_2^2 \geq 2(\frac{d}{\beta})^2$.

Lemma 5 [3]. *Let $\mathcal{C}$ be a bounded, closed, and convex subset of $\mathcal{M}_2$ and let $\{v_n\}$ be a sequence in $\mathcal{C}$ such that $\lim_{n\to\infty} |v_n - u|_2 = d = \operatorname{diam}_{|.|_\beta}(\mathcal{C})$ for all $u \in \mathcal{C}$. Then, $\lim_{n\to\infty} \|v_n - u\|_\infty = \frac{d}{2}$ for all $u \in \mathcal{C}$.*

Lemma 6 [3]. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in ℓ_2. Suppose that $d > 0$, $y_n \rightharpoonup 0$ as $n \to \infty$, $\lim_{n\to\infty} \|y_n\|_\infty = \frac{d}{2}$ and $\lim_{n\to\infty} \|x_n\|_\infty = \frac{d}{2}$, $\|x_n - y_p\| \leq \frac{d}{2}$ for all n, p and*

$$\limsup_{n\to\infty}\left\{\limsup_{p\to\infty} \|x_n + y_p\|_2^2\right\} = d^2.$$

Then,

$$\limsup_{n\to\infty}\left\{\limsup_{p\to\infty} \|x_n + y_p\|_\infty\right\} < d.$$

Let $\{u_n\}$ be a bounded sequence in Banach space $\mathcal{M}$, and $\mathcal{C}$ be a nonempty subset of $\mathcal{M}$. The *asymptotic radius* of $\{u_n\}$ at a point x in $\mathcal{M}$ is defined by

$$r\big(x, \{u_n\}\big) := \limsup_{n\to\infty} \|u_n - x\|.$$

The asymptotic radius of $\{u_n\}$ with respect to $\mathcal{C}$ is defined by

$$r\big(\mathcal{C}, \{u_n\}\big) := \inf\big\{r\big(x, \{u_n\}\big) : x \in \mathcal{C}\big\}.$$

The asymptotic center of $\{u_n\}$ with respect to $\mathcal{C}$ is defined as

$$A\big(\mathcal{C}, \{u_n\}\big) := \{x \in \mathcal{C} : r\big(x, \{u_n\}\big) = r\big(\mathcal{C}, \{u_n\}\big).$$

Definition 7 [27]. Let $\mathcal{M}$ be a Banach space and $\mathcal{C}$ a nonempty subset of $\mathcal{M}$. A mapping $S:\mathcal{C}\to\mathcal{C}$ is said to satisfy condition (C) if

$$\frac{1}{2}\|u-S(u)\|\leq\|u-v\| \quad \text{implies } \|S(u)-S(v)\|\leq\|u-v\| \quad \forall u,v\in\mathcal{C}.$$

Definition 8 [11]. Let $\mathcal{C}$ be a nonempty subset of a Banach space $\mathcal{M}$. A mapping $S:\mathcal{C}\to\mathcal{C}$ is said to fulfill condition (E_μ) if there exists $\mu\geq 1$ such that

$$\|u-S(v)\|\leq\mu\|u-S(u)\|+\|u-v\| \quad \forall u,v\in\mathcal{C}.$$

We say that S satisfies condition (E) if it satisfies (E_μ) for some $\mu\geq 1$.

3 Class of mappings satisfying condition (L)

Llorens-Fuster and Moreno-Gálvez [21] introduced the following class of nonlinear mappings:

Definition 9 Let $\mathcal{C}$ be a nonempty subset of a Banach space $(\mathcal{M},\|.\|)$. We say that a mapping $S:\mathcal{C}\to\mathcal{C}$ satisfies condition (L), (or it is an (L)-type mapping), if the following two conditions hold:

(1) If a set $\mathcal{D}\subset\mathcal{C}$ is nonempty, closed, convex, and S-invariant, (i.e., $S(\mathcal{D})\subset\mathcal{D}$), then there exists an a.f.p.s. for S in $\mathcal{D}$.

(2) For any a.f.p.s. $\{u_n\}$ of S in $\mathcal{C}$ and each $u\in\mathcal{C}$

$$\limsup_{n\to\infty}\|u_n-S(u)\|\leq\limsup_{n\to\infty}\|u_n-u\|.$$

In [21] it is shown that the above two conditions in the definition of (L)-type mappings are independent in nature.

It is proved in [21] that the class of (L)-type mappings contains strictly the following classes:

(A) nonexpansive mappings;

(B) Suzuki generalized nonexpansive mappings (cf. [27]);

(C) generalized nonexpansve in many cases, see [21];

(D) The class of mappings satisfying condition (E) that in turn satisfy condition (1) in the Definition 9 (cf. [11]).

Now, we consider a subclass of class of (L)-type mappings.

Definition 10 Let $\mathcal{C}$ be a nonempty subset of a Banach space $(\mathcal{M},\|.\|)$ and a mapping $S:\mathcal{C}\to\mathcal{C}$ satisfies condition (L-1), (or it is an (L-1)-type mapping), if the following two conditions hold:

(1) If a set $\mathcal{D}\subset\mathcal{C}$ is nonempty, closed, convex, and S-invariant, (i.e., $S(\mathcal{D})\subset\mathcal{D}$), then there exists an a.f.p.s. for S in $\mathcal{D}$.

(2) For any a.f.p.s. $\{u_n\}$ of S in $\mathcal{C}$, there exists a sequence $\{c_n\}$ in $[0,\infty)$ such that $c_n\to 0$ as $n\to\infty$ and each $u\in\mathcal{C}$, we have

$$\|u_n-S(u)\|\leq\|u_n-u\|+c_n. \tag{3.1}$$

Example 1 Let $(\ell^2,\|\cdot\|_2)$ be the Banach space of square-summable sequences endowed with its standard norm. Assume that $B[0_\mathcal{M},1]$ is a unit ball centered at $0_\mathcal{M}$ (zero element).

Suppose that $S : B[0_{\mathcal{M}}, 1] \to B[0_{\mathcal{M}}, 1]$ is the mapping given by the following definition:

$$S(u) = \begin{cases} \frac{1}{2}\frac{u}{\|u\|} & u \in B[0_{\mathcal{M}}, 1] \backslash B[0_{\mathcal{M}}, \frac{1}{2}], \\ 0_{\mathcal{M}} & u \in B[0_{\mathcal{M}}, \frac{1}{2}]. \end{cases}$$

In fact, the unique fixed point of S is $0_{\mathcal{M}}$. We can have a.f.p.s. $\{u_n\}$ given by $u_n \equiv 0$. Suppose $\{c_n\} = \{\frac{1}{n}\}$ in $[0, \infty)$, then $c_n \to 0$ as $n \to \infty$. Then, if $u \in B[0_{\mathcal{M}}, \frac{1}{2}]$

$$\left\|u_n - S(u)\right\| = \left\|0_{\mathcal{M}} - S(u)\right\| \le \|0_{\mathcal{M}} - u\| \le \|u_n - u\| + c_n.$$

Again, if $u \in B[0_{\mathcal{M}}, 1] \backslash B[0_{\mathcal{M}}, \frac{1}{2}]$

$$\left\|0_{\mathcal{M}} - S(u)\right\| = \left\|\frac{1}{2}\frac{u}{\|u\|}\right\| = \frac{1}{2} \le \|u\| = \|0_{\mathcal{M}} - u\| \le \|u_n - u\| + c_n.$$

On the other hand, $u \in B[0_{\mathcal{M}}, 1]$ with $\|u\| = \frac{1}{2}$ and $v := \frac{3}{2}u$, The mapping S is not nonexpansive.

Proposition 1 *Let $S : \mathcal{C} \to \mathcal{C}$ be a mapping satisfying condition* (L-1), *then S is a mapping satisfying condition* (L).

Proof The first conditions in both mappings are the same. Hence, we only compare the second conditions. Since mapping S is a mapping satisfying condition (L-1), then for any a.f.p.s. $\{u_n\}$ of S in $\mathcal{C}$, there exists a sequence $\{c_n\}$ in $[0, \infty)$ such that $c_n \to 0$ as $n \to \infty$ and each $u \in \mathcal{C}$

$$\left\|u_n - S(u)\right\| \le \|u_n - u\| + c_n.$$

Taking lim sup on both sides, we obtain the desired result. □

In the next theorem, we present the structure of the fixed-point set of class of (L-1)-type mappings.

Theorem 4 *Let $\mathcal{C}$ be a nonempty, closed subset of a Banach space $\mathcal{M}$ and $S : \mathcal{C} \to \mathcal{C}$ a mapping satisfying condition* (L-1) *with $F(S) \neq \emptyset$. Then, the following implications hold*:

(i) *$F(S)$ is closed in $\mathcal{C}$*;
(ii) *If $\mathcal{C}$ is convex and $\mathcal{M}$ is strictly convex then $F(S)$ is convex.*

Proof

(i) Let $\{w_n\} \subseteq F(S)$ such that $w_n \to w \in \mathcal{C}$ as $n \to \infty$. Thus, $S(w_n) = w_n$ and $\{w_n\}$ is an a.f.p.s. for S in $\mathcal{C}$. Since S is a (L-1)-type mapping, we have

$$\left\|w_n - S(w)\right\| \le \|w_n - w\| + c_n,$$

making $n \to \infty$, which implies that $S(w) = w$ and $F(S)$ is closed.

(ii) See [8, Theorem 1]. □

4 Demiclosedness principle in uniformly convex spaces

In this section, we present some results concerning the demiclosedness principle of a mapping satisfying condition (L-1).

Lemma 7 *Suppose $\mathcal{C}$ is a bounded convex subset of a uniformly convex Banach space $\mathcal{M}$ and $S : \mathcal{C} \to \mathcal{M}$ is a mapping satisfying condition (L-1). If $\{u_n\}$ and $\{v_n\}$ are approximate fixed-point sequences, then $\{w_n\} = \{\frac{1}{2}(u_n + v_n)\}$ is an approximate fixed-point sequence too.*

Proof Suppose the assertion of the lemma is false. Then, there exist sequences $\{u_n\}$ and $\{v_n\}$ satisfying $\lim_{n\to\infty} \|u_n - S(u_n)\| = 0$ and $\lim_{n\to\infty} \|v_n - S(v_n)\| = 0$ such that $\|w_n - S(w_n)\| \geq \varepsilon$ for some $\varepsilon > 0$ and every $n \in \mathbb{N}$. We can assume by passing to a subsequence that

$$\lim_{n\to\infty} \|u_n - v_n\| = 2r > 0.$$

It follows that

$$\lim_{n\to\infty} \|u_n - w_n\| = \lim_{n\to\infty} \|v_n - w_n\| = r.$$

By the definition of mapping S, for a.f.p.s. $\{u_n\}$ of S in $\mathcal{C}$, there exists a sequence $\{c_{n,1}\}$ in $[0,\infty)$ such that $c_{n,1} \to 0$ as $n \to \infty$, we have

$$\left\| u_n - S(w_n) \right\| \leq \|u_n - w_n\| + c_{n,1}. \tag{4.1}$$

Similarly,

$$\left\| v_n - S(w_n) \right\| \leq \|v_n - w_n\| + c_{n,2},$$

where $c_{n,2} \to 0$ as $n \to \infty$. Choose $s > 0$ such that $s < \frac{\varepsilon}{r}$. Hence, for sufficiently large n, we have

$$s < \frac{\varepsilon}{c_{n,1} + \|u_n - w_n\|} \tag{4.2}$$

and

$$s < \frac{\varepsilon}{c_{n,2} + \|v_n - w_n\|}.$$

Now,

$$\left\| u_n - \frac{1}{2}\big(w_n + S(w_n)\big) \right\| = \left\| \frac{u_n - S(w_n) + \frac{(u_n - v_n)}{2}}{2} \right\|$$

and it can be seen that

$$\left\| u_n - S(w_n) \right\| \leq \|u_n - w_n\| + c_{n,1}.$$

Now,

$$\|u_n - w_n\| = \left\| u_n - \frac{1}{2}(u_n + v_n) \right\| = \frac{1}{2}\|u_n - v_n\|.$$

Thus,

$$\left\| \frac{(u_n - v_n)}{2} \right\| \leq \|u_n - w_n\| + c_{n,1}$$

and $\|w_n - S(w_n)\| \geq \varepsilon$. By the uniform convexity of $\mathcal{M}$ (see Theorem 1), we have

$$\left\| u_n - \frac{1}{2}\big(w_n + S(w_n)\big) \right\| \leq \left(1 - \delta\left(\frac{\varepsilon}{c_{n,1} + \|u_n - w_n\|}\right)\right)\big(c_{n,1} + \|u_n - w_n\|\big).$$

It is noted that the modulus of convexity, $\delta(\varepsilon)$, is a nondecreasing function of ε, it follows that

$$\left\| u_n - \frac{1}{2}\big(w_n + S(w_n)\big) \right\| \leq \big(1 - \delta(s)\big)\big(c_{n,1} + \|u_n - w_n\|\big). \tag{4.3}$$

Similarly,

$$\begin{aligned}\left\| v_n - \frac{1}{2}\big(w_n + S(w_n)\big) \right\| &\leq \left(1 - \delta\left(\frac{\varepsilon}{c_{n,2} + \|v_n - w_n\|}\right)\right) \\ &\quad \times \big(c_{n,2} + \|v_n - w_n\|\big) \\ &\leq \big(1 - \delta(s)\big)\big(c_{n,2} + \|v_n - w_n\|\big). \end{aligned} \tag{4.4}$$

By the triangle inequality, (4.3), and (4.4), we obtain

$$\begin{aligned}\|u_n - v_n\| &\leq \left\| u_n - \frac{1}{2}(w_n + S(w_n) \right\| + \left\| v_n - \frac{1}{2}(w_n + S(w_n) \right\| \\ &\leq \big(1 - \delta(s)\big)\big\{\big(c_{n,1} + \|u_n - w_n\|\big) + \big(c_{n,2} + \|v_n - w_n\|\big)\big\}.\end{aligned}$$

Letting $n \to \infty$, we obtain $2r \leq 2r(1 - \delta(s))$, a contradiction and this completes the proof. □

Proposition 2 *Suppose $\mathcal{C}$ is a bounded, closed, and convex subset of a uniformly convex space. Let $S : \mathcal{C} \to \mathcal{C}$ be a mapping satisfying condition $(L\text{-}1)$. Then, S has a fixed point.*

Proof See [21, Theorem 4.4]. □

Theorem 5 (Demiclosedness principle). *Suppose $\mathcal{C}$ is a closed, convex subset of a uniformly convex space. Let $S : \mathcal{C} \to \mathcal{C}$ be a mapping satisfying condition $(L\text{-}1)$. Then, the mapping $G = I - S$ is demiclosed on $\mathcal{C}$.*

Proof Let $\{u_n\}$ be a sequence in $\mathcal{C}$ such that $\{u_n\}$ converges weakly to $u^\dagger$ and $\lim_{n\to\infty} \|u_n - S(u_n) - w\| = 0$. Without loss of generality, we assume $w = 0$, as limits are preserved under the translation. Define $\mathcal{C}_n = \overline{\text{conv}}\{u_n, u_{n+1}, \dots\}$, using Proposition 2 on set $\mathcal{C}_n$, there exists $y_n \in \mathcal{C}_n$ such that $S(y_n) = y_n$. Since any weak subsequential limit of y_n lies in $\bigcap_{n=1}^{\infty} \mathcal{C}_n = \{u^\dagger\}$, it implies that y_n converges weakly to $u^\dagger$. Therefore, $u^\dagger$ is in the weak closure of the fixed-point set $F(S)$. Since $\mathcal{M}$ is uniformly convex, $\mathcal{M}$ is both reflexive and strictly convex. From Theorem 4, fixed-point set $F(S)$ is closed and convex, so weakly closed and $u^\dagger \in F(S)$. This completes the proof. □

5 Some fixed-point theorems

In this section, we present some fixed-point results for the class of mappings satisfying condition (L-1).

Theorem 6 *Suppose $\mathcal{C}$ is a closed, convex subset of a uniformly convex space. Let $S : \mathcal{C} \to \mathcal{C}$ be a mapping satisfying condition (L-1). If $\{u_n\}$ is an a.f.p.s. for S such that it converges weakly to $u^\dagger \in \mathcal{C}$, then $u^\dagger$ is a fixed point of S.*

Proof It can be easily seen from Theorem 5 that mapping $I - S$ is demiclosed at 0. From the demiclosedness principle it follows that $u^\dagger$ is a fixed point of S. □

Remark 1 The above theorem should be compared with [21, Theorem 4.6] that asserts the same conclusion in view of the Opial property.

Theorem 7 *Let $\mathcal{C}$ be a nonempty bounded, closed, and convex subset of $\mathcal{M}_2$ and $S : \mathcal{C} \to \mathcal{C}$ a mapping satisfying condition (L-1). Assume the following conditions hold:*

(1) *If $\mathcal{D}$ is minimal with respect to S, and there is an a.f.p.s. $\{u_n\}$ in $\mathcal{D}$, then $u_n - u_{n+1} \to 0$ as $n \to \infty$;*

(2) *If $\mathcal{D}$ is minimal with respect to S, and $\{u_n\}$ is an a.f.p.s. in $\mathcal{D}$, then $|u_n - u|_2 \to d = \operatorname{diam}_{|.|_\beta}(\mathcal{C})$ for all $u \in \mathcal{D}$.*

Then, S has a fixed point.

Proof By the application of Zorn's lemma there is a nonempty, bounded, closed, convex, and S-invariant subset $\mathcal{D}$ of $\mathcal{C}$ with no proper subsets, so $\mathcal{D}$ is minimal with respect to S. Let $d = \operatorname{diam}_{|.|_\beta}(\mathcal{C})$ and assume, for a contradiction, that $d > 0$. Let $\{u_n\}$ be an a.f.p.s. in $\mathcal{D}$ such that $u_n - u_{n+1} \to 0$ as $n \to \infty$. Let $w \in \mathcal{D}$ denote the $\|.\|_2$-asymptotic center of $\{u_n\}$ in $\mathcal{D}$. By Lemma 4, we have

$$\limsup_{n\to\infty} \|u_n - w\|_2^2 \geq \frac{d^2}{2}. \tag{5.1}$$

Without loss of generality, we may assume there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $u_{n_k} \rightharpoonup u \in \mathcal{D}$ and

$$\lim_{k\to\infty} \|u_{n_k} - w\|_2^2 = \limsup_{n\to\infty} \|u_n - w\|_2^2.$$

Again, take a subsequence $\{u_{m_k}\}$ of $\{u_n\}$ with $u_{n_k} \rightharpoonup v \in \mathcal{D}$ and

$$\lim_{k\to\infty} \|u_{m_k} - u\|_2^2 = \limsup_{n\to\infty} \|u_n - u\|_2^2.$$

By (5.1) and Lemma 3(i), we obtain

$$d^2 \geq \lim_{k\to\infty} \left\{ \lim_{p\to\infty} \|u_{n_k} - u_{m_p}\|_2^2 \right\}$$

$$
\begin{aligned}
&= \limsup_{n\to\infty} \|u_n - u\|_2^2 + \limsup_{n\to\infty} \|u_n - w\|_2^2 - \|w - u\|_2^2 \\
&\geq 2 \limsup_{n\to\infty} \|u_n - w\|_2^2 \geq d^2.
\end{aligned}
$$

From the above inequalities, we have the following:

$$
\limsup_{n\to\infty} \|u_n - w\|_2^2 = \frac{d^2}{2} \tag{5.2}
$$

and

$$
\lim_{k\to\infty} \left\{ \lim_{p\to\infty} \|u_{n_k} - u_{m_p}\|_2^2 \right\} = d^2. \tag{5.3}
$$

Now, we show that

$$
\limsup_{k\to\infty} \left\{ \limsup_{p\to\infty} \left\| \frac{1}{2}(u_{n_k} + u_{m_p}) - u \right\|_\infty \right\} = \frac{d}{2} \quad \text{for all } u \in \mathcal{C}.
$$

Take $\Gamma_k = u_{n_k}$ and $\Delta_k = u_{m_k}$. From Lemma 2, for $k, p \in \mathbb{N}$, we have

$$
\begin{aligned}
&\left\| S\left(\frac{1}{2}(\Gamma_k + \Delta_p)\right) - \frac{1}{2}(\Gamma_k + \Delta_p) \right\|_2^2 + \frac{1}{4}\|\Gamma_k - \Delta_p\|_2^2 \\
&\quad = \frac{1}{2} \left\| S\left(\frac{1}{2}(\Gamma_k + \Delta_p)\right) - \Gamma_k \right\|_2^2 \frac{1}{2} \left\| S\left(\frac{1}{2}(\Gamma_k + \Delta_p)\right) - \Delta_p \right\|_2^2.
\end{aligned} \tag{5.4}
$$

Now,

$$
\left\| S\left(\frac{1}{2}(\Gamma_k + \Delta_p)\right) - \Gamma_k \right\|_2 \leq \left| S\left(\frac{1}{2}(\Gamma_k + \Delta_p)\right) - \Gamma_k \right|_2.
$$

Since $\{\Gamma_k\}$ is a.f.p.s for S, from the definition of condition (L-1), we have

$$
\left| S\left(\frac{1}{2}(\Gamma_k + \Delta_p)\right) - \Gamma_k \right|_2 \leq \left| \left(\frac{1}{2}(\Gamma_k + \Delta_p)\right) - \Gamma_k \right|_2 + c_{n,1},
$$

where $c_{n,1} \to 0$ as $n \to \infty$. From the above inequality, we obtain

$$
\begin{aligned}
\left\| S\left(\frac{1}{2}(\Gamma_k + \Delta_p)\right) - \Gamma_k \right\|_2 &\leq \frac{1}{2}|\Gamma_k - \Delta_p|_2 + c_{n,1} \\
&\leq \frac{d}{2} + c_{n,1}
\end{aligned} \tag{5.5}
$$

and, similarly,

$$
\left\| S\left(\frac{1}{2}(\Gamma_k + \Delta_p)\right) - \Delta_p \right\|_2^2 \leq \frac{d}{2} + c_{n,2}, \tag{5.6}
$$

where $c_{n,2} \to 0$ as $n \to \infty$. Using (5.5) and (5.6) in (5.4), we have

$$
\left\| S\left(\frac{1}{2}(\Gamma_k + \Delta_p)\right) - \frac{1}{2}(\Gamma_k + \Delta_p) \right\|_2^2 + \frac{1}{4}\|u_k - v_p\|_2^2 \leq \frac{1}{2}\left(\frac{d}{2} + c_{n,1}\right)^2 + \frac{1}{2}\left(\frac{d}{2} + c_{n,2}\right)^2.
$$

Since $\lim_{k\to\infty}\{\lim_{p\to\infty}\|\Gamma_k-\Delta_p\|_2^2\}=d^2$, from the above inequality, we obtain

$$\limsup_{k\to\infty}\left\{\limsup_{p\to\infty}\left\|S\left(\frac{1}{2}(\Gamma_k+\Delta_p)\right)-\frac{1}{2}(\Gamma_k+\Delta_p)\right\|_2^2\right\}=0.$$

Thus,

$$\limsup_{k\to\infty}\left\{\limsup_{p\to\infty}\left\|S\left(\frac{1}{2}(\Gamma_k+\Delta_p)\right)-\frac{1}{2}(\Gamma_k+\Delta_p)\right\|_\infty\right\}=0. \tag{5.7}$$

Assume there exists $u\in\mathcal{C}$ such that

$$\limsup_{k\to\infty}\left\{\limsup_{p\to\infty}\left\|\frac{1}{2}(\Gamma_k+\Delta_p)-u\right\|_\infty\right\}<\Omega<\frac{d}{2}. \tag{5.8}$$

From (5.7) and (5.8), we can choose subsequences $\{\Gamma_{k_q}\}$ and $\{\Gamma_{p_q}\}$ such that for $q\in\mathbb{N}$:

$$\left|S(z_q)-z_q\right|_2\le\frac{2}{q}\quad\text{and}\quad\|z_q-u\|_\infty\le\Omega,\quad\text{where } z_q=\frac{1}{2}(\Gamma_{k_q}+\Delta_{p_q}).$$

Therefore, $\lim_{q\to\infty}|S(z_q)-z_q|_2=0$ and $\limsup_{q\to\infty}\|z_q-u\|_\infty\le\Omega<\frac{d}{2}$, which contradicts Lemma 5 by assumption (2). Hence,

$$\limsup_{k\to\infty}\left\{\limsup_{p\to\infty}\left\|\frac{1}{2}(u_{n_k}+u_{m_p})-u\right\|_\infty\right\}=\frac{d}{2}\quad\text{for all } u\in\mathcal{C}. \tag{5.9}$$

In particular, it yields,

$$\limsup_{k\to\infty}\left\{\limsup_{p\to\infty}\left\|\frac{1}{2}(u_{n_k}+u_{m_p})-\frac{1}{2}(u+v)\right\|_2^2\right\}\ge\frac{d^2}{4}.$$

From Lemma 2, it follows that

$$\begin{aligned}\frac{d^2}{4}&\le\limsup_{k\to\infty}\left\{\limsup_{p\to\infty}\left\|\frac{1}{2}(u_{n_k}+u_{m_p})-\frac{1}{2}(u+v)\right\|_2^2\right\}\\&=\limsup_{k\to\infty}\left\{\limsup_{p\to\infty}\left\|\frac{1}{2}(u_{n_k}-u)+\frac{1}{2}(u_{m_p}-v)\right\|_2^2\right\}\\&=\frac{1}{4}\lim_{k\to\infty}\|u_{n_k}-u\|_2^2+\frac{1}{4}\lim_{p\to\infty}\|u_{m_p}-v\|_2^2.\end{aligned} \tag{5.10}$$

Since $u_{n_k}\rightharpoonup u\in\mathcal{D}$ as $k\to\infty$, then for each $k\in\mathbb{N}$,

$$\begin{aligned}\|u_{n_k}-w\|_2^2&=\|u_{n_k}-u+u-w\|_2^2\\&=\|u_{n_k}-u\|_2^2+2\langle u_{n_k}-u,u-w\rangle+\|u-w\|_2^2.\end{aligned}$$

From (5.2), we have

$$\lim_{k\to\infty}\|u_{n_k}-u\|_2^2=\frac{d^2}{2}-\|w-u\|_2^2. \tag{5.11}$$

Similarly,

$$\lim_{p\to\infty} \|u_{m_p} - v\|_2^2 = \frac{d^2}{2} - \|w - v\|_2^2. \tag{5.12}$$

Using (5.11) and (5.12) in (5.10) it follows that

$$\begin{aligned} \frac{d^2}{4} &\le \frac{1}{4}\left(\frac{d^2}{2} - \|w-u\|_2^2\right) + \frac{1}{4}\left(\frac{d^2}{2} - \|w-v\|_2^2\right) \\ &\le \frac{d^2}{4} - \frac{1}{4}\left(\|w-u\|_2^2 + \|w-v\|_2^2\right) \end{aligned}$$

and it proves that $u = v = w$. Take $\sigma_k = u_{n_k} - w$ and $\varrho_k = u_{m_k} - w$. Since $\{u_{m_k}\}$ converges weakly to $v = w$,

$$\varrho_k \rightharpoonup 0 \quad \text{as } k \to \infty. \tag{5.13}$$

Since $|u_{m_k} - w|_2 \to d$ and $|u_{n_k} - w|_2 \to d$ as $k \to \infty$, from Lemma 5, the following hold:

$$\|\varrho_k\|_\infty \to \frac{d}{2} \quad \text{and} \quad \|\sigma_k\|_\infty \to \frac{d}{2} \quad \text{as } k \to \infty. \tag{5.14}$$

By the definition of d, the following condition is satisfied:

$$\text{for each } k, p \in \mathbb{N}, \quad \|\sigma_k - \varrho_k\|_\infty \le \frac{d}{2}. \tag{5.15}$$

From (5.3), we have

$$\begin{aligned} \lim_{k\to\infty}\left\{\lim_{p\to\infty} \|u_{n_k} - u_{m_p}\|_2^2\right\} &= \lim_{k\to\infty}\left\{\lim_{p\to\infty} \left\|(u_{n_k} - w) - (u_{m_p} - w)\right\|_2^2\right\} \\ &= \lim_{k\to\infty}\left\{\lim_{p\to\infty} \|\sigma_k - \varrho_p\|_2^2\right\} = d^2. \end{aligned} \tag{5.16}$$

From (5.9), we obtain $\frac{1}{2}\limsup_{k\to\infty}\{\limsup_{p\to\infty} \|(u_{n_k} - w) + (u_{m_p} - w)\|_\infty\} = \frac{d}{2}$ and it follows that

$$\limsup_{k\to\infty}\left\{\limsup_{p\to\infty} \|\sigma_k + \varrho_p\|_\infty\right\} = d. \tag{5.17}$$

From Lemma 2, for all $k, p \in \mathbb{N}$, we have

$$\|\sigma_k + \varrho_p\|_2^2 = 2\|\sigma_k\|_2^2 + \|\varrho_p\|_2^2 - \|\sigma_k - \varrho_p\|_2^2.$$

In view of (5.13), (5.14), (5.15), (5.16), (5.17), and Lemma 6, this implies

$$\lim_{k\to\infty}\left\{\lim_{p\to\infty} \|\sigma_k + \varrho_p\|_2^2\right\} = d^2,$$

which is impossible. This completes the proof. □

Theorem 8 *Let $\mathcal{C}$ be a nonempty, bounded, closed (resp., weak* closed), and convex subset of a reflexive Banach space (resp., the conjugate of a separable Banach space) $\mathcal{M}$. Let $S : \mathcal{C} \to \mathcal{C}$ be a mapping satisfying condition (L-1). Suppose that the relation $\perp$ is uniformly approximately symmetric (resp., uniformly weak* approximately symmetric) in $\mathcal{M}$, then $F(S) \neq \emptyset$.*

Proof By the application of Zorn's lemma there exists a nonempty, bounded, closed, convex, and S-invariant subset $\mathcal{D}$ of $\mathcal{C}$ with no proper subsets, so $\mathcal{D}$ is minimal with respect to S. Since S satisfies condition (L-1), there exists an a.f.p.s. $\{u_n\}$ for S in $\mathcal{D}$. By the reflexiveness of $\mathcal{M}$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that u_{n_k} converges weakly to $u^\dagger$. After possible extraction of a subsequence, if necessary, we assume that $\lim_{k\to\infty} \|x_{n_k} - x^\dagger\| = \Theta$. Take $v = u^\dagger - S(u^\dagger)$. If $\Theta = 0$ or $v = 0$, then $S(u^\dagger) = u^\dagger$ and the proof is completed. Therefore, we assume that $\Theta > 0$ and $v \neq 0$. Following largely the same argument in [18, Theorem 1] let $\varepsilon = \frac{1}{2\Theta}$. By the assumptions, there exists a closed (resp., weak* closed) linear subspace $\mathcal{Y}$ such that conditions (i) in Definition 4 and (iii) in Definition 6 are satisfied. This implies that there exists a $\delta > 0$ such that

$$|\mu| \leq \|v + \mu u\| - |\mu|\delta \tag{5.18}$$

for every $u \in \mathcal{Y}$, $\|u\| = 1$ and each μ with $|\mu| \leq 2\Theta$. Further, the subspace spanned by $\mathcal{Y}$ and v has a finite-dimensional complement $\mathcal{Z}$. Therefore, for each $k \in \mathbb{N}$, $\sigma_{n_k} \in \mathcal{Y}$ and $\varrho_{n_k} \in \mathcal{Z}$, we have

$$u_{n_k} - u^\dagger = \mu_{n_k} v + \sigma_{n_k} + \varrho_{n_k}. \tag{5.19}$$

Since $\mathcal{Z}$ is a finite-dimensional space and noting the convergence of $u_{n_k} - u^\dagger$, it follows that $\mu_{n_k} \to 0$ and $\|\varrho_{n_k}\| \to 0$ as $k \to \infty$. Thus, $\|\sigma_{n_k}\| \to \Theta$ and for sufficiently large k, $\frac{\|\sigma_{n_k}\|}{(1+\mu_{n_k})} \leq 2\Theta$. From (5.18) and (5.19), we have

$$\begin{aligned}
\|u_{n_k} - S(u^\dagger)\| &= \|u_{n_k} - u^\dagger + u^\dagger - S(u^\dagger)\| = \|(1+\mu_{n_k})y + \sigma_{n_k} + \varrho_{n_k}\| \\
&\geq \|(1+\mu_{n_k})v + \sigma_{n_k}\| - \|\varrho_{n_k}\| \\
&\geq |1+\mu_{n_k}| \left\| v + \left(\frac{\|\sigma_{n_k}\|}{(1+\mu_{n_k})}\right)\frac{\sigma_{n_k}}{\|\sigma_{n_k}\|}\right\| - \|\varrho_{n_k}\| \\
&\geq \|\sigma_{n_k}\|(1+\delta) - \|\varrho_{n_k}\|.
\end{aligned} \tag{5.20}$$

Since the mapping S satisfies condition (L-1), we have

$$\|u_{n_k} - S(u^\dagger)\| \leq \|u_{n_k} - u^\dagger\| + c_k, \tag{5.21}$$

where $c_k \to 0$ as $k \to \infty$. Making $k \to \infty$, $\|u_{n_k} - S(u^\dagger)\| \to \Theta$. From (5.20), noting that $\|u_{n_k}\| \to \Theta$ and $\|v_{n_k}\| \to 0$ as $k \to \infty$ we obtain the following inequality

$$\Theta \geq (1+\delta)\Theta,$$

which is a contradiction. Therefore, $\Theta = 0$, and this completes the proof. □

Corollary 1 *Let $\mathcal{C}$ be a convex, bounded, and weak* closed subset of ℓ_1 or the James space J_0. Let $S:\mathcal{C}\to\mathcal{C}$ be a mapping satisfying condition $(L\text{-}1)$. Then, S has a fixed point in $\mathcal{C}$.*

We conclude the paper by posing the following interesting problem.

Kassay [19] showed that the converse of the above theorem is also true. More precisely, a reflexive Banach space having normal structure can be characterized by the fixed-point property for Jaggi-nonexpansive mappings.

5.1 Problem

Can a reflexive Banach space having asymptotic normal structure be characterized by the fixed-point property for mapping satisfying condition (L-1)?

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