Fixed Point Theory and Algorithms for Sciences and Engineering

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$\mathcal{F}$-Contraction of Hardy–Rogers type in supermetric spaces with applications

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Abstract
This article focuses on studying some fixed-point results via $\mathcal{F}$-contraction of Hardy–Rogers type in the context of supermetric space and ordered supermetric space. We also introduced rational-type *z*-contraction on supermetric space. For authenticity, some illustrative examples and applications have been included.

Keywords: F-contraction; Fixed point; Supermetric space; Hardy–Rogers type; Integral contraction; Ordered supermetric space

1 Introduction

Metric fixed-point theory is one of the most beneficial and appealing areas of nonlinear functional analysis. In light of Banach's remarkable fixed-point theorem [1] [BCP], several findings and publications on the topic have been made during the past decade. Fundamentally, there are two widely recognized perspectives on the advancement of the metric fixed point: one involves weakening or changing the conditions on the contraction mappings, and the other involves variation in the abstract structure. Metric spaces, in many directions, have been met with many generalizations and extensions. These include the quasimetric space, the b-metric space, the symmetric space, the dislocated metric space, the partial metric space, the modular metric space, the cone metric space, the ultrametric space, and a variety of other combinations of these.

Another important generalization of Banach's remarkable contraction was made by Wardowski [2], who proposed a new contraction termed F-contraction via an auxiliary function F, satisfying certain requirements, and also proved a result for a fixed point. In this direction, various scholars have carried out several noteworthy modifications and expansions dealing with the initial discoveries of Wardowski. The readers may refer to [3–10] for more details on F-contraction. In addition, rational contractions are also an important part of fixed-point theory, especially when looking into the existence and uniqueness of mathematical problems. Rational contractions are a powerful notion in fixed-point theory, offering a framework for examining the behavior of mappings and the existence of fixed points. They enable researchers to extend and generalize existing results and provide a flexible tool for numerous mathematical applications see [11, 12].

In the same direction, researchers presented Hardy–Rogers-type (HR-type) results for self-mappings on complete usual and partial metric spaces, notably Cosentino and Vetro in [13] and Huang and coauthors in [14]. In addition, Abbas with coauthors presented results for T-HR-type contraction in partially ordered and partial ordered metric spaces [15]. Likewise, Karapinar in [16] studied contractions of interpolative HR-type in complete metric and partial metric spaces. For more details on HR-type contraction see [17–19]

Similarly, one of the most recent generalizations is the supermetric space [20]. All three axioms of a supermetric space are satisfied by metric spaces, although the triangle inequality is not always true in supermetric spaces. Supermetric spaces can be a useful tool in a variety of applications that include fixed-point theory, functional analysis, and optimization.

In the framework of supermetric spaces, this paper explores the ideas of HR-type contraction and a contraction of rational-type z-contraction. This idea will provide a comprehensive framework for research, and help in examining diverse mathematical processes and structures. In addition, this manuscript will contribute to an understanding of the interplay between supermetric spaces and $\mathcal{F}$-contractions, shedding light on the properties of self-mappings in the context of supermetric spaces. For the validity of the presented results, illustrative examples have been discussed.

2 Preliminaries

For the following, the concepts and notations listed below are necessary. In the next sections we will use an abbreviation (S-Map) in place of self-mappings. First, we recall the basic properties of the supermetric.

Definition 2.1 [21] For a nonempty set $\mho$. Define $m_s : \mho \times \mho \to [0, +\infty)$ is termed as a supermetric if these conditions hold:

1. If $m_s(\upsilon, \omega) = 0$, then $\upsilon = \omega$ for all $\upsilon, \omega \in \mho$.
2. $m_s(\upsilon, \omega) = m_s(\omega, \upsilon)$ for all $\upsilon, \omega \in \mho$.
3. There, we have $s \geq 1$ such that for all $\omega \in \mho$, there exist distinct sequences (υ_η), $(\omega_\eta) \subset \mho$ with $m_s(\upsilon_\eta, \omega_\eta) \to 0$ as η tends to infinity, such that

$$\limsup_{\eta \to +\infty} m_s(\omega_n, \omega) \leq s \limsup_{\eta \to +\infty} m_s(\upsilon_\eta, \omega).$$

Then, we call $(\mho, m_s)$ a supermetric space.

Example 1 [20] Let $\mho = [0, +\infty]$ and define

$$m_s(\upsilon, \omega) = \begin{cases} \frac{\upsilon+\omega}{1+\upsilon+\omega} & \text{if } \upsilon \neq \omega, \upsilon \neq 0, \omega \neq 0, \\ 0 & \text{if } \upsilon = \omega, \\ \max\left(\frac{\upsilon}{2}, \frac{\omega}{2}\right) & \text{otherwise.} \end{cases}$$

Then, we can say $(\mho, m_s)$ forms a supermetric space.

Example 2 [20] Let $\mho = [2, 3]$ and define $m_s(\upsilon, \omega) = \begin{cases} \upsilon\omega & \text{if } \upsilon \neq \omega, \\ 0 & \text{if } \upsilon = \omega. \end{cases}$

Let (υ_η), (ω_η) be two distinct sequences such that $m_s(\upsilon_\eta, \omega_\eta) \to 0$ as $\eta \to +\infty$. As we have the distinct sequences thus, $m_s(\upsilon_\eta, \omega_\eta) = \upsilon_\eta \omega_\eta \to 0$. It can be chosen that $\omega_\eta \to 0$ and $\upsilon_\eta \to t$ as $\eta \to +\infty$, where $t \in \mho$. Furthermore, for any $\omega \in \mho$,

$$\limsup_{\eta\to+\infty} m_s(\omega_\eta, \omega) = \limsup_{\eta\to+\infty} \omega_\eta \omega = 0 \leq \limsup_{\eta\to+\infty} m_s(\upsilon_\eta, \omega) = \limsup_{\eta\to+\infty} \upsilon_\eta \omega = t \cdot \omega,$$

hence, this leads to $(\mho, m_s)$ being a supermetric space.

Example 3 [22] Let $\mho = [0,1]$ with $s = 1$, and define $m_s : \mho \times \mho \to [0, +\infty)$ as follows:

$$m_s(\upsilon, \omega) = \upsilon\omega, \quad \text{for all } \upsilon \neq \omega,\ \upsilon, \omega \in (0,1),$$

$$m_s(\upsilon, \omega) = 0, \quad \text{for all } \upsilon = \omega,\ \upsilon, \omega \in [0,1],$$

$$m_s(0, \omega) = m_s(\omega, 0) = \omega, \quad \text{for all } \omega \in (0,1],$$

$$m_s(1, \omega) = m_s(\omega, 1) = 1 - \frac{\omega}{2}, \quad \text{for all } \omega \in [0,1).$$

Then, m_s defines a supermetric.

Definition 2.2 [20]

1. For a supermetric space $(\mho, m_s)$ a sequence (υ_η) in $\mho$ converges to ω in $\mho$, if and only if $m_s(\upsilon_\eta, \omega)$ tends to 0, as η goes to $+\infty$.
2. For a supermetric space $(\mho, m_s)$ a sequence (υ_η) in $\mho$ can be claimed as a Cauchy sequence in $\mho$, if and only if $\lim_{\eta\to+\infty} \sup m_s(\upsilon_\eta, \upsilon_m) : m > \eta = 0$.
3. A space $(\mho, m_s)$ can be claimed as a complete supermetric space if and only if, every Cauchy sequence in $\mho$ converges.

Here, we recall $\digamma$-contraction.

Definition 2.3 [2] Let $\digamma : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:

- ($\mathfrak{F}1$) For all $\upsilon, \omega \in \mathbb{R}^+$ $\upsilon < \omega$, implies to $\digamma(\upsilon) < \digamma(\omega)$, that it $\digamma$ is strictly increasing.
- ($\mathfrak{F}2$) For every $\{\upsilon_\eta\}_{\eta\in\mathbb{N}}$ a positive term sequence, $\lim_{\eta\to+\infty} \upsilon_\eta = 0$ if and only if $\lim_{n\to+\infty} \digamma(\upsilon_\eta) = -\infty$.
- ($\mathfrak{F}3$) There exists $t \in (0,1)$ such that $\lim_{\upsilon\to 0^+} \upsilon^t \digamma(\upsilon) = 0$.

We will use the notion $\daleth$ for the family of all those functions that met the requirements ($\mathfrak{F}1$) – ($\mathfrak{F}3$).

Definition 2.4 [2] A S-Map $S : \mho \to \mho$ is referred to as an $\digamma$-contraction if for some $\tau > 0$ such that for all $\upsilon, \omega \in \mho$:

$$(m(S\upsilon, S\omega) > 0) \Rightarrow (\tau + \digamma(m(S\upsilon, S\omega)) \leq \digamma(m(\upsilon, \omega))), \tag{1}$$

where $m(\cdot,\cdot)$ is the distance metric.

Also, we recall $\digamma$-contraction of HR type.

Definition 2.5 [13] Suppose $(\mho, m)$ is a metric space. a S-Map S is said to be an $\digamma$-contraction of HR type on $\mho$ if there is $\digamma \in \daleth$ and $\tau \in \mathbb{R}^+$ such that;

$$\begin{aligned} \tau + \digamma(m(S\upsilon, S\omega)) \leq \digamma(&\alpha m(\upsilon,\omega) + \beta m(\upsilon, S\upsilon) + \gamma m(\omega, S\omega) \\ &+ \delta m(\upsilon, S\omega) + Lm(\omega, S\upsilon)), \end{aligned} \tag{2}$$

for all $\upsilon, \omega \in \mho$ with $m(S\upsilon, S\omega) > 0$, where $\alpha + 2\delta + \beta + \gamma = 1$, $L \geq 0$ and $\gamma \neq 1$.

Remark 1 [13] From ($\mathfrak{F}1$) and (1), it can be derived that, in every $\digamma$-contraction S is a contractive mapping, that is, $m(S\upsilon, S\omega) < m(\upsilon, \omega)$, for all $\upsilon, \omega \in \mho$, $S\upsilon \neq S\omega$.

Likewise, ($\mathfrak{F}1$) and (2) yields:

$$m(S\upsilon, S\omega)) < \alpha m(\upsilon,\omega) + \beta m(\upsilon, S\upsilon) + \gamma m(\omega, S\omega) + \delta m(\upsilon, S\omega) + Lm(\omega, S\upsilon), \tag{3}$$

for all $\upsilon, \omega \in \mho$, $S\upsilon \neq S\omega$, where $\alpha + \beta + \gamma + 2\delta = 1$, $L \geq 0$, and $\gamma \neq 1$, which means that every $\digamma$-contraction of HR type, S satsifies the above condition.

Now, we introduce the definition of $\digamma$-contraction of HR-type on a supermetric space.

Definition 2.6 A S-Map $S : \mho \to \mho$ is referred to as an $\digamma$-contraction in the context of supermetric if for some $\tau > 0$ such that for all $\upsilon, \omega \in \mho$:

$$(m_s(S\upsilon, S\omega) > 0) \Rightarrow (\tau + \digamma(m_s(S\upsilon, S\omega)) \leq \digamma(m_s(\upsilon, \omega))), \tag{4}$$

where $m_s(\cdot,\cdot)$ is the distance supermetric.

Definition 2.7 Suppose $(\mho, m_s)$ is a supermetric space. A S-Map S on $\mho$ is said to be an $\digamma$-contraction of HR-type if there exist $\digamma \in \daleth$ and $\tau \in \mathbb{R}^+$ such that;

$$\begin{aligned} \tau + \digamma(m_s(S\upsilon, S\omega)) \leq \digamma(&\alpha m_s(\upsilon,\omega) + \beta m_s(\upsilon, S\upsilon) + \gamma m_s(\omega, S\omega) \\ &+ \delta m_s(\upsilon, S\omega) + Lm_s(\omega, S\upsilon)), \end{aligned} \tag{5}$$

for all $\upsilon, \omega \in \mho$ with $m_s(S\upsilon, S\omega) > 0$, where $\alpha + 2\delta + \beta + \gamma = 1$, $L \geq 0$ and $\gamma \neq 1$.

Example 4 [2] Let $\digamma : \mathbb{R}^+ \to \mathbb{R}$ be defined as $\digamma(\upsilon) = \ln \upsilon$. Certainly, $\digamma$ satisfies ($\mathfrak{F}1$) – ($\mathfrak{F}3$) and ($\mathfrak{F}3$) for any $t \in (0, 1)$.

Remark 2 Every mapping $S : \mho \to \mho$ satisfying (4) is an $\digamma$-contraction with

$$m_s(S\upsilon, S\omega) \leq e^{-\tau} m_s(\upsilon, \omega),$$

for all $\upsilon, \omega \in \mho$, $S\upsilon \neq S\omega$.

Remark 3 From ($\mathfrak{F}1$) and (4), it can be derived that, in every $\digamma$-contraction S is a contractive mapping, that is, $m_s(S\upsilon, S\omega) < m_s(\upsilon, \omega)$, for all $\upsilon, \omega \in \mho$, $S\upsilon \neq S\omega$.

Likewise, ($\mathfrak{F}1$) and (5) yields:

$$m_s(S\upsilon, S\omega)) < \alpha m_s(\upsilon,\omega) + \beta m_s(\upsilon, S\upsilon) + \gamma m_s(\omega, S\omega) + \delta m_s(\upsilon, S\omega) + Lm_s(\omega, S\upsilon), \tag{6}$$

for all $\upsilon, \omega \in \mho$, $S\upsilon \neq S\omega$, where $\alpha + \beta + \gamma + 2\delta = 1$, $L \geq 0$, and $\gamma \neq 1$, which means that for every F-contraction of HR type in the context of supermetric, S satisfies the above condition.

Theorem 2.8 [13] *For a complete metric space $(\mho, m_s)$ and a S-Map S on $\mho$. Assuming there exists $F \in \daleth$ and $\tau > 0$ such that S is a generalized F-contraction of HR-type, that is,*

$$\tau + F(m_s(S\upsilon, S\omega)) \leq F(\alpha m_s(\upsilon, \omega) + \beta m_s(\upsilon, S\upsilon) + \gamma m_s(\omega, S\omega) + \delta m_s(\upsilon, S\omega) + Lm_s(\omega, S\upsilon)),$$

for all $\upsilon, \omega \in \mho$, $S\upsilon \neq S\omega$, where $\alpha + \gamma + \beta + 2\delta = 1$, $\gamma \neq 1$, and $L \geq 0$. Then, S possesses a fixed point. Furthermore, if $\alpha + \delta + L \leq 1$, then S possesses a unique fixed point.

Definition 2.9 [13] For a metric space $(\mho, m)$ and partially ordered $(\mho, \precsim)$ on a nonempty set $\mho$, $(\mho, m, \precsim)$ is termed as an ordered metric space. Now, if either of $\vartheta \precsim \psi$ or $\psi \precsim \vartheta$ holds, then $\vartheta, \psi \in \mho$ will be comparable.

Definition 2.10 [13] For a partially ordered set $(\mho, \precsim)$, a S-Map S is referred to as nondecreasing if $S\vartheta \precsim S\psi$, whenever $\vartheta \precsim \psi$ for all $\vartheta, \psi \in \mho$.

Definition 2.11 [13] For an ordered metric space $(\mho, m, \precsim)$, $\mho$ is termed regular, for a nondecreasing sequence $\{\mu_\eta\}$ in $\mho$ with respect to $\precsim$ if, $\lim_{\eta \to +\infty} \mu_\eta = \mu \in \mho$, then $\mu_\eta \precsim \mu$ for all $\eta \in \mathbb{N} \cup \{0\}$.

Theorem 2.12 [13] *For a self- and nondecreasing mapping S on $\mho$ and an ordered complete metric space $(\mho, m_s, \precsim)$. Assuming $F \in \daleth$ and $\tau > 0$ such that S is a generalized F-contraction of HR-type, that is,*

$$\tau + F(m_s(S\upsilon, S\omega)) \precsim F(\alpha m_s(\upsilon, \omega) + \beta m_s(\upsilon, S\upsilon) + \gamma m_s(\omega, S\omega) + \delta m_s(\upsilon, S\omega) + Lm_s(\omega, S\upsilon)),$$

for all $\upsilon, \omega \in \mho$ comparable, $S\upsilon \neq S\omega$, where $\alpha + \gamma + \beta + 2\delta = 1$, $\gamma \neq 1$, and $L \geq 0$. If these requirements are true:

(C1) *There is $\upsilon \in \mho$ such that $\upsilon \precsim S\upsilon$;*

(C2) *$\mho$ is regular;*

then S will have a fixed point. Furthermore, if $\alpha + L + \delta \leq 1$, then the set of fixed points of S will be well ordered, if and only if S possesses a unique fixed point.

3 Main result

This article focuses on examining fixed-point results using an F – *contraction* of Hardy–Rogers type in the context of supermetric space and ordered supermetric space. The paper contains several examples as well as interesting applications of the obtained theoretical results.

Theorem 3.1 *Let $(\mho, m_s)$ be a complete supermetric space with S a self-map on $\mho$. Assuming there is $F \in \daleth$ and $\tau > 0$ such that S is a generalized F-contraction of HR-type, that* *is,*

$$\tau + F(m_s(S\upsilon, S\omega)) \leq F(\alpha m_s(\upsilon,\omega) + \beta m_s(\upsilon, S\upsilon) + \gamma m_s(\omega, S\omega) + \delta m_s(\upsilon, S\omega) + L m_s(\omega, S\upsilon)), \tag{7}$$

for all $\upsilon, \omega \in \mho$, $S\upsilon \neq S\omega$, where $\alpha + \beta + \gamma = 1$, $\delta = 0$, $\gamma < 1$, and $L \geq 0$. Then, S ensures a fixed point. Furthermore, if $\alpha + \delta + L \leq 1$, then S ensures a unique fixed point.

Proof Supposing υ_0 is an arbitrary point in $\mho$, with $\{\upsilon_n\}$ the Picard sequence and initial point υ, that is, $\upsilon_\eta = S\upsilon_{\eta-1} = S^\eta(\upsilon_0)$.

If $\upsilon_\eta = \upsilon_{\eta-1}$ for some $\eta \in \mathbb{N}$, then υ_η is a fixed point of S.

Now, let $m_\eta = m_s(\upsilon_\eta, \upsilon_{\eta+1})$ for all $\eta \in \mathbb{N} \cup \{0\}$.

Also, $\upsilon_\eta \neq \upsilon_{\eta+1}$, means that $S\upsilon_{\eta-1} \neq S\upsilon_\eta$ for all $\eta \in \mathbb{N}$, utilizing (7) such that $\upsilon = \upsilon_{\eta-1}$ and $\omega = \upsilon_\eta$, it can be obtained that

$$\begin{aligned}
\tau + F(m_\eta) &= \tau + F(m_s(\upsilon_\eta, \upsilon_{\eta+1})) = \tau + F(m_s(S\upsilon_{\eta-1}, S\upsilon_\eta)) \\
&\leq F(\alpha m_s(\upsilon_{\eta-1}, \upsilon_\eta) + \beta m_s(\upsilon_{\eta-1}, S\upsilon_{\eta-1}) + \gamma m_s(\upsilon_\eta, S\upsilon_\eta) + L m_s(\upsilon_\eta, S\upsilon_{\eta-1})) \\
&= F(\alpha m_s(\upsilon_{\eta-1}, \upsilon_\eta) + \beta m_s(\upsilon_{\eta-1}, \upsilon_\eta) + \gamma m_s(\upsilon_\eta, \upsilon_{\eta+1}) + L_s m(\upsilon_\eta, \upsilon_\eta)) \\
&= F(\alpha m_{\eta-1} + \beta m_{\eta-1} + \gamma m_\eta) \\
&\leq F((\alpha + \beta) m_{\eta-1} + \gamma m_\eta.
\end{aligned}$$

Now, since $F \in \daleth$, it is strictly increasing,

$$m_\eta < (\alpha + \beta) m_{\eta-1} + \gamma m_\eta \quad \text{for all } \eta \in \mathbb{N}.$$

Focusing on $\alpha + \beta + \gamma = 1$ with $\gamma \neq 1$, one would have $1 - \gamma > 0$.

$$m_\eta < \frac{\alpha + \beta}{1 - \gamma} m_{\eta-1} = m_\eta, \quad \text{for all } \eta \in \mathbb{N}.$$

As a result,

$$\tau + F(m_\eta) \leq F(m_{\eta-1}) \quad \text{for all } \eta \in \mathbb{N},$$

which means that

$$F(m_\eta) \leq F(m_{\eta-1}) - \tau \leq \cdots \leq F(m_0) - \eta\tau, \quad \text{for all } \eta \in \mathbb{N}.$$

Now, $\lim_{\eta\to+\infty} F(m_\eta) = -\infty$. By ($\mathfrak{F}2$) of 2.3, one can obtain $m_\eta \to 0$ as $\eta \to +\infty$.

This implies that

$$\lim_{\eta\to+\infty} m_s(\upsilon_\eta, \upsilon_{\eta+1}) = 0.$$

Now, suppose that $\kappa, \eta \in \mathbb{N}$ and $\kappa > \eta$. If $\upsilon_\eta = \upsilon_\kappa$, we have

$$S^\kappa(\upsilon_0) = S^\eta(\upsilon_0).$$

Hence,

$$S^{\kappa-\eta}(S^{\eta}(\upsilon_0)) = S^{\eta}(\upsilon_0).$$

Thus, $S^{\eta}(\upsilon_0)$ is the fixed point of $S^{\kappa-\eta}$.

Also,

$$S(S^{\kappa-\eta}(S^{\eta}(\upsilon_0))) = S^{\kappa-\eta}(S(S^{\eta}(\upsilon_0))) = S(S^{\eta}(\upsilon_0)).$$

This means that $S(S^{\eta}(\upsilon_0))$ is the fixed point of $S^{\kappa-\eta}$ as well.

Thus, $S(S^{\eta}(\upsilon_0)) = S^{\eta}(\upsilon_0)$.

Hence, $S^{\eta}(\upsilon_0)$ is the fixed point of S.

Hence, maintaining generality, it can be supposed that, $\upsilon_\eta \neq \upsilon_\kappa$.

Therefore, $\limsup_{\eta\to+\infty} m_s(\upsilon_\eta, \upsilon_{\eta+2}) \leq \limsup_{\eta\to+\infty} m_s(\upsilon_{\eta+1}, \upsilon_{\eta+2})$.

Thus, as $\limsup_{\eta\to+\infty} m_s(\upsilon_\eta, \upsilon_{\eta+2}) = 0$, we have,

$$\limsup_{\eta\to+\infty} m_s(\upsilon_\eta, \upsilon_{\eta+3}) \leq \limsup_{\eta\to+\infty} m_s(\upsilon_{\eta+2}, \upsilon_{\eta+3}) = 0.$$

Inductively, it can be concluded that

$$\limsup_{\eta\to+\infty}\{m_s(\upsilon_\eta, \upsilon_\kappa) : \kappa > \eta\} = 0.$$

This leads to the fact that sequence $\{\upsilon_\eta\}$ is Cauchy. Provided that $\mho$ is a complete supermetric space, there must be $u \in \mho$ such that $\upsilon_\eta \to u$. The proof is completed if, $u = Su$. Assuming $u \neq Su$. If $S\upsilon_\eta = Su$ for infinitely many values of $\eta \in \mathbb{N} \cup \{0\}$, then $\{\upsilon_\eta\}$ ensures having a convergent subsequence that converges to Su, and since the limit is unique, it ensures that $u = Su$.

Hence, it can be assumed that $S\upsilon_\eta \neq Su$ for all $\eta \in \mathbb{N} \cup \{0\}$.

Further noting,

$$m_s(\upsilon_{\eta+1}, Su) = m_s(S\upsilon_\eta, Su),$$

by (6), we have

$$\begin{aligned} m_s(\upsilon_{\eta+1}, Su) &\leq \alpha m_s(\upsilon_\eta, u) + \beta m_s(\upsilon_\eta, S\upsilon_\eta) + \gamma m_s(u, Su) + L m_s(u, S\upsilon_\eta) \\ &= m_s(u, \upsilon_{\eta+1}) + \alpha m_s(\upsilon_\eta, u) + \beta m_s(\upsilon_\eta, \upsilon_{\eta+1}) + \gamma m_s(u, Su) + L m_s(u, \upsilon_{\eta+1}), \end{aligned}$$

which leads to a contradiction, thus $u = Su$.

Uniqueness

Now, supposing another fixed point $q \in \mho$ is not equal to u, of S to justify the uniqueness. This leads to $m_s(u, w) > 0$. Inserting $\upsilon = u$ and $\omega = q$ into (7), we have

$$\begin{aligned} \tau + \mathcal{F}(m_s(u, q)) &= \tau + \mathcal{F}(m_s(Su, Sq)) \\ &\leq \mathcal{F}(\alpha m_s(u, q) + \beta m_s(u, Su) + \gamma m_s(q, Sq) + L m_s(q, Su)) \end{aligned}$$

$$= F((\alpha + L)m_s(u,q)),$$

which leads to a contradiction, for choosing, $\alpha + L \leq 1$, and hence $u = w$. □

The above result can yield the following corollaries:

Corollary 1 *For a complete supermetric space $(\mho, m_s)$. If a S-Map S on $\mho$ is an F-contraction, then S has a unique fixed point.*

Proof Setting, $\beta = L = \gamma = \delta = 0$ with $\alpha = 1$ in Theorem (3.1), we obtain the result. □

Corollary 2 *For a complete supermetric space $(\mho, m_s)$ and a S-Map S on $\mho$. Assume there exist $F \in \daleth$ with $\tau \in \mathbb{R}^+$ such that,*

$$\tau + F(m_s(S\upsilon, S\omega)) \leq F(\beta m_s(\upsilon, S\upsilon) + \gamma m_s(\omega, S\omega)),$$

for all $\upsilon, \omega \in \mho$, $S\upsilon \neq S\omega$, where $\gamma + \beta = 1$, with $\gamma < 1$. Then, S possesses a unique fixed point in $\mho$.

Proof Taking $\alpha = \delta = L = 0$ in Theorem 3.1, one can obtain the result. □

Corollary 3 *For a complete supermetric space $(\mho, m_s)$ and a S-Map S on $\mho$. Assume there exist $F \in \daleth$ with $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(m_s(S\upsilon, S\omega)) \leq F(\alpha m_s(\upsilon, \omega) + \beta m_s(\upsilon, S\upsilon) + \gamma m_s(\omega, S\omega)),$$

for all $\upsilon, \omega \in \mho$, $S\upsilon \neq S\omega$, where $\alpha + \gamma + \beta = 1$, $\gamma < 1$. Then, S ensures a unique fixed point in $\mho$.

Proof Taking $\delta = L = 0$ in Theorem 3.1, we obtain the result. □

Remark 4 Corollary 1 is actually the main result of Wardowski [2] in the framework of supermetric space.

Remark 5 Corollary 2 is the version of Kannan's result [23] in the framework of supermetric space.

Remark 6 Corollary 3 is a Reich [24] type result in the setting of a supermetric space.

Now, we will move on to our next result of this section that is a rational-type F-contraction in the setting of supermetric space.

Theorem 3.2 *Let $(\mho, m_s)$ be a complete supermetric space and $S : \mho \to \mho$ be a generalized F-contraction, such that,*

$$\tau + F(m_s(S\upsilon, S\omega)) \leq F\left[k \max\left\{m_s(\upsilon, \omega), \frac{m_s(\upsilon, S\upsilon)m_s(\omega, S\omega)}{1 + m_s(\upsilon, \omega)}\right\}\right], \tag{8}$$

holds for all $\upsilon, \omega \in \mho$, with $k \in [0, 1)$, $S\upsilon \neq S\omega$, $\tau > 0$ a constant and $F \in \daleth$.

Then, S has a unique fixed point.

Proof Using (8) and $\mathfrak{F}1$, we have:

$$m_s(S\upsilon, S\omega) \le k \max\left\{m_s(\upsilon,\omega)\frac{m_s(\upsilon, S\upsilon)m_s(\omega, S\omega)}{1+m_s(\upsilon,\omega)}\right\}. \tag{9}$$

Let $\upsilon_0 \in \mho$ and let $\upsilon_1 = S\upsilon_0$. If $\upsilon_0 = \upsilon_1$ then there is nothing to prove, the proof is completed.

Assuming $\upsilon_0 \neq \upsilon_1$. Thus, $m_s(\upsilon_0, \upsilon_1) > 0$. Thus, maintaining generality, we can define the Picard sequence, $S\upsilon_\eta = \upsilon_{\eta+1}$ with $\upsilon_\eta \neq \upsilon_{\eta+1}$.

Hence, $m_s(\upsilon_\eta, \upsilon_{\eta+1}) > 0$, for all $\eta \in \mathbb{N}$.

From (9),

$$\begin{aligned} m_s(\upsilon_\eta, \upsilon_{\eta+1}) &= m_s(S\upsilon_{\eta-1}, S\upsilon_\eta) \\ &\le k \max\left\{m_s(\upsilon_{\eta-1}, \upsilon_\eta), \frac{m_s(\upsilon_{\eta-1}, S\upsilon_{\eta-1})m_s(\upsilon_\eta, S\upsilon_\eta)}{1+m_s(\upsilon_{\eta-1}, \upsilon_\eta)}\right\} \\ &= k \max\left\{m_s(\upsilon_{\eta-1}, \upsilon_\eta), \frac{m_s(\upsilon_{\eta-1}, \upsilon_\eta)m_s(\upsilon_\eta, \upsilon_{\eta+1})}{1+m_s(\upsilon_{\eta-1}, \upsilon_\eta)}\right\} \\ &\le k \max\{m_s(\upsilon_{\eta-1}, \upsilon_\eta), m_s(\upsilon_\eta, \upsilon_{\eta+1})\}. \end{aligned}$$

For the case of choosing

$$\max\{m_s(\upsilon_{\eta-1}, \upsilon_\eta), m_s(\upsilon_\eta, \upsilon_{\eta+1})\} = m_s(\upsilon_\eta, \upsilon_{\eta+1}),$$

we meet a contradiction.

This implies that

$$\max\{m_s(\upsilon_{\eta-1}, \upsilon_\eta), m_s(\upsilon_\eta, \upsilon_{\eta+1})\} = m_s(\upsilon_{\eta-1}, \upsilon_\eta).$$

Thus, it can be concluded that

$$m_s(\upsilon, \upsilon_{\eta+1}) \le k m_s(\upsilon_\eta, \upsilon_{\eta-1}) \le k^2 m_s(\upsilon_{\eta_2}, \upsilon_{\eta_1}) \le \cdots \le k^n m_s(\upsilon_0, \upsilon_1)$$

and taking the limit, we obtain,

$$\lim_{\eta\to+\infty} m_s(\upsilon_\eta, \upsilon_{\eta+1}) = 0.$$

Now, suppose that $\kappa, \eta \in \mathbb{N}$ and $\kappa > \eta$. If $\upsilon_\eta = \upsilon_\kappa$, we have

$$S^\kappa(\upsilon_0) = S^\eta(\upsilon_0).$$

Hence, we have

$$S^{\kappa-\eta}(S^\eta(\upsilon_0)) = S^\eta(\upsilon_0).$$

Thus, $S^\eta(\upsilon_0)$ is the fixed point of $S^{\kappa-\eta}$.

Also,

$$S(S^{\kappa-\eta}(S^{\eta}(\upsilon_0))) = S^{\kappa-\eta}(S(S^{\eta}(\upsilon_0))) = S(S^{\eta}(\upsilon_0)).$$

This means that $S(S^{\eta}(\upsilon_0))$ is also the fixed point of $S^{\kappa-\eta}$.

Thus, $S(S^{\eta}(\upsilon_0)) = S^{\eta}(\upsilon_0)$.

Hence, $S^{\eta}(\upsilon_0)$ is the fixed point of S.

Hence, maintaining generality, it can be supposed that, $\upsilon_\eta \neq \upsilon_\kappa$.

Therefore, $\limsup_{\eta\to+\infty} m_s(\upsilon_\eta, \upsilon_{\eta+2}) \leq \limsup_{\eta\to+\infty} m_s(\upsilon_{\eta+1}, \upsilon_{\eta+2})$.

Thus, as $\limsup_{\eta\to+\infty} m_s(\upsilon_\eta, \upsilon_{\eta+2}) = 0$, we have,

$$\limsup_{\eta\to+\infty} m_s(\upsilon_\eta, \upsilon_{\eta+3}) \leq \limsup_{\eta\to+\infty} m_s(\upsilon_{\eta+2}, \upsilon_{\eta+3}) = 0.$$

Inductively, it can be concluded that

$$\limsup_{\eta\to+\infty} \{m_s(\upsilon_\eta, \upsilon_\kappa) : \kappa > \eta\} = 0.$$

This justifies that $\{\upsilon_\eta\}$ is Cauchy.

Since $(\mho, m_s)$ is supposed to be a complete supermetric space, $\{\upsilon_\eta\}$ converges to $u \in \mho$. We claim that u is the fixed point of S. On the contrary, assume $m_s(u, Su) > 0$.

Note that

$$m_s(\upsilon_{n+1}, Su) = m_s(S\upsilon_\eta, Su) \leq k \max\left\{m_s(\upsilon_\eta, \omega), \frac{m_s(\upsilon_\eta, S\upsilon_\eta) m_s(\omega, S\omega)}{1 + m_s(\upsilon_\eta, \omega)}\right\}. \tag{10}$$

Thus, $\lim_{\eta\to+\infty} m_s(\upsilon_{\eta+1}, Su) \leq k \lim_{\eta\to+\infty} m_s(\upsilon_\eta, u) = 0$.

If there exists some $N > 0$ such that for all $\eta > N$, $\upsilon_{N+1} = u$, equation (10) concludes that $m_s(u, Su) = 0$ and so u is the fixed point for S. Otherwise, assuming for all $\eta \in \mathbb{N}$, $\upsilon_n \neq u$. Thus, we have

$$m_s(u, Su) \leq s \lim_{\eta\to+\infty} \sup m_s(\upsilon_\eta, u).$$

Also, it can be concluded that $m_s(u, Su) = 0$, which leads to a contradiction. As a result, u is the fixed point of S in $\mho$.

Uniqueness

For instance, letting another point $p \in \mho$ such that $Sp = p \neq u = Su$. Then, by (8), we obtain

$$\tau + F(m_s(Sp, Su)) \leq F\left[k \max\left\{m_s(p, u), \frac{m_s(p, Sp) m_s(u, Su)}{m_s(p, u) + 1}\right\}\right].$$

Thus,

$$\tau \leq F\{m_s(p, u)\} - F\{m_s(p, u)\} = 0,$$

which is a contradiction. □

Example 5 Let $\mho = [2,3]$ with supermetric be defined as

$$m_s(\upsilon,\omega) = \begin{cases} \upsilon\omega, & \upsilon \neq \omega, \\ 0, & \upsilon = \omega. \end{cases}$$

Now, consider $S : \mho \times \mho$ as follows,

$$S(\upsilon) = \begin{cases} 2, & \upsilon \neq 3, \\ \frac{5}{2}, & \upsilon = 3 \end{cases}$$

and define

$$\mathfrak{F}(t) = ln(t), \quad t > 0.$$

Then, proving the following is not tedious

$$\tau + F(m_s(S\upsilon, Sg)) \leq F(m_s(\upsilon, g)),$$

for some $\tau \geq 0$.

All the requirements of the Corollary 1 are fulfilled. Thus, the fixed point of S will certainly be unique.

Example 6 Let $\mho = [0,1]$ with $s = 1$, and define $m_s : \mho \times \mho \to [0,+\infty)$:

$$\begin{aligned} &m_s(\upsilon,\omega) = \upsilon\omega, \quad \text{for all } \upsilon \neq \omega,\ \upsilon,\omega \in (0,1), \\ &m_s(\upsilon,\omega) = 0, \quad \text{for all } \upsilon = \omega,\ \upsilon,\omega \in [0,1], \\ &m_s(0,\omega) = m_s(\omega,0) = \omega, \quad \text{for all } \omega \in (0,1], \\ &m_s(1,\omega) = m_s(\omega,1) = 1 - \frac{\omega}{2}, \quad \text{for all } \omega \in [0,1),\ \text{is a supermetric.} \end{aligned}$$

Now, consider, $S : \mho \to \mho$ as follows:

$$S(\upsilon) = \begin{cases} \frac{\upsilon}{4}, & \text{if } \upsilon \in [0,1), \\ \frac{1}{8}, & \text{if } \upsilon = 1 \end{cases}$$

and define

$$F(t) = \frac{-1}{\sqrt{t}}, \quad t > 0.$$

For $k = \frac{1}{2}$, we check the mapping for the following cases:

1. If $\upsilon = 0$, $\omega \in (0,1)$,

$$m_s(0,\omega) = \omega, \quad m_s(S0, S\omega) = m_s(\frac{\omega}{4}).$$

Also, we can have $F(\frac{1}{2}m_s(\upsilon,\omega)) = \frac{-\sqrt{2}}{\sqrt{\omega}}$, and $F(m_s(S\upsilon,S\omega)) = \frac{-2}{\sqrt{\omega}}$.

Thus,

$$\tau + \digamma(m_s(S\upsilon, S\omega)) \leq \digamma\left[k \max\left\{m_s(\upsilon,\omega), \frac{m_s(\upsilon,S\upsilon)m_s(\omega,S\omega)}{1+m_s(\upsilon,\omega)}\right\}\right].$$

Now, for the other two cases,

$$\text{If } \upsilon = 0,\ \omega = 1, \quad \text{and,} \quad \upsilon = 1,\ \omega \in (0,1).$$

Following the same argument we meet the same consequence. Thus, S will have a unique fixed point.

4 Fixed-point results in the context of ordered supermetric spaces

In the study of results of fixed points, which have vital applications in functional analysis, operator theory, and nonlinear analysis, partial ordered metric spaces have been employed extensively. These theorems are crucial in demonstrating the existence and uniqueness of solutions of certain mathematical and engineering problems. In 2004, Reurings and Ran [25] initiated the concept of how fixed points of self-mappings exist in ordered sets. This plays a significant role in the ordered theoretic approach. This study was further continued by Nieto and Rodriguez-Lopez [26]. In the same direction, many more valuable results have been established [27, 28].

This section comprises HR-type contractions in the context of ordered supermetric spaces.

Theorem 4.1 *Let $(\mho, m_s, \precsim)$ be a complete supermetric space and with S a self- and non-decreasing mapping on $\mho$. Assuming $\digamma \in \daleth$ and $\tau > 0$ such that S is a generalized $\digamma$-contraction of HR-type, that is,*

$$\begin{aligned}\tau + \digamma(m_s(S\upsilon, S\omega)) \precsim \digamma(&\alpha m_s(\upsilon,\omega) + \beta m_s(\upsilon,S\upsilon) + \gamma m_s(\omega,S\omega)\\ &+ \delta m_s(\upsilon,S\omega) + L m_s(\omega,S\upsilon)),\end{aligned} \tag{11}$$

for all $\upsilon,\omega \in \mho$ comparable, $S\upsilon \neq S\omega$, where $\alpha+\gamma+\beta = 1$, $\delta = 0$, $\gamma < 1$, and $L \geq 0$. If the conditions below are true:

(C1) *There is $\upsilon \in \mho$ such that $\upsilon \precsim S\upsilon$;*

(C2) *$\mho$ is regular;*

then S will have a fixed point. Furthermore, if $\alpha + L \precsim 1$, then S will have a well-ordered set of fixed points, if and only if S possesses a unique fixed point.

Proof Let $\upsilon_0 \in \mho$ and $\upsilon_0 \precsim S\upsilon_0$, with $\{\upsilon_\eta\}$ a Picard sequence possessing initial point υ, that is, $\upsilon_\eta = S\upsilon_{\eta-1} = S^\eta(\upsilon_0)$. If $\upsilon_\eta = \upsilon_{\eta-1}$ for some $\eta \in \mathbb{N}$, then υ_η S has υ_η as a fixed point.

Next, letting $m_\eta = m_s(\upsilon_\eta, \upsilon_{\eta+1})$ for all $\eta \in \mathbb{N}\cup\{0\}$ and suppose $\upsilon_\eta \neq \upsilon_{\eta-1}$ for each $\eta \in \mathbb{N}$.

Having S nondecreasing with $\upsilon_0 \precsim S\upsilon_0$, it can be obtained that:

$$\upsilon_0 \prec \upsilon_1 \prec \cdots \prec \upsilon_n \prec \cdots, \tag{12}$$

which means that υ_η and $\upsilon_{\eta+1}$ can be compared and $S\upsilon_{\eta-1} \neq S\upsilon_\eta$ for every $\eta \in \mathbb{N}\cup\{0\}$.

Following the same direction as in the proof of (3.1), we obtain $\{\upsilon_\eta\}$ as a Cauchy sequence. The completeness of $\mho$ means there must be some $u \in \mho$ such that $\upsilon_\eta \to u$. If $u = Su$ there is nothing to prove.

Let us assume $u \neq Su$.

As $\mho$ is regular and from (12) it can be deduced that υ_η and u are comparable and $S\upsilon_\eta \neq Su$ for all $\eta \in \mathbb{N} \cup \{0\}$.

Noting that

$$m_s(\upsilon_{\eta+1}, Su) = m_s(S\upsilon_\eta, Su)$$

by (6) in partial order, we have

$$\begin{aligned} m_s(\upsilon_{\eta+1}, Su) &\precsim \alpha m_s(\upsilon_\eta, u) + \beta m_s(\upsilon_\eta, S\upsilon_\eta) + \gamma m_s(u, Su) + L m_s(u, S\upsilon_\eta) \\ &= m_s(u, \upsilon_{\eta+1}) + \alpha m_s(\upsilon_\eta, u) + \beta m_s(\upsilon_\eta, \upsilon_{\eta+1}) + \gamma m_s(u, Su) + L m_s(u, \upsilon_{\eta+1}). \end{aligned}$$

Letting $\eta \to +\infty$ in the above inequality, we obtain,

$$m_s(u, Su) \precsim \gamma m_s(u, Su) \prec d(u, Su),$$

which leads to a contradiction, thus $u = Su$.

Now, we assume that $\alpha + L \leq 1$ and S has a well-ordered set of fixed points. We declare that S possesses a unique fixed point. Assuming contrarily that $\mho$ has another fixed point of S that is g, such that $u \neq g$. Using (11), with $\upsilon = u$ and $\omega = g$, we obtain

$$\begin{aligned} \tau + \digamma(m_s(u,g)) &\precsim \digamma(\alpha m_s(u,g) + \beta m_s(u, Su) + \gamma m_s(g, Sg) + L m_s(g, Su)) \\ &= \digamma((\alpha + L) m_s(u,g)) \\ &\precsim \digamma(m_s(u,g)), \end{aligned}$$

which leads to a contradiction. Thus, $u = g$. Conversely, if S possesses a unique fixed point, then being a singleton set, the set of fixed points of S is well ordered. □

Remark 7 Choosing $\digamma(\upsilon) = ln\upsilon$ in Theorem (4.1) and putting $\gamma = \beta = L = \delta = 0$ and $\tau = 1$, then the Theorem 2.2 of [26] in the frame of supermetric space can be obtained.

Theorem 4.2 *Let* $(\mho, m_s, \precsim)$ *be a complete ordered supermetric space and let* S *be a S-Map and nondecreasing on* $\mho$. *Assuming* $\digamma \in \daleth$ *and* $\tau > 0$ *such that* S *is a generalized* $\digamma$*-contraction of HR type with* $\delta = \gamma = 0$, *if the conditions below are true*:

(U1) *There is* $\upsilon \in \mho$ *such that* $\upsilon \precsim S\upsilon$,

(U2) $\mho$ *is regular, then* S *possesses a fixed point. Furthermore, if* $\alpha + L < 1$ *and,*

(U3) *For all* $u, g \in \mho$ *there exists* $t \in \mho$ *such that* t *and* u *are comparable and* g *and* t *are comparable*;

then S *possesses a unique fixed point*.

Proof As a consequence of Theorem 4.1 S ensures a fixed point. Now, let $u \in \mho$ be a fixed point of S. For all $t \in \mho$ comparable with u such that $Su \neq St$, we have

$$\begin{aligned}\tau + F(m_s(Su,St)) &\precsim F(\alpha m_s(u,t) + \beta m_s(u,Su) + Lm_s(t,Su)) \\ &\precsim F(\alpha m_s(u,t) + Lm_s(t,u)) \\ &= F((\alpha + L)m_s(u,t)).\end{aligned}$$

Due to the fact that F is strictly increasing, it can be obtained that

$$m_s(u,St) < (\alpha + L)m_s(u,t).$$

As S is nondecreasing, it can be achieved that u and $S^\eta t$ can be compared for all $\eta \in \mathbb{N}$. If $u \neq S^\eta t$ for all $\eta \in \mathbb{N}$, then

$$m_s(u,S^\eta t) < \lambda^\eta m_s(u,t), \quad \text{for all } \eta \in \mathbb{N},$$

where $\lambda = (\alpha + L) < 1$. From the previous inequality, we achieve $m_s(u,S^\eta t) \to 0$ as $\eta \to +\infty$.

Now, if u and g are both the fixed points of S, by (U3), there must be some $t \in \mho$ such that u and t can be compared and g and t are comparable.

If $u = S^\eta t$ or $g = S^\eta t$ for some $\eta \in \mathbb{N}$, then u and g can be compared, and the uniqueness of the fixed point implies since S is an F-contraction of HR-type.

Supposing $u \neq S^\eta t$ and $g \neq S^\eta t$ for all $\eta \in \mathbb{N}$. Thus,

$$\begin{aligned}\tau + F(m_s(u,g)) = F(m_s(Su,Sg)) &\precsim F(\alpha m_s(u,g) + \beta m_s(u,Su) + Lm_s(g,Su)) \\ &\precsim F(\alpha m_s(u,g) + \beta m_s(u,Su) + Lm_s(g,u)),\end{aligned}$$

$$\tau \precsim (\alpha + L - 1)m_s(u,g),$$

which is a contradiction because, $\alpha + L < 1$ and hence $m_s(u,g) = 0$, that is, $u = g$. □

Example 7 Let $\mho = [2,3]$ and define,

$$m_s(\upsilon,\omega) = \begin{cases} \upsilon\omega & \text{if } \upsilon \neq \omega, \\ 0 & \text{if } \upsilon = \omega \end{cases}$$

is clearly a supermetric space, see Example 2.

Let us define a self-mapping S on $\mho$ such that,

$$S\upsilon = \frac{\upsilon^2}{2}.$$

Choosing $F(\upsilon) = \ln \upsilon$ and consider partial order $\precsim$ on $\mho$ as

$$\upsilon_m \precsim \upsilon_\eta \quad \text{if } (m \leq \eta).$$

Now, we need to justify that S is an ordered F-contraction of HR-type with

$$\delta = \beta = \gamma = L = 0 \quad \text{and} \quad \tau = \alpha = 1, S\upsilon \neq S\omega.$$

Since, for $\digamma(\upsilon) = \ln \upsilon$, the HR-type contraction is equivalent to:

$$\frac{m_s(S\upsilon, S\omega)}{m_s(\upsilon, \omega)} \precsim e^{-\tau}.$$

To justify our stance, consider the following calculations:

$$\frac{m_s(S\upsilon, S\omega)}{m_s(\upsilon, \omega)} = \frac{\upsilon\omega}{4} \precsim e^{-1}.$$

Now, if $\{\upsilon_\eta\}$ is a convergent nondecreasing sequence, then $\upsilon_m = \upsilon_\eta$ and since $\mho$ is regular, as $\upsilon_0 \precsim S\upsilon_0$, all the requirements of Theorem 4.1 are fulfilled. Thus, S ensures a fixed point.

5 Applications

This section will utilize the fixed-points results obtained in the previous sections to demonstrate the existence of unique fixed points for certain integral-type contractions.

First, we shall establish a precise definition for an alternating distance function.

Definition 5.1 A function $\urcorner : [0, +\infty) \to [0, +\infty)$ is termed an alternating distance function if it fulfills these requirements:

(a) $\urcorner$ possesses continuity and is nondecreasing;

(b) $\urcorner(\ell) = 0$ if and only if $\ell = 0$.

Let us now propose the following definition.

Definition 5.2 Let $\aleph$ be the set of functions $\mathbb{T} : [0, +\infty) \to [0, +\infty)$ that fulfill these requirements:

(i) $\mathbb{T}$ on each compact subset of $[0, +\infty)$ is Lebesgue integrable.

(ii) $\int_0^z \mathbb{T}(\ell) d\ell > 0$ for all $z > 0$.

It is quite simple to show whether the mapping $\rho : [0, +\infty) \to [0, +\infty)$ defined as

$$\rho(z) = \int_0^z \mathbb{T}(\ell) d\ell > 0$$

is an alternating distance function.

Next, the first new outcome of this section is put forward.

Theorem 5.3 *Let $(\mho, m_s, \precsim)$ be a complete supermetric space, and let S be a S-Map on $\mho$. Assuming there exist $\digamma \in \daleth$ and $\tau > 0$ such that S is a generalized $\digamma$-contraction of HR-type, that is,*

$$\begin{aligned} \tau + \int_0^{\digamma(m_s(S\upsilon_1, S\omega))} \mathbb{T}(\ell) d\ell \preceq \digamma \Bigg\{ &\alpha \int_0^{m_s(\upsilon,\omega)} \mathbb{T}(\ell) d\ell + \beta \int_0^{m_s(\upsilon, S\upsilon)} \mathbb{T}(\ell) d\ell \\ &+\gamma \int_0^{m_s(\omega, S\omega)} \mathbb{T}(\ell) d\ell + L \int_0^{m_s(\omega, S\upsilon)} \mathbb{T}(\ell) d\ell \Bigg\} \end{aligned}$$

for $S\upsilon \neq S\omega$, where $\alpha + \beta + \gamma = 1$, $\delta = 0$, $\gamma < 1$, and $L \geq 0$. then S will have a fixed point. Furthermore, if $\alpha + L \leq 1$, then the fixed point will be unique.

Proof In Theorem 3.1, considering $\rho(z) = \int_0^z \mathbb{T}(\ell)d\ell$, we have the conclusion. □

Theorem 5.4 *Let $(\mho, m_s, \precsim)$ be a complete supermetric space, and let S be a S-Map on $\mho$. Assuming there exist $F \in \daleth$ and $\tau > 0$ such that S is a generalized F-contraction of HR-type, that is,*

$$\tau + \int_0^{F(m_s(S\upsilon_1, S\omega))} \mathbb{T}(\ell)d\ell \preceq F\left\{\alpha \int_0^{m_s(\upsilon,\omega)} \mathbb{T}(\ell)d\ell\right\}$$

for $S\upsilon \neq S\omega$, then S possesses a unique fixed point.

Proof In Corollary 1, considering $\rho(z) = \int_0^z \mathbb{T}(\ell)d\ell$, we can have the conclusion. □

Theorem 5.5 *Let $(\mho, m_s, \precsim)$ be a complete supermetric space, and let S be a S-Map on $\mho$. Assuming there exist $F \in \daleth$ and $\tau > 0$ such that S is a generalized F-contraction of HR-type, that is,*

$$\tau + \int_0^{F(m_s(S\upsilon_1, S\omega))} \mathbb{T}(\ell)d\ell \preceq F\left\{\beta \int_0^{m_s(\upsilon,S\upsilon)} \mathbb{T}(\ell)d\ell + \gamma \int_0^{m_s(\omega,S\omega)} \mathbb{T}(\ell)d\ell\right\}$$

for $S\upsilon \neq S\omega$, where $\beta + \gamma = 1$, $\gamma < 1$, then S ensures a unique fixed point.

Proof In Corollary 2, considering $\rho(z) = \int_0^z \mathbb{T}(\ell)d\ell$, we can have the conclusion. □

Theorem 5.6 *Let $(\mho, m_s, \precsim)$ be a complete supermetric space, and let S be a S-Map on $\mho$. Assuming there exist $F \in \daleth$ and $\tau > 0$ such that S is a generalized F-contraction of HR-type, that is,*

$$\tau + \int_0^{F(m_s(S\upsilon_1, S\omega))} \mathbb{T}(\ell)d\ell \preceq F\left\{\alpha \int_0^{m_s(\upsilon,\omega)} \mathbb{T}(\ell)d\ell + \beta \int_0^{m_s(\upsilon,S\upsilon)} \mathbb{T}(\ell)d\ell + \gamma \int_0^{m_s(\omega,S\omega)} \mathbb{T}(\ell)d\ell\right\}$$

for all $\upsilon, \omega \in \mho$ comparable, $S\upsilon \neq S\omega$, where $\alpha + \beta + \gamma = 1$, $\gamma < 1$, then S possesses a unique fixed point.

Proof In Corollary 3, considering $\rho(z) = \int_0^z \mathbb{T}(\ell)d\ell$, we can have the conclusion. □

Theorem 5.7 *Let $(\mho, m_s)$ be a complete supermetric space, and $S : \mho \to \mho$ be a generalized F-contraction, such that,*

$$\tau + \int_0^{F(m_s(S\upsilon_1, S\omega))} \mathbb{T}(\ell)d\ell \precsim k \int_0^{F[k(M_s(\upsilon,\omega))]} \mathbb{T}(\ell)d\ell,$$

where

$$M_s(\upsilon,\omega) = \max\{m_s(\upsilon,\omega), \frac{m_s(\upsilon,S\upsilon)m_s(\omega,S\omega)}{1 + m_s(\upsilon,\omega)}\}.$$

Then, S will ensure a unique fixed point.

Proof In Theorem 3.2, taking $\rho(z) = \int_0^z \mathbb{T}(\ell)d\ell$, we obtain the conclusion. □

Theorem 5.8 *Let $(\mho, m_s, \precsim)$ be an ordered complete supermetric space, and let S be a S-Map on $\mho$. Assuming there exist $\digamma \in \daleth$ and $\tau > 0$ such that S is a generalized $\digamma$-contraction of HR type, that is,*

$$\tau + \int_0^{\digamma(m_s(S\upsilon_1,S\omega))} \mathbb{T}(\ell)d\ell \preceq \digamma\left\{\alpha \int_0^{m_s(\upsilon,\omega)} \mathbb{T}(\ell)d\ell + \beta \int_0^{m_s(\upsilon,S\upsilon)} \mathbb{T}(\ell)d\ell + \gamma \int_0^{m_s(\omega,S\omega)} \mathbb{T}(\ell)d\ell + L \int_0^{m_s(\omega,S\upsilon)} \mathbb{T}(\ell)d\ell\right\}$$

for all $\upsilon, \omega \in \mho$ comparable, $S\upsilon \neq S\omega$, where $\alpha + \beta + \gamma = 1$, $\delta = 0$, $\gamma < 1$, and $L \geq 0$. If the following conditions are true:

(Q1) *There is $\upsilon \in \mho$ such that $\upsilon \precsim S\upsilon$;*

(Q2) *$\mho$ is regular;*

then S will have a fixed point. Furthermore, if $\alpha + L \leq 1$, then S will have the set of fixed points well ordered, if and only if S has a unique fixed point.

Proof In Theorem 4.1, considering $\rho(z) = \int_0^z \mathbb{T}(\ell)d\ell$, we can have the conclusion. □

6 Conclusion

The concepts of $\digamma$-contraction of Hardy–Rogers type and rational-type z-contraction have been studied and generalized in many contexts because of many generalized structure of metric spaces. In this paper, we investigate the $\digamma$-contraction of Hardy–Rogers type and rational-type z-contraction in the setting of supermetric and ordered supermetric space. The study provides a deeper understanding of the behavior of mappings in the framework of supermetric space and its applications.

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