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On some common fixed point results for two infinite families of uniformly L-Lipschitzian total asymptotically quasi-nonexpansive mappings

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Abstract

In this manuscript, we introduce a new class of uniformly L-Lipschitzian and total asymptotically quasi-nonexpansive mappings, which is significantly more general than many known nonexpansive mappings that appeared before. We establish some convergence and Δ-convergence theorems for two infinite families of such mappings in the setting of CAT(κ) spaces. Our results refine, generalize, and improve several corresponding results in the existing literature.

Keywords: Common fixed point; Δ-Convergence; Asymptotically quasi-nonexpansive map; Total asymptotically nonexpansive mappings; CAT(0) space; Hyperbolic space; CAT(κ) space

1 Introduction

For the last ten decades researchers have paid a lot of attention to the development of fixed point theory in various classes of maps in different spaces. Different algorithms for approximation of fixed points have been investigated by a number of mathematicians (see [9, 12, 13, 20, 24] and the references therein).

In 2012, Khan et al. [12] introduced an implicit algorithm for two finite families of nonexpansive maps in a more general setting of hyperbolic spaces. Recently, Nuntadilok et al. [22] established common fixed point theorems of two finite families of asymptotically quasi-nonexpansive mappings in hyperbolic spaces. Their results were a refinement and generalization of several recent results in CAT(0) spaces and uniformly convex Banach spaces.

Let $(\mathcal{M},\mathfrak{D})$ be a metric space, where $\mathfrak{D}:\mathcal{M}\times\mathcal{M}\to\mathbb{R}^+$ is a metric. Let $\mathcal{N}$ be a subset of $\mathcal{M}$, and $\mathcal{U}:\mathcal{N}\to\mathcal{N}$ be a mapping. Denote by $F(\mathcal{U})=\{x\in\mathcal{M}:\mathcal{U}x=x\}$ the set of all fixed points of the mapping $\mathcal{U}$. Here $\mathbb{R}^+=[0,\infty)$ and $\mathbb{N}$ = the set of positive integers.

Recall that a mapping $\mathcal{U}:\mathcal{N}\to\mathcal{N}$ is said to be:

(i) *nonexpansive* if

$$\mathfrak{D}(\mathcal{U}x,\mathcal{U}y)\le\mathfrak{D}(x,y),\ \forall x,y\in\mathcal{N};$$

(ii) *quasi-nonexpansive* if $F(\mathcal{U}) \neq \emptyset$ and

$$\mathfrak{D}(\mathcal{U}x, p) \leq \mathfrak{D}(x, p), \forall p \in F(\mathcal{U});$$

(iii) *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\mathfrak{D}(\mathcal{U}^n x, \mathcal{U}^n y) \leq k_n \mathfrak{D}(x, y), \forall x, y \in \mathcal{N} \text{ and } \forall n \in \mathbb{N};$$

(iv) *asymptotically quasi-nonexpansive* if $F(\mathcal{U}) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\mathfrak{D}(\mathcal{U}^n x, p) \leq k_n \mathfrak{D}(x, p), \forall x, y \in \mathcal{N}, \forall p \in F(\mathcal{U}) \text{ and } \forall n \in \mathbb{N};$$

(v) The mapping $\mathcal{U}$ is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\mathfrak{D}(\mathcal{U}^n x, \mathcal{U}^n y) \leq L\mathfrak{D}(x, y), \ \forall x, y \in \mathcal{N}.$$

Remark 1 One can easily see that if $F(\mathcal{U}) \neq \emptyset$, then nonexpansive mapping, quasi-nonexpansive mapping, asymptotically nonexpansive mapping all are asymptotically quasi-nonexpansive mappings, but the converse is not true in general.

In 1993, Bruck et al. [4] introduced the following definition.

Definition 1 [4] A mapping $\mathcal{U} : \mathcal{N} \to \mathcal{N}$ is said to be *asymptotically nonexpansive in the intermediate sense* if $\mathcal{U}$ is uniformly continuous and

$$\limsup_{n\to\infty} \mathfrak{D}(\mathcal{U}^n x, \mathcal{U}^n y) - \mathfrak{D}(x, y) \leq 0, \forall x, y \in \mathcal{N}, \ n \geq 1. \tag{1}$$

We note that the class of asymptotically nonexpansive mappings in the intermediate sense is more general than the class of asymptotically nonexpansive mappings.

Definition 2 [1] A mapping $\mathcal{U} : \mathcal{N} \to \mathcal{N}$ is said to be $(\{\gamma_n\}, \{\mu_n\}, \zeta)$-total asymptotically nonexpansive if there exist nonnegative sequences $\{\gamma_n\}, \{\mu_n\}$ with $\lim_{n\to\infty} \gamma_n = 0 = \lim_{n\to\infty} \mu_n$ and a strictly increasing continuous function $\zeta : [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ such that

$$\mathfrak{D}(\mathcal{U}^n x, \mathcal{U}^n y) \leq \mathfrak{D}(x, y) + \gamma_n \zeta\left(\rho(x, y)\right) + \mu_n, \forall x, y \in \mathcal{N}, n \geq 1. \tag{2}$$

Remark 2 Note that the notion of total asymptotically nonexpansive mappings is more general than that of asymptotically nonexpansive mappings in the intermediate sense (see [6]).

In recent years, CAT(0) spaces have played a very significant role in different aspects of geometry [8]. Kirk [15, 16] showed that a nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space has a fixed point.

In 2012, Chang et al. [7] studied the demiclosedness principle and Δ-convergence theorems for total asymptotically nonexpansive mappings in the setting of CAT(0) spaces. Since then the convergence of several iteration procedures for this type of mappings has been rapidly developed, and many articles have appeared (see, e.g., [2, 5, 14, 18, 21, 22, 25, 27–29]).

Let $\mathcal{N}$ be a nonempty closed convex subset of a CAT(0) space $\mathcal{M}$ and $\mathcal{U}:\mathcal{N}\to\mathcal{N}$ be a total asymptotically nonexpansive mapping defined by (2). Given $x_1\in\mathcal{N}$, let $\{x_n\}\subset\mathcal{N}$ be defined by

$$x_{n+1}=(1-\alpha_n)x_n\bigoplus\alpha_n\mathcal{U}^n\bigl((1-\beta_n)x_n\bigoplus\mathcal{U}^nx_n\bigr),\ n\in\mathbb{N},\tag{3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. In 2014, under some suitable assumptions, Karapinar et al. [11] obtained the demiclosedness principle, fixed point theorems, and convergence theorems for the iteration (3).

It is well known that any CAT(κ) space is a CAT(κ_1) space for $\kappa\le\kappa_1$. Thus, all results for CAT(0) spaces immediately contain any CAT(κ) space with $\kappa\le 0$.

In 2014, Panyanak [23] obtained the demiclosedness principle, fixed point theorems, and convergence theorems for total asymptotically nonexpansive mappings on a CAT(κ) space with $\kappa>0$, which generalizes the results of Chang et al. [7], Karapinar et al. [11], and Tang et al. [25],

Inspired and motivated by the work going on in this direction, Chang et al. [6] studied the strong convergence of a sequence generated by an infinite family of total asymptotically nonexpansive mappings in CAT(κ) spaces with $\kappa>0$. Their results are extensions and improvements of the corresponding results of Chang et al. [7], Hea et al. [10], Karapinar et al. [11], Tang et al. [25], Panyanak [23], and many others.

The purpose of this manuscript is to investigate the existence of common fixed points of two infinite families of uniformly L-Lipschitzian and $(\{\gamma_n\},\{\mu_n\},\zeta)$-total asymptotically quasi-nonexpansive mappings, a more general class of mappings, in the setting of CAT(κ) spaces.

2 Preliminaries

Let $(\mathcal{M},\mathfrak{D})$ be a metric space. A geodesic path joining x to y for $x,y\in\mathcal{M}$ is a mapping $\omega:[0,l]\to\mathcal{M}$ such that $\omega(0)=x,\omega(l)=y$, and $\mathfrak{D}(\omega(t),\omega(t'))=|t-t'|$ for all $t,t'\in[0,l]\subset\mathbb{R}$. In particular, ω is an isometry and $\mathfrak{D}(x,y)=l$. The image $\omega([0,l])$ of ω is called a geodesic segment joining x and y. This geodesic segment is denoted by $[x,y]$, when it is unique. Then $z\in[x,y]$ if and only if there exists $\alpha\in[0,1]$ such that

$$\mathfrak{D}(x,z)=(1-\alpha)\mathfrak{D}(x,y)\quad\text{and}\quad\mathfrak{D}y,z)=\alpha\mathfrak{D}(x,y).$$

We denote such z by $(1-\alpha)x\bigoplus\alpha y$. That is, $z=(1-\alpha)x\bigoplus\alpha y$.

A metric space $(\mathcal{M},\mathfrak{D})$ is said to be a geodesic space (D-geodesic space) if every two points of $\mathcal{M}$ are joined by a geodesic, and $\mathcal{M}$ is said to be uniquely geodesic (D-uniquely geodesic) if there is exactly one geodesic joining x and y for each $x,y\in\mathcal{M}$ (i.e. for any $x,y\in\mathcal{M}$ with $\mathfrak{D}(x,y)<D$). A subset $\mathcal{N}$ of $\mathcal{M}$ is said to be convex if $\mathcal{N}$ includes every geodesic segment $[x,y]$ for any $x,y\in\mathcal{N}$.

A geodesic triangle $\Delta(x,y,z)$ in a geodesic space $(\mathcal{M},\rho)$ consists of three points x,y,z in $\mathcal{M}$ (the vertices of Δ) and three geodesic segments between each pair of vertices (the edges of Δ). A comparison triangle for a geodesic triangle $\Delta(x,y,z)$ in $(\mathcal{M},\mathfrak{O})$ is a triangle $\bar{\Delta}(\bar{x},\bar{y},\bar{z})$ in $\mathbb{M}_\kappa^2$ such that

$$\mathfrak{O}(x,y) = \mathfrak{O}_{\mathbb{M}_\kappa^2}(\bar{x},\bar{y}),\ \mathfrak{O}(y,z) = \mathfrak{O}_{\mathbb{M}_\kappa^2}(\bar{y},\bar{z}),\ \mathfrak{O}(z,x) = \mathfrak{O}_{\mathbb{M}_\kappa^2}(\bar{z},\bar{x}).$$

If $\kappa < 0$, then such a comparison triangle always exists in $\mathbb{M}_\kappa^2$. If $\kappa > 0$, then such a triangle exists whenever $\mathfrak{O}(x,y) + \mathfrak{O}(y,z) + \mathfrak{O}(z,x) < 2D_\kappa$, where $D_\kappa = \dfrac{\pi}{\sqrt{\kappa}}$.

A point $\bar{p} \in [\bar{x},\bar{y}]$ is called a comparison point for $p \in [x,y]$ if $\mathfrak{O}(x,p) = \mathfrak{O}_{\mathbb{M}_\kappa^2}(\bar{x},\bar{p})$. A geodesic triangle $\Delta(x,y,z)$ in $\mathcal{M}$ is said to satisfy the CAT(κ) inequality if for any $p,q \in \Delta(x,y,z)$ and for their comparison points $\bar{p},\bar{q} \in \bar{\Delta}(\bar{x},\bar{y},\bar{z})$ one has

$$\mathfrak{O}(p,q) \le \mathfrak{O}_{\mathbb{M}_\kappa^2}(\bar{p},\bar{q}).$$

Remark 3 For more details on the model spaces $\mathbb{M}_\kappa^n$, we refer readers to [8, 27].

Definition 3 A metric space $(\mathcal{M},\mathfrak{O})$ is called a CAT(0) space if $\mathcal{M}$ is a geodesic space such that all of its geodesic triangles satisfy the CAT(κ) inequality.

Note that $\mathcal{M}$ is called a CAT(κ) space with $\kappa > 0$ if $\mathcal{M}$ is D_κ-geodesic and any geodesic triangle $\Delta(x,y,z)$ in $\mathcal{M}$ with $\mathfrak{O}(x,y)+\mathfrak{O}(y,z)+\mathfrak{O}(z,x) < 2D_\kappa$ satisfies the CAT(κ) inequality.

Let $\{x_n\}$ be a bounded sequence in a CAT(κ) space $(\mathcal{M},\mathfrak{O})$. For $x \in \mathcal{M}$, we set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} \mathfrak{O}(x,x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is defined by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in \mathcal{M}\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in \mathcal{M} : r(x,\{x_n\}) = r(\{x_n\})\}.$$

The asymptotic radius $r(\{x_n\})$ with respect to $\mathcal{N} \subseteq \mathcal{M}$ of $\{x_n\}$ is given by

$$r_{\mathcal{N}}(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in \mathcal{N}\}.$$

The asymptotic center $A_{\mathcal{N}}(\{x_n\})$ with respect to $\mathcal{N} \subseteq \mathcal{M}$ of $\{x_n\}$ is the set

$$A_{\mathcal{N}}(\{x_n\}) = \{x \in \mathcal{N} : r(x,\{x_n\}) = r(\{x_n\})\}.$$

We now recall the concept of Δ-convergence and some of its basic properties.

Definition 4 [17, 19] A sequence $\{x_n\}$ in $\mathcal{M}$ is said to Δ-converge to $x \in \mathcal{M}$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$, and x is called the Δ-limit of $\{x_n\}$.

Lemma 1 [6] *Let* $(\mathcal{M},\mathfrak{D})$ *be a complete CAT*(κ) *space with* $\kappa > 0$ *and* $diam(\mathcal{M}) \leq \frac{\frac{\pi}{2}-\epsilon}{\sqrt{\kappa}}$, $\epsilon \in (0,\frac{\pi}{2})$. *Then the following statements hold*:

(i) ([29], *Corollary* 4.4) *Every sequence in* $\mathcal{M}$ *has a* Δ*-convergence subsequence*.

(ii) ([29], *Proposition* 4.5) *If* $\{x_n\} \subset \mathcal{M}$ *and* $\Delta\text{-}\lim_{n\to\infty} x_n = x$, *then*

$$x \in \cap_{n=1}^{\infty}\overline{conv}\{x_n, x_{n+1}, \dots\},$$

where $\overline{conv}(D) = \{E : D \subseteq E \text{ and } E \text{ is closed and convex}\}$.

By the uniqueness of asymptotic centers, Chang et al. [6] obtained the following lemma.

Lemma 2 [6] *Let* $(\mathcal{M},\mathfrak{D})$ *be a complete CAT*(κ) *space with* $\kappa > 0$ *and* $diam(\mathcal{M}) \leq \frac{\frac{\pi}{2}-\epsilon}{\sqrt{\kappa}}$, $\epsilon \in (0,\frac{\pi}{2})$. *If* $\{x_n\}$ *is a sequence in* $\mathcal{M}$ *with* $A(\{x_n\}) = \{x\}$ *and if* $\{u_n\}$ *is a subsequence of* $\{x_n\}$ *with* $A(\{u_n\}) = \{u\}$ *and the sequence* $\{\mathfrak{D}(x_n,u)\}$ *converges, then* $x = u$.

The following lemma is due to Bridson and Haefliger [3].

Lemma 3 [3] *Let* $(\mathcal{M},\mathfrak{D})$ *be a complete CAT*(κ) *space with* $\kappa > 0$ *and* $diam(\mathcal{M}) \leq \frac{\frac{\pi}{2}-\epsilon}{\sqrt{\kappa}}$ *for some* $\epsilon \in (0,\frac{\pi}{2})$. *Then*

$$\mathfrak{D}\big((1-\alpha)x \bigoplus \alpha y, z\big) \leq (1-\alpha)\mathfrak{D}(x,z) + \alpha\mathfrak{D}(y,z)$$

for all $x,y,z \in \mathcal{M}$ *and* $\alpha \in [0,1]$.

Proposition 1 [6] *Let* $(\mathcal{M},\mathfrak{D})$ *be a complete uniformly convex CAT*(κ) *space with* $\kappa > 0$ *and* $diam(\mathcal{M}) \leq \frac{\frac{\pi}{2}-\epsilon}{\sqrt{\kappa}}$ *for some* $\epsilon \in (0,2)$. *Let* $x \in \mathcal{M}$ *be a given point and* $\{\alpha_n\}$ *be a sequence in* $[a,b]$ *with* $a,b \in (0,1)$ *and* $0 < b(1-a) \leq \frac{1}{2}$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be any sequences in* $\mathcal{M}$ *such that*

$$\limsup_{n\to\infty}\mathfrak{D}(x_n,x) \leq r, \quad \limsup_{n\to\infty}\mathfrak{D}(y_n,x) \leq r \quad \text{and}$$
$$\lim_{n\to\infty}\mathfrak{D}\big((1-\alpha_n)x_n \bigoplus \alpha_n y_n, x\big) = r \text{ for some } r \geq 0.$$

Then $\lim_{n\to\infty}\mathfrak{D}(x_n,y_n) = 0$.

The following lemmas are essential.

Lemma 4 *Let* $\{a_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ *and* $\{\delta_n\}_{n=1}^{\infty}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + \gamma_n,\ n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty}\gamma_n < \infty$ *and* $\sum_{n=1}^{\infty}\delta_n < \infty$, *then*

(i) $\lim_{n\to\infty} a_n$ exists.

(ii) *In particular, if* $\{a_n\}_{n=1}^{\infty}$ *has a subsequence that converges strongly to* 0*, then* $\lim_{n\to\infty} a_n = 0$.

Lemma 5 [26] *For each positive integer* $n \geq 1$*, then*

(1) *the unique solutions* $i(n)$ *and* $l(n)$ *with* $l(n) \geq i(n)$ *to the following positive integer equation*

$$n = i(n) + \frac{(l(n)-1)l(n)}{2} \tag{4}$$

are as follows:

$$i(n) = n - \frac{(l(n)-1)l(n)}{2},$$
$$l(n) = \Big[\frac{1}{2} + \sqrt[2]{2n - \frac{7}{4}}\Big],\ l(n) \geq i(n)$$

and $l(n) \to \infty$ (*as* $n \to \infty$)*, where* $[x]$ *denotes the maximal integer that is not larger than* x.

(2) *For each* $i \geq 1$*, denote*

$$\Gamma_i = \{n \in \mathbb{N} : n = i + \frac{(l(n)-1)l(n)}{2},\ l(n) \geq i\} \text{ and}$$
$$\mathcal{K}_i = \{l(n) : n \in \Gamma_i, n = i + \frac{(l(n)-1)l(n)}{2},\ l(n) \geq i\},$$

then $l(n) + 1 = l(n+1), \forall n \in \Gamma_i$.

Let $\mathcal{U}_i : \mathcal{N} \to \mathcal{N}$ be uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive mappings defined by (2). For each positive integer $n \geq 1$, let $i(n)$ and $l(n)$ be the unique solutions of the positive integer equation (4). Recently, Chang et al. [6] proved a strong convergence theorem for total asymptotically nonexpansive mappings in $CAT(\kappa)$ spaces via the following iterative scheme.

For $x_1 \in \mathcal{N}$, define a sequence $\{x_n\}$ as follows:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n \bigoplus \alpha_n \mathcal{U}_{i(n)}^{l(n)} y_n, \\ y_n &= (1-\beta_n)x_n \bigoplus \beta_n \mathcal{U}_{i(n)}^{l(n)}, n \geq 1, \end{aligned} \tag{5}$$

where $\mathcal{N}$ is a nonempty closed and convex subset of a complete CAT(κ) space $\mathcal{M}$ with $\kappa > 0$.

More precisely, Chang et al. [6] obtained the following result.

Theorem 2 [6] *Let* $\mathcal{N}$ *be a nonempty closed and convex subset of a complete uniformly convex* $CAT(\kappa)$ *space* $(\mathcal{M}, \mathfrak{D})$ *with* $\kappa > 0$ *and* $diam(\mathcal{M}) \leq \frac{\frac{\pi}{2} - \epsilon}{\sqrt{\kappa}}$ *for some* $\epsilon \in (0, \frac{\pi}{2})$*. And, for each* $i \geq 1$*, let* $\mathcal{U}_i : \mathcal{N} \to \mathcal{N}$ *be a uniformly* L_i*-Lipschitzian and* $(\{\gamma_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$*-total asymptotically nonexpansive mapping defined by* (2) *such that*

(i) $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \mu_n^{(i)} < \infty$;

(ii) *there exists a constant* $\mathfrak{M} > 0$ *such that* $\zeta^{(i)}(\theta) \leq \mathfrak{M} \cdot \theta, \forall \theta \geq 0, i = 1, 2, 3, \ldots$;

(iii) *there exist constants* $a, b \in (0,1)$ *with* $0 < b(1-a) \leq \frac{1}{2}$ *such that* $\{\alpha_n\}, \{\beta_n\} \subset [a,b]$.
If $\mathcal{F} := \cap_{i=1}^{\infty} F(\mathcal{U}_i) \neq \emptyset$ *and there exist a mapping* $\mathcal{U}_{n_0} \in \{\mathcal{U}_i\}_{i=1}^{\infty}$ *and a nondecreasing function* $h : [0,\infty) \to [0,\infty)$ *with* $h(0) = 0$ *and* $h(r) > 0, \forall r > 0$ *such that*

$$h(\rho(x_n, \mathcal{F})) \leq \mathfrak{O}(x_n, \mathcal{U}_{n_0} x_n), \forall n \geq 1, \tag{6}$$

then the sequence $\{x_n\}$ *defined by* (5) *converges strongly to some point* $x* \in \mathcal{F}$.

In this manuscript, inspired and motivated by the works of Chang et al. in [6] and some related papers, we establish common fixed point theorems for two infinite families of uniformly L-Lipschitzian and $(\{\gamma_n\}, \{\mu_n\}, \zeta)$-total asymptotically quasi-nonexpansive mappings in the setting of CAT(κ) spaces. Our results significantly refine and generalize the works of Chang et al. [6] as well as many other comparable results in the literature.

3 Convergence and Δ-convergence theorems

We first introduce the following definition.

Definition 5 A mapping $\mathcal{U} : \mathcal{N} \to \mathcal{N}$ is said to be $(\{\gamma_n\}, \{\mu_n\}, \zeta)$-total asymptotically quasi-nonexpansive if $F(\mathcal{U}) \neq \emptyset$ and there exist nonnegative sequences $\{\mu_n\}, \{\gamma_n\}$ and a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \gamma_n$ and a strictly increasing continuous function $\zeta : [0,\infty) \to [0,\infty)$ with $\zeta(0) = 0$ such that

$$\mathfrak{O}(\mathcal{U}^n x, p) \leq k_n \mathfrak{O}(x,p) + \gamma_n \zeta\left(\mathfrak{O}(x,p)\right) + \mu_n, \forall x, y \in \mathcal{D},\ n \geq 1 \text{ and } p \in F(\mathcal{U}). \tag{7}$$

Remark 4 It is easy to see that every $(\{\gamma_n\}, \{\mu_n\}, \zeta)$-total asymptotically nonexpansive mapping defined by (2) is $(\{\gamma_n\}, \{\mu_n\}, \zeta)$-total asymptotically quasi-nonexpansive mapping, but the converse is not true in general.

For each $i \geq 1$, let $\mathcal{U}_i, \mathcal{V}_i : \mathcal{N} \to \mathcal{N}$ be two uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically quasi-nonexpansive mappings defined by (7), and for each positive integer $n \geq 1$, $i(n)$ and $l(n)$ are the unique solutions of the positive integer equation (4). In this section, we will prove strong convergence and Δ-convergence theorems of two infinite families of two uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically quasi-nonexpansive mappings $\{\mathcal{U}_i : i \geq 1\}$ and $\{\mathcal{V}_i : i \geq 1\}$ via the following iterative scheme. For $x_1 \in \mathcal{N}$, define the sequence $\{x_n\}$ as follows:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n \bigoplus \alpha_n \mathcal{U}_{i(n)}^{l(n)} y_n, \\ y_n &= (1-\beta_n)x_n \bigoplus \beta_n \mathcal{V}_{i(n)}^{l(n)}, n \geq 1, \end{aligned} \tag{8}$$

where $\mathcal{N}$ is a nonempty closed and convex subset of a complete uniformly convex CAT(κ) space $\mathcal{M}$ with $\kappa > 0$.

Remark 5 If $\mathcal{U}_i = \mathcal{V}_i$ for each $i \geq 1$, then the sequence defined by (8) reduces to sequence (5).

We denote $\mathbb{F} = \cap_{i=1}^{\infty} (F(\mathcal{U}_i) \cap F(\mathcal{V}_i)), i \geq 1$. We first prove the following two technical lemmas, which will be useful in the proof of our main results.

Lemma 6 *Let $\mathcal{N}$ be a nonempty closed and convex subset of a complete uniformly convex CAT(κ) space $(\mathcal{M}, \mathfrak{O})$ with $\kappa > 0$ and diam$(\mathcal{M}) \leq \frac{\frac{\pi}{2} - \epsilon}{\sqrt{\kappa}}$ for some $\epsilon \in (0, \frac{\pi}{2})$. And, for each $i \geq 1$, let $\mathcal{U}_i, \mathcal{V}_i : \mathcal{N} \to \mathcal{N}$ be two uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically quasi-nonexpansive mappings defined by (7) such that*

(i) $\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu_n^{(i)} < \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n k_n < \infty$, $\sum_{n=1}^{\infty} \beta_n k_n < \infty$ *and* $\alpha_n \to 0$, $\beta_n \to 0$;

(iii) *there exists a constant* $\mathfrak{M} > 0$ *such that* $\zeta^{(i)}(\theta) \leq \mathfrak{M} \cdot \theta, \forall \theta \geq 0, i = 1, 2, 3, \ldots$;

(iv) *there exist constants* $a, b \in (0, 1)$ *with* $0 < b(1 - a) \leq \frac{1}{2}$ *such that* $\{\alpha_n\}, \{\beta_n\} \subset [a, b]$.

If $\{x_n\}$ is the sequence defined by (8), then $\lim_{n\to\infty} \mathfrak{O}(x_n, \mathbb{F})$ and $\lim_{n\to\infty} \mathfrak{O}(x_n, p), p \in \mathbb{F}$, exist.

Proof Let $p \in \mathbb{F}$. First we note that for each $i \geq 1, \mathcal{U}_i, \mathcal{V}_i : \mathcal{N} \to \mathcal{N}$ are $(\{\gamma_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically quasi-nonexpansive mappings. By condition (iii), for each $n \geq 1$ and any $x, y \in \mathcal{N}$, we have

$$\mathfrak{O}(\mathcal{U}_i^n x, p) \leq \big(k_n + \gamma_n^{(i)} \mathfrak{M}\big) \mathfrak{O}(x, p) + \mu_n^{(i)} \tag{9}$$

and

$$\mathfrak{O}(\mathcal{V}_i^n x, p) \leq \big(k_n + \gamma_n^{(i)} \mathfrak{M}\big) \mathfrak{O}(x, p) + \mu_n^{(i)}. \tag{10}$$

We will prove that $\lim_{n\to\infty} \mathfrak{O}(x_n, \mathbb{F})$ and $\lim_{n\to\infty} \mathfrak{O}(x_n, p)$ exist for each $p \in \mathbb{F}$.

In fact, since $p \in \mathbb{F}$ and $\mathcal{U}_i, \mathcal{V}_i, i \geq 1$, are total asymptotically quasi-nonexpansive mappings, it follows from Lemma 3, (9), and (10) that

$$\begin{aligned}
\mathfrak{O}(y_n, p) &= \mathfrak{O}\big((1 - \beta_n) x_n\big) \bigoplus \beta_n \mathcal{V}_{i(n)}^{l(n)} x_n, p\big) \\
&\leq (1 - \beta_n) \mathfrak{O}(x_n, p) + \beta_n \mathfrak{O}(\mathcal{V}_{i(n)}^{l(n)} x_n, p) \\
&\leq (1 - \beta_n) \mathfrak{O}(x_n, p) + \beta_n \big[k_n \mathfrak{O}(x_n, p) + \gamma_{l(n)}^{i(n)} \zeta^{i(n)} \big(\mathfrak{O}(x_n, p)\big) + \mu_{l(n)}^{i(n)})\big] \\
&\leq \mathfrak{O}(x_n, p) + \big(\beta_n k_n + \beta_n \gamma_{l(n)}^{i(n)} \mathfrak{M}\big) \mathfrak{O}(x_n, p) + \beta_n \mu_{l(n)}^{i(n)} \\
&\leq \big(1 + \beta_n k_n + \beta_n \gamma_{l(n)}^{i(n)} \mathfrak{M}\big) \mathfrak{O}(x_n, p) + \beta_n \mu_{l(n)}^{i(n)}
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
\mathfrak{O} x_{n+1}, p) &= \mathfrak{O}\big((1 - \alpha_n) x_n \bigoplus \alpha_n \mathcal{U}_{i(n)}^{l(n)} y_n, p\big) \\
&\leq (1 - \alpha_n) \mathfrak{O}(x_n, p) + \alpha_n \mathfrak{O}(\mathcal{U}_{i(n)}^{l(n)} y_n, p) \\
&\leq (1 - \alpha_n) \mathfrak{O}(x_n, p) + \alpha_n \big[k_n \rho(y_n, p) + \gamma_{l(n)}^{i(n)} \zeta^{i(n)} \big(\mathfrak{O}(y_n, p)\big) + \mu_{l(n)}^{i(n)}\big] \\
&\leq \mathfrak{O}(x_n, p) + \alpha_n \big(k_n + \gamma_{l(n)}^{i(n)} \mathfrak{M}\big) \mathfrak{O}(y_n, p) + \alpha_n \mu_{l(n)}^{i(n)}.
\end{aligned} \tag{12}$$

Substituting (11) in (12), we get

$$\begin{aligned}
\mathfrak{O}(x_{n+1}, p) \leq{}& \mathfrak{O}(x_n, p) + \alpha_n \big(k_n + \gamma_{l(n)}^{i(n)} \mathfrak{M}\big) \Big[(1 + \beta_n k_n + \beta_n \gamma_{l(n)}^{i(n)} \mathfrak{M}) \mathfrak{O}(x_n, p) + \beta_n \mu_{l(n)}^{i(n)}\Big] \\
&+ \alpha_n \mu_{l(n)}^{i(n)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \Big(1+\alpha_n\Big(k_n+\gamma_{l(n)}^{i(n)}\mathfrak{M}+\beta_n k_n^2+2\beta_n k_n\gamma_{l(n)}^{i(n)}\mathfrak{M}+\beta_n(\gamma_{l(n)}^{i(n)}\mathfrak{M})^2\Big)\Big)\mathfrak{D}(x_n,p)\\
&\qquad\qquad +\alpha_n\beta_n\big(k_n+\gamma_{l(n)}^{i(n)}\mathfrak{M}\big)\mu_{l(n)}^{i(n)}+\alpha_n\mu_{l(n)}^{i(n)} \qquad (13)\\
&\leq (1+\sigma_n)\mathfrak{D}(x_n,p)+\xi_n, \forall n\geq 1 \text{ and } p\in\mathbb{F},
\end{aligned}
$$

where $\sigma_n = b\Big(k_n + \beta_n k_n^2 + \gamma_{l(n)}^{i(n)}\mathfrak{M} + 2\beta_n k_n\gamma_{l(n)}^{i(n)}\mathfrak{M} + \beta_n(\gamma_{l(n)}^{i(n)}\mathfrak{M})^2\Big)$, $\xi_n = b\big(\beta_n k_n + \beta_n\gamma_{l(n)}^{i(n)}\mathfrak{M}\big)\mu_{l(n)}^{i(n)} + \mu_{l(n)}^{i(n)}$ (because $\alpha_n,\beta_n\in[a,b]$). Therefore

$$
\mathfrak{D}(x_{n+1},p)\leq(1+\sigma_n)\mathfrak{D}(x_n,\mathbb{F})+\xi_n,\forall n\geq 1. \qquad (14)
$$

By using conditions (i) and (ii), we have

$$
\sum_{n=1}^{\infty}\sigma_n<\infty \quad \text{and} \quad \sum_{n=1}^{\infty}\xi_n<\infty. \qquad (15)
$$

By Lemma 4, $\lim_{n\to\infty}\mathfrak{D}(x_n,\mathbb{F})$ and $\lim_{n\to\infty}\mathfrak{D}(x_n,p)$ exist for each $p\in\mathbb{F}$. □

Lemma 7 *Let $\mathcal{N}$ be a nonempty closed and convex subset of a complete uniformly convex CAT(κ) space $(\mathcal{M},\mathfrak{D})$ with $\kappa>0$ and diam($\mathcal{M}$) $\leq \frac{\frac{\pi}{2}-\epsilon}{\sqrt{\kappa}}$ for some $\epsilon\in(0,\frac{\pi}{2})$. And, for each $i\geq 1$, let $\mathcal{U}_i,\mathcal{V}_i:\mathcal{N}\to\mathcal{N}$ be two uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\},\{\mu_n^{(i)}\},\zeta^{(i)})$-total asymptotically quasi-nonexpansive mappings defined by (7) such that*

- (i) $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\gamma_n^{(i)}<\infty$, $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\mu_n^{(i)}<\infty$;
- (ii) $\sum_{n=1}^{\infty}\alpha_n k_n<\infty$, $\sum_{n=1}^{\infty}\beta_n k_n<\infty$ *and* $\alpha_n\to 0$, $\beta_n\to 0$;
- (iii) *there exists a constant* $\mathfrak{M}>0$ *such that* $\zeta^{(i)}(\theta)\leq\mathfrak{M}\cdot\theta,\forall\theta\geq 0, i=1,2,3,\dots$;
- (iv) *there exist constants* $a,b\in(0,1)$ *with* $0<b(1-a)\leq\frac{1}{2}$ *such that* $\{\alpha_n\},\{\beta_n\}\subset[a,b]$.

Suppose that $\{x_n\}$ is a sequence defined by (8), then

$$
\lim_{n\to\infty}\mathfrak{D}(x_n,\mathcal{U}_{i(n)}^{l(n)}x_n)=0 \quad \textit{and} \quad \lim_{n\to\infty}\mathfrak{D}(x_n,\mathcal{V}_{i(n)}^{l(n)}x_n)=0.
$$

In particular, we have

$$
\lim_{m(\in\Gamma_i)\to\infty}\mathfrak{D}(x_m,\mathcal{U}_i x_m)=0 \quad \textit{and} \quad \lim_{m(\in\Gamma_i)\to\infty}\mathfrak{D}(x_m,\mathcal{V}_i x_m)=0
$$

for $i\geq 1$.

Proof (1). Firstly, we will prove that

$$
\lim_{n\to\infty}\mathfrak{D}(x_n,\mathcal{U}_{i(n)}^{l(n)}x_n)=0 \quad \text{and} \quad \lim_{n\to\infty}\mathfrak{D}(x_n,\mathcal{V}_{i(n)}^{l(n)}x_n)=0.
$$

In fact, it follows from Lemma 6 that for each given $p\in\mathbb{F}$, the $\lim_{n\to\infty}\mathfrak{D}(x_n,p)$ exists. Without loss of generality, we can assume that

$$
\lim_{n\to\infty}\mathfrak{D}(x_n,p)=r\geq 0. \qquad (16)
$$

From (11) we have

$$
\limsup_{n\to\infty}\mathfrak{D}(y_n,p)\leq\lim_{n\to\infty}\{\big(1+\beta_n k_n+\beta_n u_{l(n)}^{i(n)}\mathfrak{M}\big)\mathfrak{D}(x_n,p)+\beta_n\mu_{l(n)}^{i(n)}\}=r. \qquad (17)
$$

Since

$$\begin{aligned}\mathfrak{D}(\mathcal{U}_{i(n)}^{k(n)}y_n,p) &\le k_n\mathfrak{D}(y_n,p)+\gamma_{l(n)}^{i(n)}\zeta^{i(n)}\big(\mathfrak{D}(y_n,p)\big)+\mu_{k(n)}^{i(n)}\\ &\le (k_n+\gamma_{l(n)}^{i(n)}\mathfrak{M})\mathfrak{D}(y_n,p)+\mu_{l(n)}^{i(n)},\quad \forall n\ge 1,\end{aligned}$$

from (17) we get

$$\limsup_{n\to\infty}\mathfrak{D}(\mathcal{U}_{i(n)}^{l(n)}y_n,p)\le r. \tag{18}$$

In addition, it follows from (13) that

$$\begin{aligned}\mathfrak{D}(x_{n+1},p) &= \mathfrak{D}((1-\alpha_n)x_n\bigoplus\alpha_n\mathcal{U}_{i(n)}^{l(n)}y_n,p)\\ &\le \Big(1+\alpha_n\Big(k_n+\gamma_{l(n)}^{i(n)}\mathfrak{M}+\beta_nk_n^2+2\beta_nk_n\gamma_{l(n)}^{i(n)}\mathfrak{M}+\beta_n(\gamma_{l(n)}^{i(n)}\mathfrak{M})^2\Big)\Big)\mathfrak{D}(x_n,p)\\ &\quad +\alpha_n\beta_n(k_n+\gamma_{l(n)}^{i(n)}\mathfrak{M})\mu_{l(n)}^{i(n)}+\alpha_n\mu_{l(n)}^{i(n)},\ \forall n\ge 1,\ p\in\mathbb{F}.\end{aligned} \tag{19}$$

This implies that

$$\lim_{n\to\infty}\mathfrak{D}((1-\alpha_n)x_n\bigoplus\alpha_n\mathcal{U}_{i(n)}^{l(n)}y_n,p)=r. \tag{20}$$

From (16), (18), (20), and Proposition 1, we have

$$\lim_{n\to\infty}\mathfrak{D}(x_n,\mathcal{U}_{i(n)}^{l(n)}y_n)=0. \tag{21}$$

Since

$$\begin{aligned}\mathfrak{D}(x_n,p) &\le \mathfrak{D}(x_n,\mathcal{U}_{i(n)}^{l(n)}y_n)+\mathfrak{D}(\mathcal{U}_{i(n)}^{l(n)}y_n,p)\\ &\le \mathfrak{D}(x_n,\mathcal{U}_{i(n)}^{l(n)}y_n)+\{k_n\mathfrak{D}(y_n,p)+\gamma_{l(n)}^{i(n)}\zeta^{i(n)}\big(\mathfrak{D}(y_n,p)\big)+\mu_{l(n)}^{i(n)}\}\\ &\le \mathfrak{D}(x_n,\mathcal{U}_{i(n)}^{l(n)}y_n)+(k_n+\gamma_{l(n)}^{i(n)}\mathfrak{M})\mathfrak{D}(y_n,p)+\mu_{l(n)}^{i(n)}.\end{aligned} \tag{22}$$

Taking lim inf on both sides of the above inequality and using (21), we have

$$r\le\liminf_{n\to\infty}\mathfrak{D}(y_n,p).$$

From this together with (17), we obtain

$$\lim_{n\to\infty}\mathfrak{D}(y_n,p)=r. \tag{23}$$

Using (11) and (23), we have

$$\begin{aligned}r=\lim_{n\to\infty}\mathfrak{D}(y_n,p) &= \lim_{n\to\infty}\{\mathfrak{D}\big((1-\beta_n)x_n\bigoplus\beta_n\mathcal{U}_{i(n)}^{l(n)}x_n,p\big)\}\\ &\le \lim_{n\to\infty}[\big(1+\beta_nk_n+\beta_n\gamma_{l(n)}^{i(n)}\mathfrak{M}\big)\mathfrak{D}(x_n,p)+\beta_n\mu_{l(n)}^{i(n)}]\\ &= r.\end{aligned} \tag{24}$$

This yields

$$\lim_{n\to\infty}\{\mathfrak{D}((1-\beta_n)x_n \bigoplus \beta_n \mathcal{U}_{i(n)}^{l(n)}x_n, p)\} = r. \tag{25}$$

Now, we have

$$\begin{aligned}\limsup_{n\to\infty}\rho(\mathcal{U}_{i(n)}^{l(n)}x_n,p) &\leq \limsup_{n\to\infty}[k_n\mathfrak{D}(x_n,p) + \gamma_{l(n)}^{i(n)}\zeta^{i(n)}\big(\mathfrak{D}(x_n,p)\big) + \mu_{l(n)}^{i(n)}] \\ &\leq \limsup_{n\to\infty}[\big(k_n + \gamma_{k_n}^{i(n)}\mathfrak{M}\big)\mathfrak{D}(x_n,p) + \mu_{l(n)}^{i(n)}] = r.\end{aligned} \tag{26}$$

Applying (16), (25), (26) and Proposition 1, we obtain

$$\lim_{n\to\infty}\mathfrak{D}(x_n,\mathcal{U}_{i(n)}^{l(n)}x_n) = 0. \tag{27}$$

Similarly, one can show that

$$\lim_{n\to\infty}\mathfrak{D}(x_n,\mathcal{V}_{i(n)}^{l(n)}x_n) = 0. \tag{28}$$

(2). Secondly, we show that for each $i \geq 1$ there exists some subsequence $\{x_{m(\in\Gamma_i)}\} \subset \{x_n\}$ such that

$$\lim_{m(\in\Gamma_i)\to\infty}\mathfrak{D}(x_m,\mathcal{U}_i x_m) = 0 \quad \textit{and} \lim_{m(\in\Gamma_i)\to\infty}\mathfrak{D}(x_m,\mathcal{V}_i x_m) = 0, \tag{29}$$

where Γ_i is the set of positive integers defined by Lemma 5(2). By using (27), we get

$$\begin{aligned}\mathfrak{D}(x_n,y_n) &= \rho\big(x_n,(1-\beta_n)x_n \bigoplus \mathcal{U}_{i(n)}^{l(n)}x_n\big) \\ &\leq \beta_n\rho(x_n,\mathcal{U}_{i(n)}^{l(n)}x_n) \to 0 \ \ (\text{as}\, n\to\infty).\end{aligned} \tag{30}$$

Furthermore, it follows from (21) that

$$\begin{aligned}\mathfrak{D}(x_{n+1},x_n) &= \rho\big((1-\alpha_n)x_n \bigoplus \mathcal{U}_{i(n)}^{l(n)}y_n,x_n\big) \\ &\leq \alpha_n\rho(\mathcal{U}_{i(n)}^{l(n)}y_n,x_n) \to 0 \ (\text{as}\, n\to\infty).\end{aligned} \tag{31}$$

From (30) and (31), we get

$$\mathfrak{D}(x_{n+1},y_n) \leq \mathfrak{D}(x_{n+1},x_n) + \mathfrak{D}(x_n,y_n) \to 0 \ \ (\text{as}\, n\to\infty). \tag{32}$$

From (21), (27), (31), (32), and Lemma 5, for each given positive integer $i \geq 1$, there exist subsequences $\{x_m\}_{m\in\Gamma_i},\{y_m\}_{m\in\Gamma_i}$, and $\{l(m)\}_{m\in\Gamma_i} \subset \mathcal{K}_i := \{l(m) : m \in \Gamma_i, m = i + \dfrac{(l(m)-1)l(m)}{2}, l(m) \geq i\}$, we have that

$$\begin{aligned}\mathfrak{D}(x_m,\mathcal{U}_i x_m) &\leq \mathfrak{D}(x_m,\mathcal{U}_i^{l(m)}x_m) + \mathfrak{D}(\mathcal{U}_i^{l(m)}x_m,\mathcal{U}_i^{l(m)}y_{m-1}) + \mathfrak{D}(\mathcal{U}_i^{l(m)}y_{m-1},\mathcal{U}_i x_m) \\ &\leq \mathfrak{D}(x_m,\mathcal{U}_i^{l(m)}x_m) + L_i\mathfrak{D}(x_m,y_{m-1}) + L_i\mathfrak{D}(\mathcal{U}_i^{l(m)-1}y_{m-1},x_m) \\ &\leq \mathfrak{D}(x_m,\mathcal{U}_i^{l(m)}x_m) + L_i\mathfrak{D}(x_m,y_{m-1}) + L_i\mathfrak{D}(\mathcal{U}_i^{l(m)-1}y_{m-1},x_{m-1}) \\ &\quad + L_i\mathfrak{D}(x_{m-1},x_m).\end{aligned} \tag{33}$$

This yields

$$\lim_{m(\in\Gamma_i)\to\infty} \mathfrak{D}(x_m,\mathcal{U}_i x_m) = 0. \tag{34}$$

Similarly, one can show that

$$\lim_{m(\in\Gamma_i)\to\infty} \mathfrak{D}(x_m,\mathcal{V}_i x_m) = 0. \tag{35}$$

This completes our proof of Lemma 7. □

Theorem 3 *Let $\mathcal{N}$ be a nonempty, closed, and convex subset of a complete uniformly convex CAT(κ) space $(\mathcal{M},\mathfrak{D})$ with $\kappa > 0$ and diam$(\mathcal{M}) \leq \frac{\frac{\pi}{2}-\epsilon}{\sqrt{\kappa}}$ for some $\epsilon \in (0,\frac{\pi}{2})$. And, for each $i \geq 1$, let $\mathcal{U}_i,\mathcal{V}_i : \mathcal{N}\to\mathcal{N}$ be two uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\},\{\mu_n^{(i)}\},\zeta^{(i)})$-total asymptotically quasi-nonexpansive mappings defined by (7) such that*

(i) $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\gamma_n^{(i)} < \infty$, $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\mu_n^{(i)} < \infty$;

(ii) $\sum_{n=1}^{\infty}\alpha_n k_n < \infty$. $\sum_{n=1}^{\infty}\beta_n k_n < \infty$, *and* $\alpha_n \to 0$, $\beta_n \to 0$;

(iii) *there exists a constant* $\mathfrak{M} > 0$ *such that* $\zeta^{(i)}(\theta) \leq \mathfrak{M}\cdot\theta, \forall\theta \geq 0, i = 1,2,3,\ldots$;

(iv) *there exist constants* $a,b \in (0,1)$ *with* $0 < b(1-a) \leq \frac{1}{2}$ *such that* $\{\alpha_n\},\{\beta_n\} \subset [a,b]$.

If $\mathbb{F} := \cap_{i=1}^{\infty}\big(F(\mathcal{U}_i)\cap F(\mathcal{V}_i)\big) \neq \emptyset$ and there exist mappings $\mathcal{U}_{n_0} \in \{\mathcal{U}_i\}_{i=1}^{\infty}$, $\mathcal{V}_{n_0} \in \{\mathcal{V}_i\}_{i=1}^{\infty}$ and a nondecreasing function $h : [0,\infty) \to [0,\infty)$ with $h(0) = 0$ and $h(r) > 0, \forall r > 0$ such that

$$h(\mathfrak{D}(x_n,\mathbb{F})) \leq \mathfrak{D}(x_n,\mathcal{U}_{n_0}x_n),\ \forall n \geq 1, \tag{36}$$

and

$$h(\mathfrak{D}(x_n,\mathbb{F})) \leq \mathfrak{D}(x_n,\mathcal{V}_{n_0}x_n), \forall n \geq 1. \tag{37}$$

Then the sequence $\{x_n\}$ defined by (8) converges strongly to a common fixed point $x* \in \mathbb{F}$.

Proof In fact, it follows from Lemma 7 that for given mappings $\mathcal{U}_{n_0},\mathcal{V}_{n_0}$ there exists some subsequence $\{x_m\}_{m\in\Gamma_{n_0}}$ of $\{x_n\}$ such that

$$\lim_{m(\in\Gamma_i)\to\infty} \mathfrak{D}(x_m,\mathcal{U}_{n_0}x_m) = 0 \quad \text{and} \lim_{m(\in\Gamma_i)\to\infty} \mathfrak{D}(x_m,\mathcal{V}_{n_0}x_m) = 0. \tag{38}$$

By (36) and (37) we have

$$h(\mathfrak{D}(x_m,\mathbb{F})) \leq \mathfrak{D}(x_m,\mathcal{U}_{n_0}x_m),\ \forall m \geq 1, \tag{39}$$

and

$$h(\mathfrak{D}(x_m,\mathbb{F})) \leq \mathfrak{D}(x_m,\mathcal{V}_{n_0}x_m),\ \forall m \geq 1. \tag{40}$$

Taking the limit as $m \to \infty$ on the above inequalities, we have $\lim_{m\to\infty} h(\mathfrak{D}(x_m,\mathbb{F})) = 0$. This implies that

$$\lim_{m(\in\Gamma_{n_0})\to\infty} \mathfrak{D}(x_m,\mathbb{F})) = 0. \tag{41}$$

Next we show that $\{x_m\}_{m\in\Gamma_{n_0}}$ is a Cauchy sequence in $\mathcal{N}$. In fact, it follows from (13) that for any $p \in \mathbb{F}$,

$$\mathfrak{D}(x_{m+1},p) \le (1+\sigma_m)\mathfrak{D}(x_m,p) + \xi_m, \forall m \ge 1, \ \forall m \in \Gamma_{n_0}, \tag{42}$$

where $\sum_{m=1}^{\infty}\sigma_m < \infty$ and $\sum_{m=1}^{\infty}\xi_m < \infty$. For any positive integers $j,n \in \Gamma_{n_0}, n > j$, let $n = m+j$ for some positive integer m, and since $1+x \le e^x$ for each $x \ge 0$, by following the same line of proof of Theorem 2 (i.e. Theorem 3.2 in [6]), we will obtain

$$\begin{aligned}\mathfrak{D}(x_n,x_j) = \mathfrak{D}(x_{m+j},x_j) &\le (1+K)\mathfrak{D}(x_j,p) + K\sum_{i=j}^{m+j-1}\xi_i \\ &\le (1+K)\mathfrak{D}(x_j,p) + K\sum_{i=j}^{n-1}\xi_i \quad \text{for each } p \in \mathbb{F}.\end{aligned}$$

This implies

$$\mathfrak{D}(x_n,x_j) \le (1+K)\mathfrak{D}(x_j,\mathbb{F}) + K\sum_{i=j}^{n-1}\xi_i,$$

where $K = e^{(\sum_{i=1}^{\infty}\sigma_i)} < \infty$. From (13) and (41), we have

$$\mathfrak{D}(x_n,x_j) \le (1+K)\mathfrak{D}(x_j,\mathbb{F}) + K\sum_{i=j}^{n-1}\xi_i \to 0\,(\text{as } n,j(\in \Gamma_{n_0}) \to \infty).$$

Therefore the subsequence $\{x_m\}_{m\in\Gamma_{n_0}} \subset \mathcal{N}$ is a Cauchy sequence. We deduce that $\mathcal{N}$ is complete since it is a closed subset in a complete CAT(κ) space $\mathcal{M}$. Therefore, we can assume that the subsequence $\{x_m\}$ converges strongly to some common fixed point $x^* \in \mathcal{N}$. We know that $\mathbb{F}$ is a closed subset in $\mathcal{N}$ and that $\lim_{m\to\infty}\mathfrak{D}(x_m,\mathbb{F}) = 0$, so $x^* \in \mathbb{F}$. By Lemma 6 and Lemma 4, we can conclude that the whole sequence $\{x_n\}$ converges strongly to a common fixed point $x^* \in \mathbb{F}$. Our proof is finished. □

Remark 6 Note that we add one more condition (i.e. condition (ii)) in Theorem 3, which is different from the work of Chang et al. [6] (see Theorem 2 above), to obtain our main result.

We now prove the Δ-convergence result.

Theorem 4 *Let $\mathcal{N}$ be a nonempty closed and convex subset of a complete uniformly convex CAT(κ) space $(\mathcal{M},\mathfrak{D})$ with $\kappa > 0$ and diam($\mathcal{M}$) $\le \frac{\frac{\pi}{2}-\epsilon}{\sqrt{\kappa}}$ for some $\epsilon \in (0,\frac{\pi}{2})$. And, for each $i \ge 1$, let $\mathcal{U}_i, \mathcal{V}_i : \mathcal{N} \to \mathcal{N}$ be two uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\},\{\mu_n^{(i)}\},\zeta^{(i)})$-total asymptotically quasi-nonexpansive mappings defined by (7) such that*

(i) $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\gamma_n^{(i)} < \infty, \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\mu_n^{(i)} < \infty$;
(ii) $\sum_{n=1}^{\infty}\alpha_n k_n < \infty, \sum_{n=1}^{\infty}\beta_n k_n < \infty$, *and* $\alpha_n \to 0$, $\beta_n \to 0$;
(iii) $0 < L_i < 1$, *for each* i;
(iv) *there exists a constant* $\mathfrak{M} > 0$ *such that* $\zeta^{(i)}(\theta) \le \mathfrak{M}\cdot\theta, \forall\theta \ge 0, i = 1,2,\dots$;

(v) *there exist constants* $a, b \in (0,1)$ *with* $0 < b(1-a) \leq \frac{1}{2}$ *such that* $\{\alpha_n\}, \{\beta_n\} \subset [a,b]$.
Then the sequence $\{x_n\}$ *defined by* (8) Δ*-converges strongly to a common fixed point of* $\{\mathcal{U}_i : i \geq 1\}$ *and* $\{\mathcal{V}_i : i \geq 1\}$.

Proof It follows from Lemma 6 that $\{x_n\}$ is bounded. Therefore, by Lemma 2, $\{x_n\}$ has a unique asymptotic center. Assume that $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. From Lemma 7 we have $\lim_{n\to\infty} \mathfrak{D}(u_n, \mathcal{U}_i u_n) = 0 = \lim_{n\to\infty} \mathfrak{D}(u_n, \mathcal{V}_i u_n)$ for each $i \geq 1$.

We will prove that u is a common fixed point of $\{\mathcal{U}_i : i \geq 1\}$ and $\{\mathcal{V}_i : i \geq 1\}$. For each given positive integer $i \geq 1$, and $\{l(m)\}_{m\in\Gamma_i} \subset \mathcal{K}_i := \{l(m) : m \in \Gamma_i, m = i + \frac{(l(m)-1)l(m)}{2}, l(m) \geq i\}$, we define a sequence $\{z_m\}$ in $\mathcal{N}$ by $z_m = \mathcal{U}_i^{l(m)} u$. Observe that

$$\begin{aligned}
\mathfrak{D}(z_m, u_n) &\leq \mathfrak{D}(\mathcal{U}_i^{l(m)} u, \mathcal{U}_i^{l(m)} u_n) + \mathfrak{D}(\mathcal{U}_i^{l(m)} u_n, \mathcal{U}_i^{l(m)-1} u_n) + \cdots + \mathfrak{D}(\mathcal{U}_i u_n, u_n) \\
&\leq L_i \mathfrak{D}(u, u_n) + L_i \mathfrak{D}(\mathcal{U}_i u_n, u_n) + \cdots + \mathfrak{D}(\mathcal{U}_i u_n, u_n) \\
&\leq L_i \mathfrak{D}(u, u_n) + K\mathfrak{D}(\mathcal{U}_i u_n, u_n) \\
&\leq \mathfrak{D}(u, u_n) + K\mathfrak{D}(\mathcal{U}_i u_n, u_n), (\text{some constant } K \text{ and } 0 < L_i < 1).
\end{aligned}$$

This implies

$$r(z_m, \{u_n\}) = \limsup_{n\to\infty} \mathfrak{D}(z_m, u_n) \leq \limsup_{n\to\infty} \mathfrak{D}(u, u_n) = r(u, \{u_n\}).$$

Hence $|r(z_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. It follows from Lemma 2 that $\mathcal{U}_i u = u$. Hence u is the common fixed point of $\{\mathcal{U}_i : i \geq 1\}$. Similarly, one can show that u is a common fixed point of $\{\mathcal{V}_i : i \geq 1\}$. Therefore u is a common fixed point of $\{\mathcal{U}_i : i \geq 1\}$ and $\{\mathcal{V}_i : i \geq 1\}$. Note that $\lim_{n\to\infty} \mathfrak{D}(x_n, u)$ exists by Lemma 6.

Suppose $x \neq u$. By the uniqueness of asymptotic centers, and following the same method as in the proof of Theorem 3.4 in [22], we can reach a contradiction. Hence $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ-converges to a common fixed point of $\{\mathcal{U}_i : i \geq 1\}$ and $\{\mathcal{V}_i : i \geq 1\}$. □

4 Conclusions

In this manuscript, we establish new results concerning two infinite families of uniformly L-Lipschitzian and total asymptotically quasi-nonexpansive mappings in CAT(κ) spaces with $\kappa > 0$. These mappings are essentially more general than nonexpansive mappings, asymptotically quasi-nonexpansive mapping, and total asymptotically nonexpansive mappings in the intermediate sense. Our results are a refinement, generalization of the result recently obtained by Chang et al. [6]. Besides, we also establish a Δ-convergence result for such mappings. As a further development, one can use the background of CAT(κ) spaces with $\kappa > 0$ and other more general metric spaces to study implicit type contractive conditions inspired by those from the current work.

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Author contributions

BN contributed to the conceptualization, formal analysis, methodology, writing, editing, and approving the manuscript. PK involved in formal analysis, methodology and writing the original draft. JN involved in formal analysis, editing and approving the manuscript. KS involved in formal analysis and and approving the manuscript. All authors read and approved the final manuscript.

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Data Availability

No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate

Not applicable.

Consent for publication

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Competing interests

The authors declare no competing interests.

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References

1. Alber, Y.I., Chidume, C.E., Zegeye, H.: Approximating fixed points of total asymptotically nonexpansive mappings. Fixed Point Theory Appl. **2006**, Article ID 10673 (2006)
2. Basarir, M., Sahin, A.: On the strong and convergence for total asymptotically nonexpansive mappings on a CAT(0) space. Carpath. Math. Publ. **5**, 170–179 (2013)
3. Bridson, M., Haefliger, A.: Metric Spaces of Non-Positive Curvature. Springer, Berlin (1999)
4. Bruck, R.E., Kuczumow, Y., Reich, S.: Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property. Colloq. Math. **65**(2), 169?179 (1993)
5. Chang, S.S., Wang, L., Lee, H.W.J., Chan, C.K.: Strong and convergence for mixed type total asymptotically nonexpansive mappings in CAT(0) spaces. Fixed Point Theory Appl. **2013**, Article ID 122 (2013)
6. Chang, S., Zhao, L., Liu, M., Tang, J.: Convergence theorems for total asymptotically nonexpansive mappings in CAT(κ) spaces. Fixed Point Theory Algorithms Sci. Eng. **2023**, Article ID 2 (2023). https://doi.org/10.1186/s13663-022-00739-2
7. Chang, S.S., Wang, L., Lee, H.W.J., Chan, C.K., Yang, L.: Demiclosed principle and Δ-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces. Appl. Math. Comput. **219**, 2611–2617 (2012)
8. Espinola, R., Nicolae, A.: Geodesic Ptolemy spaces and fixed points. Nonlinear Anal. **74**(1), 27–34 (2011)
9. Fukhar-ud-din, H., Khan, A.R.: Convergence of implicit iterates with errors for mappings with unbounded domain in Banach spaces. Int. J. Math. Math. Sci. **10**, 1643–1653 (2005)
10. Hea, J.S., Fang, D.H., Lopez, G., Li, C.: Mann's algorithm for nonexpansive mappings in CAT(κ) spaces. Nonlinear Anal. **75**, 445–452 (2012)
11. Karapinar, E., Salahifard, H., Vaezpour, S.M.: Demiclosedness principle for total asymptotically nonexpansive mappings in CAT(0) spaces. J. Appl. Math. **2014**, Article ID 738150 (2014)
12. Khan, A.R., Fukhar-ud-din, H., Khan, M.A.A.: An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces. Fixed Point Theory Appl. **2012**, 54 (2012)
13. Khan, A.R., Khamsi, M.A., Fukhar-ud-din, H.: Strong convergence of a general iteration scheme in CAT(0) spaces. Nonlinear Anal. **74**, 783–791 (2011)
14. Kingkam, P., Nantadilok, J.: Iterative process for finding fixed points of quasi-nonexpansive multimaps in CAT(0) spaces. Korean J. Math. **31**(1), 35–48 (2023). https://doi.org/10.11568/kjm.2023.31.1.35
15. Kirk, W.A.: Geodesic geometry and fixed point theory. In: Seminar of Math. Anal. (Malaga/Seville, 2002/2003). Colecc. Abierta, vol. 64, pp. 195–225. Univ. Sevilla Secr. Pub., Seville (2003)
16. Kirk, W.A.: Geodesic geometry and fixed point theory II. In: International Conference on Fixed Point Theo. Appl., pp. 113–142. Yokohama Pub., Yokohama (2004)
17. Kirk, W.A., Panyanak, B.: A concept of convergence in geodesic spaces. Nonlinear Anal. **68**, 3689–3696 (2008)
18. Kratuloek, K., Kumam, P., Amnuaykarn, K., Nantadilok, J., Salisu, S.: Iterative approximation of common fixed points of two nonself asymptotically nonexpansibve mappings in CAT(0) spaces with numerical examples. Math. Methods Appl. Sci., 1–19 (2023)
19. Lim, T.C.: Remarks on some fixed point theorems. Proc. Am. Math. Soc. **60**, 179–182 (1976)
20. Liu, J.A.: Some convergence theorems of implicit iterative process for nonexpansive mappings in Banach spaces. Math. Commun. **7**, 113–118 (2002)
21. Nantadilok, J., Nuntadilok, B.: Common fixed points of asymptotically quasi-nonexpansive mappings in CAT(0) spaces. Fixed Point Theory and Chaos (2022)

22. Nuntadilok, B., Kingkam, P., Nantadilok, J.: Common fixed point theorems of two finite families of asymptotically quasi-nonexpansive mappings in hyperbolic spaces. J. Nonlinear Funct. Anal. **2023**, 27 (2023)
23. Panyanak, B.: On total asymptotically nonexpansive mappings in CAT(κ) spaces. J. Inequal. Appl. (2014)
24. Sun, Z.H., He, C., Ni, Y.Q.: Strong convergence of an implicit iteration process for nonexpansive mappings in Banach space. Nonlinear Funct. Anal. Appl. **8**(4), 595–602 (2003)
25. Tang, J.F., Chang, S.S., Lee, H.W.J., Chan, C.K.: Iterative algorithm and convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces. Abstr. Appl. Anal. **2012**, Article ID 965751 (2012)
26. Tang, Y.K., Chang, S.S., Wang, L., Xu, Y.G., Wang, G.: Strong convergence theorems of modified Halpern-Mann-type iterations for quasi-ϕ-asymptotically nonexpansive mappings and applications. Acta Math. Sci. Ser. A **34**(5), 1151–1160 (2014)
27. Wang, L., Chang, S.S., Ma, Z.: Convergence theorems for total asymptotically nonexpansive non-selfmappings in CAT(0) spaces. J. Inequal. Appl. **2013**, Article ID 135 (2013)
28. Yang, L., Zhao, F.H.: Strong and convergence theorems for total asymptotically nonexpansive non-selfmappings in CAT(0) spaces. J. Inequal. Appl. **2013**, Article ID 557 (2013)
29. Zhao, L.C., Chang, S.S., Wang, X.R.: Convergence theorems for total asymptotically nonexpansive mappings in hyperbolic spaces. J. Appl. Math. **2013**, Article ID 689765 (2013)

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