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RESEARCH

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Weak and strong convergence theorems for a new class of enriched strictly pseudononspreading mappings in Hilbert spaces

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Abstract

Let Ω be a nonempty closed convex subset of a real Hilbert space $\mathfrak{H}$. Let $\Im$ be a nonspreading mapping from Ω into itself. Define two sequences $\{\psi_n\}_{n=1}^{\infty}$ and $\{\phi_n\}_{n=1}^{\infty}$ as follows:

$$\begin{cases} \psi_{n+1} = \pi_n\psi_n + (1-\pi_n)\Im\psi_n, \\ \phi_n = \frac{1}{n}\sum_{t=1}^{n}\psi_t, \end{cases}$$

for $n \in N$, where $0 \le \pi_n \le 1$, and $\pi_n \to 0$. In 2010, Kurokawa and Takahashi established weak and strong convergence theorems of the sequences developed from the above Baillion-type iteration method (Nonlinear Anal. 73:1562–1568, 2010). In this paper, we prove weak and strong convergence theorems for a new class of (η,β)-enriched strictly pseudononspreading ((η,β)-ESPN) maps, more general than that studied by Kurokawa and W. Takahashi in the setup of real Hilbert spaces. Further, by means of a robust auxiliary map incorporated in our theorems, the strong convergence of the sequence generated by Halpern-type iterative algorithm is proved thereby resolving in the affirmative the open problem raised by Kurokawa and Takahashi in their concluding remark for the case in which the map $\Im$ is averaged. Some nontrivial examples are given, and the results obtained extend, improve, and generalize several well-known results in the current literature.

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1 Introduction

From time immemorial, it has been an indisputable fact that the exact solutions of several physically modeled problems of the form

$$\Im\psi = \wp \tag{1.1}$$

are either very difficult to attain or relatively impossible to solve. Considering the influence of such a problem as (1.1) in human existence, it becomes necessary to seek approximate solutions. Luckily, problem (1.1) can be reduced to a fixed point (FP) problem of the form

$$\Im\psi = \psi. \tag{1.2}$$

Research findings show that the solution of (1.2) is achievable through approximate FP theorems, which does not only provide vital information on the existence of such FP but also on its uniqueness.

Throughout this paper, we make the following assumptions: $\mathfrak{H}$ is a real Hilbert space, $\emptyset \neq \Omega \subset \mathfrak{H}$ is a closed convex set, $\mathcal{N}$, $\mathcal{R}$, and $\mathcal{F}(\Im)$ are the sets of positive integers, real numbers, and FPs of the map $\Im : \Omega \longrightarrow \Omega$, respectively.

Definition 1.1 Let $\Im : \Omega \longrightarrow \Omega$ be a nonlinear map. Recall that

1. $\Im$ is called nonexpansive if

$$\|\Im\psi - \Im\wp\| \leq \|\psi - \wp\| \quad \forall \psi, \phi \in \Omega. \tag{1.3}$$

2. $\Im$ is called quasi-nonexpansive if $\mathcal{F}(\Im) \neq \emptyset$ and for all $(\psi, \vartheta) \in \Omega \times \mathcal{F}(\Im)$,

$$\|\Im\psi - \vartheta\| \leq \|\psi - \vartheta\|. \tag{1.4}$$

3. $\Im$ is called nonspreading [1] if for all $\psi, \wp \in \Omega$,

$$2\|\Im\psi - \Im\wp\|^2 \leq \|\Im\psi - \wp\|^2 + \|\Im\wp - \psi\|^2. \tag{1.5}$$

It is not difficult to show that (1.5) is equivalent to

$$\|\Im\psi - \Im\wp\|^2 \leq \|\psi - \wp\|^2 + 2\langle \wp - \Im\wp, \psi - \Im\rangle. \tag{1.6}$$

4. $\Im$ is called β-strictly pseudononspreading (β-SPN) [2] if there exists $\beta \in [0, 1)$ such that for all $\psi, \wp \in \Omega$,

$$\|\Im\psi - \Im\wp\|^2 \leq \|\psi - \wp\|^2 + \beta\|\psi - \Im\psi - (\wp - \Im\wp)\| + 2\langle \wp - \Im\wp, \psi - \Im\rangle. \tag{1.7}$$

Remark 1.1 It is easy to see from Definition 1.1 [(3) and (4)] that

(a) if (1.5) holds and $\mathcal{F}(\Im) \neq \emptyset$, then (1.4) immediately follows for all $\vartheta \in \mathcal{F}(\Im)$.

(b) if (1.5) holds, then (1.7) holds with $\beta = 0$; the converse is not true from the following example.

Example 1.1 Let R be endowed with the usual norm, and let $\Im:\mathcal{R}\longrightarrow\mathcal{R}$ be defined by

$$\Im\psi=\begin{cases}\psi, & \psi\in(-\infty,0],\\ -2\psi, & \psi\in[0,\infty).\end{cases}$$

Then $\Im$ satisfies (1.7) but not (1.5). Thus the class of maps satisfying (1.7) is more general than the class of maps satisfying (1.5).

In 2011, Osilike and Isiogugu [2] initiated the concept of β-SPN maps and established weak convergence result of Bailion type similar to that obtained in [1] and [3]. In addition, using the notion of mean convergence, they obtained strong convergence results similar to that established in [1] and hence brought to rest an open problem posed by Kurokawa and Takahashi [1] for the case in which the map $\Im$ is averaged.

On the other hand, the notion of enriched nonlinear maps was first introduced by Berinde [4] (see also [5] and [6]) in the setup of a real Hilbert space. This concept was later extended to the more general Banach space by Saleem et al. [7].

Definition 1.2 A map $\Im:\Omega\longrightarrow\Omega$ is called $(\eta,\Psi_\Im)$-enriched Lipshitizian (see [7]) if for all $\psi,\phi\in\Omega$, there exist $\eta\in[0,+\infty)$ and a continuous nondecreasing function $\Psi_\Im:R^+\longrightarrow R^+$ with $\Psi_\Im(0)=0$ such that

$$\|\eta(\psi-\phi)+\Im\psi-\Im\phi\|\le(\eta+1)\Psi_\Im(\|\psi-\phi\|). \tag{1.8}$$

The following particular cases emanating from inequality (1.8) are worth mentioning:

(i) if $\eta=0$, then inequality (1.8) reduces to the class of maps known as $\Psi_\Im$-enriched Lipschitzian;

(ii) if $\eta=0$ and $\Psi_\Im(t)=Lt$ for $L>0$, then (1.8) reduces to the class of maps called L-Lipschitzian. In a more particular case where $\eta=0$ and $\Psi_\Im(t)=t$, then the $\Psi_\Im$-enriched Lipschitzian map immediately reduces to the class of nonexpansive maps on Ω;

(iii) if $\Psi_\Im(t)=1$, then (1.8) becomes

$$\|\eta(\psi-\phi)+\Im\psi-\Im\phi\|\le(\eta+1)\|\psi-\phi\| \tag{1.9}$$

and is known as an η-enriched nonexpansive map. This class of maps was first studied by Berinde [5, 6] as a generalization of the well-known class of nonexpansive maps.

Note that if $\Psi_\Im$ is not necessarily nondecreasing and satisfies the condition

$$\Psi_\Im(t)<t,\quad t>0,$$

then we have the class of η-enriched nonexpansive maps.

Definition 1.3 [7] A map $\Im$ is called (η,λ)-enriched strictly pseudocontractive ((η,λ)-ESPC) if for all $\psi,\phi\in\Omega$, there exist $\eta\in[0,+\infty)$ and $j(\psi-\phi)\in J(\psi-\phi)$ such that

$$\langle\eta(\psi-\phi)+\Im\psi-\Im\phi,j((\eta+1)(\psi-\phi))\rangle\le(\eta+1)^2\|\psi-\phi\|^2$$

$$-\lambda\|\psi-\phi-(\Im\psi-\Im\phi)\|^2, \tag{1.10}$$

where $\lambda=\frac{1}{2}(1-\beta)$ for some $\beta\in[0,1)$.

In the setup of a real Hilbert space, inequality (1.10) is equivalent to

$$\|\eta(\psi-\phi)+\Im\psi-\Im\phi\|^2\leq(\eta+1)^2\|\psi-\phi\|^2+\beta\|\psi-\phi-(\Im\psi-\Im\phi)\|^2, \tag{1.11}$$

where $\beta=1-2\lambda$.

Saleem et al. [7] established that if Ω is a bounded close convex subset of a real Banach space and $\Im:\Omega\longrightarrow\Omega$ is a finite family of (η,λ)-ESPC maps, then $\Im$ has an FP in Ω.

In view of the above results, it is pertinent to consider the following question.

Question 1.1 Is it possible to prove the results in [2] for the class of maps more general than that studied in [2] and obtain the results in [2] as particular cases?

Berinde [4] considered the class of (η,λ)-ESPC maps and proved that if Ω is a bounded close convex subset of a real Hilbert space and $\Im:\Omega\longrightarrow\Omega$ is an (η,λ)-ESPC map, then $\Im$ has an FP in Ω. Osilike and Isiogugu [2] introduced and studied the class of β-SPN maps. Apart from providing an affirmative answer to the lingering open problem raised at the concluding remark of [1], the results they obtained extended and generalized the corresponding results in [1] and several others in the existing literature. Inspired by Berinde [4] and Osilike and Isiogugu [2], in this paper, we first introduce a new class of nonlinear maps called (η,β)-ESPN and give some nontrivial examples to demonstrate its existence. Further, we study the Bailion-type and Halpern-type iterative schemes and thereafter give an affirmative answer to Question 1.1.

2 Preliminaries

The following well-known results shall be helpful in the course of establishing our main results. Let $\mathfrak{H}$ be a real Hilbert space, and let $\{\psi_n\}\in\mathfrak{H}$. We will denote the weak convergence of $\{\psi_n\}$ to a point $\psi\in\mathfrak{H}$ by $\psi_n\rightharpoonup\psi$ and the strong convergence of $\{\psi_n\}$ to a point $\psi\in\mathfrak{H}$ by $\psi_n\to\psi$ as $n\to\infty$, respectively.

Let $\mathfrak{X}$ be a real Banach space. A map $\Im$ with domain $\mathfrak{D}(\Im)$ and range $\mathfrak{R}(\Im)$ in $\mathfrak{X}$ is called demiclosed at a point ϑ (see, for instance, [8]) if whenever $\{\psi_n\}$ is a sequence in $\mathfrak{D}(\Im)$ such that $\psi_n\rightharpoonup\psi\in\mathfrak{D}(\Im)$ and $\{\Im\psi_n\}$ converges strongly to $\wp$, then $\Im\psi=\wp$.

Lemma 2.1 ([2]) *Let $\mathfrak{H}$ be a real Hilbert space. Then*

(i)

$$\|\sigma\psi+(1-\sigma)\wp\|^2=\sigma\|\psi\|^2+(1-\sigma)\|\wp\|^2-\sigma(1-\sigma)\|\psi-\wp\|^2,$$

for all $\psi,\wp\in\mathfrak{H}$ and $\sigma\in[0,1]$;

(ii)

$$\|\psi+\wp\|^2\leq\|\psi\|^2+2\langle\wp,\psi+\wp\rangle\quad\forall\psi,\wp\in\mathfrak{H};$$

(iii) *if* $\{\psi_n\}$ *is a sequence in* $\mathfrak{H}$ *such that* $\psi_n \rightharpoonup \wp \in \mathfrak{H}$, *then*

$$\limsup_{n\to\infty} \|\psi_n - \hbar\|^2 = \limsup_{n\to\infty} \|\psi_n - \wp\|^2 + \|\wp - \hbar\|^2 \quad \forall \hbar \in \mathfrak{H}.$$

Consider a real Hilbert space $\mathfrak{H}$ and a closed convex set $\emptyset \neq \Omega \subset \mathfrak{H}$. The nearest point projection $\mathcal{P}_\Omega : \mathfrak{H} \longrightarrow \Omega$ is the operator that assigns to each $\psi \in \mathfrak{H}$ its nearest point, denoted by $\mathcal{P}_\Omega\psi$, in Ω. Thus $\mathcal{P}_\Omega\psi$ is the unique point in Ω such that

$$\|\psi - \mathcal{P}_\Omega\psi\| \leq \|\psi - \hbar\| \quad \forall \hbar \in \Omega.$$

It has been established that for each $\psi \in \mathfrak{H}$,

$$\langle \psi - \mathcal{P}_\Omega\psi, \hbar - \mathcal{P}_\Omega\psi\rangle \leq 0 \quad \forall \hbar \in \Omega. \tag{2.1}$$

Lemma 2.2 ([9]) *Consider a real Hilbert space* $\mathfrak{H}$, *a closed convex set* $\emptyset \neq \Omega \subset \mathfrak{H}$, *and a metric projection* $\mathcal{P}_\Omega : \mathfrak{H} \longrightarrow \Omega$. *Let* $\{\psi_n\}$ *be a sequence in* $\mathfrak{H}$ *such that*

$$\|\psi_{n+1} - \vartheta\| \leq \|\psi_n - \vartheta\|$$

for all $\vartheta \in \Omega$ *and* $n = 0, 1, 2, \ldots$. *Then* $\{\mathcal{P}_\Omega\psi_n\}$ *converges strongly to some* $u \in \Omega$.

Lemma 2.3 ([10, 11]) *Let* $\{\pi_n\}$ *be a sequence of nonnegative real numbers satisfying*

$$\nu_{n+1} \leq (1 - \pi_n)\nu_n + \pi_n\mu_n,$$

where $\{\nu_n\}$ *and* $\{\mu_n\}$ *are real sequences such that*

(i) $\{\nu_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \nu_n = \infty$;

(ii) $\limsup_{n\to\infty} \mu_n \leq 0$.

Then $\lim_{n\to\infty} \pi_n = 0$.

Lemma 2.4 ([12]) *Consider a real Hilbert space* $\mathfrak{H}$, *a closed convex set* $\emptyset \neq \Omega \subset \mathfrak{H}$, *and a* β*-SPN map* $\Im : \Omega \longrightarrow \Omega$ *such that* $\mathcal{F}(\Im) \neq \emptyset$. *Let* $\Im_\xi = \xi I + (1 - \xi)\Im$, $\xi \in [\beta, 1)$. *Then*:

1. $\mathcal{F}(\Im) = \mathcal{F}(\Im_\xi)$;
2. *the map* $I - \Im_\xi$ *is demiclosed at zero*;
3. $\|\Im\psi - \Im\psi\|^2 \leq \|\psi - \phi\|^2 + \dfrac{2}{1-\xi}\langle \psi - \Im\psi, \phi - \Im\phi\rangle$;
4. $\Im_\xi$ *is a quasi-nonexpansive map.*

3 Results and discussion

We need the following definition.

Definition 3.1 Consider a real Hilbert space $\mathfrak{H}$. A map $\Im$ with domain $\mathfrak{D}(\Im)$ and range $\mathfrak{R}(\Im)$ in $\mathfrak{H}$ is known as (η, β)-ESPN in the sense of Browder and Petryshyn [13] if there exist $\eta \in [0, \infty)$ and $\beta \in [0, 1)$ such that for all $(\psi, \phi) \in \mathfrak{D}(\Im)$,

$$\|\eta(\psi - \phi) + \Im\psi - \Im\phi\|^2 \leq (\eta + 1)^2\|\psi - \phi\|^2 + \beta\|\psi - \Im\psi - (\phi - \Im\phi)\|^2 + 2\langle \psi - \Im\psi, \phi - \Im\phi\rangle. \tag{3.1}$$

Remark 3.1 It is easy to see that if $\eta = 0$ in (3.1), then the class of maps known as β-SPN emerges. Further, if $\beta = 1$ in (3.1), then we have the class of η-enriched pseudononspreading maps. For the particular case $\beta = 0$, we obtain the class of η-enriched nonspreading maps.

Let $\omega = \dfrac{1}{\eta+1}$. Then it is clear that $\omega \in (0,1]$. In this case, inequality (3.1) becomes

$$\begin{aligned}&\left\|\frac{(1-\omega)}{\omega}(\psi-\phi)+\Im\psi-\Im\phi\right\|^2\\ &\quad\le \frac{1}{\omega^2}\|\psi-\phi\|^2+\beta\|\psi-\Im\psi-(\phi-\Im\phi)\|^2+2\langle\psi-\Im\psi,\phi-\Im\phi\rangle,\end{aligned}$$

which, on simplification, yields

$$\begin{aligned}&\|\Im_\omega\psi-\Im_\omega\phi\|^2\\ &\quad\le \|\psi-\phi\|^2+\beta\|\psi-\Im_\omega\psi-(\phi-\Im_\omega\phi)\|^2+2\langle\psi-\Im_\omega\psi,\phi-\Im_\omega\phi\rangle.\end{aligned}\tag{3.2}$$

Inequality (3.2) is equivalently written as

$$\begin{aligned}&\langle(I-\Im_\omega)\psi-(I-\Im_\omega)\phi,\psi-\phi\rangle\\ &\quad\ge \lambda\|\psi-\Im_\omega\psi-(\phi-\Im_\omega\phi)\|^2-\langle\psi-\Im_\omega\psi,\phi-\Im_\omega\phi\rangle,\end{aligned}\tag{3.3}$$

where $\Im_\omega = (1-\omega)I+\omega\Im$, $\lambda = \frac{1}{2}(1-\beta)$, and I is the identity operator on Ω. It not difficult to see from (3.2) that the average operator $\Im_\omega$ is strictly pseudononspreading.

The following example shows that the class of (η,β)-ESPN maps is larger than that of β-SPN maps.

Example 3.1 Let $\Im : [-2,2] \longrightarrow [-2,2]$ be defined by

$$\Im\psi = -\frac{5}{3}\psi, \quad \psi\in[-2,2].$$

Then we have

$$\begin{aligned}&|\eta(\psi-\phi)+\Im\psi-\Im\phi|^2 = \left(\eta-\frac{5}{3}\right)|\psi-\phi|^2,\\ &\frac{1}{4}|\psi-\Im\psi-(\phi-\Im\phi)|^2 = \frac{1}{4}\left|\psi+\frac{5}{3}\psi-\left(\phi+\frac{5}{3}\phi\right)\right|^2 = \left(\frac{1}{4}\right)\left(\frac{64}{9}\right)|\psi-\phi|^2,\\ &\quad 2\langle\psi-\Im\psi,\phi-\Im\phi\rangle = 2\left\langle\psi+\frac{5}{3}\psi,\phi+\frac{5}{3}\phi\right\rangle = \frac{128}{9}\psi\phi.\end{aligned}$$

Thus, for $\eta = \frac{5}{3}$, $\beta = \frac{1}{4}$, and $\Phi(\psi,\phi) = (\eta+1)^2|\psi-\phi|^2 + \frac{1}{4}|\psi-\Im\psi-(\phi-\Im\phi)|^2 + 2\langle\psi-\Im\psi,\phi-\Im\phi\rangle$, we get

$$\begin{aligned}\Phi(\psi,\phi) &= \frac{64}{9}|\psi-\phi|^2+\left(\frac{1}{4}\right)\left(\frac{64}{9}\right)|\psi-\phi|^2+\frac{128}{9}\psi\phi\\ &= \frac{64}{9}[\psi^2-2\psi\phi+\phi^2]+\left(\frac{1}{4}\right)\left(\frac{64}{9}\right)|\psi-\phi|^2+\frac{128}{9}\psi\phi\end{aligned}$$

$$= \frac{64}{9}[\psi^2 + \phi^2] + \left(\frac{1}{4}\right)\left(\frac{64}{9}\right)|\psi - \phi|^2$$
$$= \left|\frac{5}{3}(\psi - \phi) - \frac{5}{3}(\psi - \phi)\right|^2$$
$$= |\eta(\psi - \phi) + \Im\psi - \Im\phi|^2 > 0.$$

Hence $\Im$ is a $(\frac{5}{3}, \frac{1}{4})$-ESPN map, but it is not β-SPN since for $\psi = \frac{3}{2}$ and $\phi = -\frac{3}{2}$, we obtain

$$\|\Im\psi - \Im\phi\|^2 = \left|\Im\left(\frac{3}{2}\right) - \Im\left(-\frac{3}{2}\right)\right|^2 = \left|-\frac{5}{3}\left(\frac{3}{2}\right) + \frac{5}{3}\left(-\frac{3}{2}\right)\right|^2 = \left|-\frac{10}{2}\right|^2 = 25,$$
$$|\psi - \phi|^2 = \left|\frac{3}{2} - \left(-\frac{3}{2}\right)\right|^2 = 9,$$
$$\beta\left|(I - \Im)\left(\frac{3}{2}\right) - (I - \Im)\left(-\frac{3}{2}\right)\right|^2 = \frac{1}{4}\left|\frac{3}{2} + \frac{5}{3}\left(\frac{3}{2}\right) - \left(\left(-\frac{3}{2}\right) + \frac{5}{3}\left(-\frac{3}{2}\right)\right)\right|^2 = \frac{1}{4}|8|^2 = 16,$$

and

$$2\left\langle(I - \Im)\left(\frac{3}{2}\right), (I - \Im)\left(-\frac{3}{2}\right)\right\rangle = 2\left\langle\frac{3}{2} + \frac{5}{3}\left(\frac{3}{2}\right), \left(-\frac{3}{2}\right) + \frac{5}{3}\left(-\frac{3}{2}\right)\right\rangle = 2(4)(-4) = -32.$$

Therefore

$$\begin{aligned}\|\Im\psi - \Im\phi\|^2 &= 25 > 9 + 16 - 32\\ &= \|\psi - \phi\|^2 + \beta\|\psi - \Im\psi - (\phi - \Im\phi)\|^2 + 2\langle\psi - \Im\psi, \phi - \Im\phi\rangle\end{aligned}$$

for $\beta = \frac{1}{4}$.

Proposition 3.1 *Let $\mathfrak{E}$ be a normed space, and let $\Im : \mathfrak{D}(\Im) \subseteq \mathfrak{E} \longrightarrow \mathfrak{E}$ be an (η, λ)-ESPC map. Then $\Im$ is an $\mathcal{L}$-Lipschitzian map.*

Proof Since $\Im$ is an (η, λ)-ESPC map, by (1.11) there exists $\beta \in [0, 1)$ such that for all $\psi, \phi \in \mathfrak{D}(\Im)$,

$$\|\eta(\psi - \phi) + \Im\psi - \Im\phi\|^2 \leq (\eta + 1)^2\|\psi - \phi\|^2 + \beta\|\psi - \Im\psi - (\phi - \Im\phi)\|^2.$$

From the above inequality,

$$\begin{aligned}&\|\eta(\psi - \phi) + \Im\psi - \Im\phi\|^2\\ &\quad\leq (\eta + 1)^2\|\psi - \phi\|^2 + \beta\|\psi - \Im\psi - (\phi - \Im\phi)\|^2\\ &\quad\leq [(\eta + 1)\|\psi - \phi\| + \sqrt{\beta}\|\psi - \Im\psi - (\phi - \Im\phi)\|]^2\\ &\quad\leq [(\eta + 1)\|\psi - \phi\| + \sqrt{\beta}\|(\eta + 1)(\psi - \phi) - [\eta(\psi - \phi) + \Im\psi - \Im\phi]\|]^2\\ &\quad\leq [(\eta + 1)\|\psi - \phi\| + \sqrt{\beta}(\eta + 1)\|\psi - \phi\| + \sqrt{\beta}\|\eta(\psi - \phi) + \Im\psi - \Im\phi\|]^2.\end{aligned}$$

Therefore

$$\|\eta(\psi - \phi) + \Im\psi - \Im\phi\| \leq \mathcal{L}\|\psi - \phi\|$$

with $\mathcal{L} = \frac{(\eta + 1)(1 + \sqrt{\beta})}{1 - \sqrt{\beta}}$. □

Proposition 3.2 *Consider a real Hilbert space* $\mathfrak{H}$, $\emptyset \neq \Omega \subset \mathfrak{H}$, *and an* (η,β)*-ESPN map* $\Im : \Omega \longrightarrow \Omega$. *Then* $(I - \Im)$ *is demiclosed at* 0.

Proof Let $\{\psi_n\}$ be a sequence in $F(\Im)$ that converges weakly to ϑ and $\{\psi_n - \Im\psi_n\}$ converges strongly to 0. We want to show that $\vartheta \in \mathcal{F}(\Im)$. Now, since $\{\psi_n\}$ converges weakly, it is bounded.

For each $\psi \in \mathfrak{H}$, define $f : \mathfrak{H} \longrightarrow [0,\infty)$ by

$$f(\psi) = \limsup_{n\to\infty} \|\psi_n - \psi\|^2.$$

Then, using Lemma 2.1(iii), we get

$$f(\psi) = \limsup_{n\to\infty} \|\psi_n - \vartheta\|^2 + \|\vartheta - \psi\|^2 \quad \forall \psi \in \mathfrak{H}.$$

Consequently,

$$f(\psi) = f(\vartheta) + \|\vartheta - \psi\|^2,$$

and

$$f(\Im_\omega) = f(\vartheta) + \|\vartheta - \Im_\omega\vartheta\|^2 = f(\vartheta) + \frac{1}{(\eta+1)^2}\|\vartheta - \Im\vartheta\|^2 \quad \forall \psi \in \mathfrak{H}. \tag{3.4}$$

Observe that

$$\begin{aligned}
f(\Im_\omega) &= \limsup_{n\to\infty} \|\psi_n - \Im_\omega\vartheta\|^2 \\
&= \limsup_{n\to\infty} \|\psi_n - \Im_\omega\psi_n + \Im_\omega\psi_n - \Im_\omega\vartheta\|^2 \\
&= \limsup_{n\to\infty} \|\psi_n - [(1-\omega)\psi_n + \omega\Im\psi_n] + (1-\omega)\psi_n + \omega\Im\psi_n - [(1-\omega)\vartheta + \omega\Im\vartheta]\|^2 \\
&= \limsup_{n\to\infty} \|\omega(\psi_n - \Im\psi_n) + (1-\omega)(\psi_n - \vartheta) + \omega(\Im\psi_n - \Im\vartheta)\|^2 \\
&= \limsup_{n\to\infty} \left\| \frac{\eta}{\eta+1}(\psi_n - \vartheta) + \frac{1}{\eta+1}(\Im\psi_n - \Im\vartheta)\right\|^2 \\
&= \frac{1}{(\eta+1)^2} \limsup_{n\to\infty} \|\eta(\psi_n - \vartheta) + \Im\psi_n - \Im\vartheta\|^2 \\
&\leq \frac{1}{(\eta+1)^2} \limsup_{n\to\infty} [(\eta+1)^2\|\psi_n - \vartheta\|^2 + \beta\|\vartheta - \Im\vartheta\|^2] \\
&= f(\vartheta) + \frac{\beta}{(\eta+1)^2}\|\vartheta - \Im\vartheta\|^2.
\end{aligned} \tag{3.5}$$

From (3.4) and (3.5) it follows that

$$(1-\beta)\|\vartheta - \Im\vartheta\| \leq 0,$$

so that $\vartheta \in \mathcal{F}(\Im)$ as required. □

Proposition 3.3 *Consider a real Hilbert space $\mathfrak{H}$, $\emptyset \neq \Omega \subset \mathfrak{H}$, and an (η, β)-ESPN map $\Im : \Omega \longrightarrow \Omega$. Then $\mathcal{F}(\Im)$ is closed and convex.*

Proof Let $\{\psi_n\}$ be a sequence in $F(\Im)$ that converges to ψ. We want to show that $\psi \in \mathcal{F}(\Im)$. Since

$$\begin{aligned}\omega\|\Im\psi - \psi\| &\leq \omega\|\Im\psi - \Im\psi_n\| + \omega\|\psi_n - \psi\| \\ &= \omega\|\eta(\psi - \psi_n) + \Im\psi - \Im\psi_n - \eta(\psi - \psi_n)\| + \omega\|\psi_n - \psi\| \\ &\leq \omega\|\eta(\psi - \psi_n) + \Im\psi - \Im\psi_n\| + \omega(\eta + 1)\|\psi_n - \psi\| \end{aligned} \tag{3.6}$$

and $\Im$ is an (η, β)-ESPN mapping, we have

$$\begin{aligned}\|\eta(\psi - \psi_n) + \Im\psi - \Im\psi_n\|^2 &\leq (\eta + 1)^2\|\psi - \psi_n\|^2 + \beta\|\psi_n - \Im\psi_n - (\psi - \Im\psi)\|^2 \\ &\quad + \langle \psi_n - \Im\psi_n, \psi - \Im\psi\rangle \\ &\leq [(\eta + 1)\|\psi - \psi_n\| + \sqrt{\beta}\|\psi - \Im\psi\|]^2. \end{aligned} \tag{3.7}$$

Using (3.6) in (3.7), we obtain

$$0 \leq \|\psi - \Im\psi\| \leq \frac{2(\eta + 1)}{1 - \sqrt{\beta}}\|\psi - \psi_n\| \to 0 \text{ as } n \to \infty.$$

Hence $\psi \in \mathcal{F}(\Im)$.

Next, let $\vartheta_1, \vartheta_2 \in \mathcal{F}(\Im)$ and $\lambda \in [0, 1]$. We prove that $\lambda\vartheta_1 + (1 - \lambda)\vartheta_2 \in \mathcal{F}(\Im)$. Set $\wp = \lambda\vartheta_1 + (1 - \lambda)\vartheta_2$. Then $\vartheta_1 - \wp = (1 - \lambda)(\vartheta_1 - \vartheta_2)$ and $\vartheta_2 - \wp = \lambda(\vartheta_2 - \vartheta_1)$. Since

$$\begin{aligned}&\omega^2\|\Im\wp - \wp\|^2 = \|\wp - \Im_\omega\wp\|^2 \\ &= \|\lambda\vartheta_1 + (1 - \lambda)\vartheta_2 - \Im_\omega\wp\|^2 \\ &= \|\lambda(\vartheta_1 - \Im_\omega\wp) + (1 - \lambda)(\vartheta_2 - \Im_\omega\wp)\|^2 \\ &= \lambda\|\vartheta_1 - \Im_\omega\wp\|^2 + (1 - \lambda)\|\vartheta_2 - \Im_\omega\wp\|^2 - \lambda(1 - \lambda)\|\vartheta_1 - \vartheta_2\|^2 \\ &= \lambda\|(1 - \omega)\vartheta_1 + \omega\Im\vartheta_1 - [(1 - \omega)\wp + \omega\Im\wp]\|^2 \\ &\quad + (1 - \lambda)\|(1 - \omega)\vartheta_2 + \omega\Im\vartheta_2 - [(1 - \omega)\wp + \omega\Im\wp]\|^2 \\ &\quad - \lambda(1 - \lambda)\|\vartheta_1 - \vartheta_2\|^2 \\ &= \|(1 - \omega)(\vartheta_1 - \wp) + \omega(\Im\vartheta_1 - \Im\wp)\|^2 \\ &\quad + (1 - \lambda)\|(1 - \omega)(\vartheta_2 - \wp) + \omega(\Im\vartheta_2 - \Im\wp)\|^2 \\ &\quad - \lambda(1 - \lambda)\|\vartheta_1 - \vartheta_2\|^2 \\ &= \frac{\lambda}{(\eta + 1)^2}\|\eta(\vartheta_1 - \wp) + \Im\vartheta_1 - \Im\wp\|^2 \\ &\quad + \frac{1 - \lambda}{(\eta + 1)^2}\|\eta(\vartheta_2 - \wp) + \Im\vartheta_2 - \Im\wp\|^2 - \lambda(1 - \lambda)\|\vartheta_1 - \vartheta_2\|^2 \\ &\leq \frac{\lambda}{(\eta + 1)^2}[(\eta + 1)^2\|\vartheta_1 - \wp\|^2 + \beta\|\wp - \Im\wp\|^2]\end{aligned}$$

$$
\begin{aligned}
&+\frac{1-\lambda}{(\eta+1)^2}[(\eta+1)^2\|\vartheta_2-\wp\|^2+\beta\|\wp-\Im\wp\|^2]\\
&-\lambda(1-\lambda)\|\vartheta_1-\vartheta_2\|^2\\
=\ &\lambda(1-\lambda)^2\|\vartheta_1-\vartheta_2\|^2+\frac{\beta}{(\eta+1)^2}\|\wp-\Im\wp\|^2\\
&+(1-\lambda)\lambda^2\|\vartheta_2-\vartheta_1\|^2-\lambda(1-\lambda)\|\vartheta_1-\vartheta_2\|^2\\
=\ &\lambda(1-\lambda)[1-\lambda+\lambda]\|\vartheta_1-\vartheta_2\|^2+\frac{\beta}{(\eta+1)^2}\|\wp-\Im\wp\|^2\\
&-\lambda(1-\lambda)\|\vartheta_1-\vartheta_2\|^2,
\end{aligned}
$$

it follows that $(1-\beta)\|\wp-\Im\wp\|\leq 0$. Therefore $\wp=\Im\wp$, and $\wp\in\mathcal{F}(\Im)$, as required. □

The examples below demonstrate the conclusion that the class of (η,λ)-ESPC maps and the class of maps studied in this paper are independent.

Example 3.2 Let $\Im:\mathcal{R}\longrightarrow\mathcal{R}$ be defined, for each $\psi\in\mathcal{R}$, by

$$
\Im\psi=\begin{cases}0 \text{ if } \psi\in(-\infty,2],\\ 1 \text{ if } \psi\in(2,\infty),\end{cases}
$$

where $\mathcal{R}$ denotes the reals with usual norm. Then, for all $\psi,\phi\in(-\infty,2]$ and $\beta\in[0,1)$, $\Im$ is an (η,β)-ESPN map with $\eta=0$ (see [2] for details). However, $\Im$ is not an (η,λ)-ESPC map since every (η,λ)-ESPC map satisfies the Lipschitz condition (see Proposition 3.1).

Example 3.3 Let $\Im:\mathcal{R}\longrightarrow\mathcal{R}$ be defined, for each $\psi\in\mathcal{R}$, by

$$
\Im\psi=-3\psi, \tag{3.8}
$$

where $\mathcal{R}$ denotes the reals with usual norm. It is shown in [2] that $\Im$ is an (η,λ)-ESPC map with $\eta=0$. Nevertheless, it is not difficult to see that $\Im$ is not an (η,β)-ESPN map. Indeed, for $\eta=0$, if $\psi=\frac{1}{2}$ and $\phi=-\frac{1}{2}$, then

$$
\begin{aligned}
|\eta(\psi-\phi)+\Im\psi-\Im\phi|^2=9(\eta+1)&\\
&=(\eta+1)|\psi-\phi|^2+|\psi-\Im\psi-(\phi-\Im\phi)|^2\\
&\quad+2\langle\psi-\Im\psi,\phi-\Im\phi\rangle\\
&>(\eta+1)|\psi-\phi|^2+\beta|\psi-\Im\psi-(\phi-\Im\phi)|^2\\
&\quad+2\langle\psi-\Im\psi,\phi-\Im\phi\rangle
\end{aligned}
$$

for all $\beta\in[0,1)$.

Theorem 3.4 *Consider a real Hilbert space* $\mathfrak{H}$, $\emptyset\neq\Omega\subset\mathfrak{H}$, *and an* (η,β)*-ESPN map* $\Im:\Omega\longrightarrow\Omega$ *such that* $\mathcal{F}(\Im)\neq\emptyset$. *Let* $\xi\in[\beta,1)$ *and* $\{\pi_n\}$ *be in* $[0,1)$ *with* $\lim_{n\to\infty}\pi_n=0$. *Let*

$\{\psi_n\}$ and $\{\phi_n\}$ be sequences in Ω developed from arbitrary $\psi_1 \in \Omega$ by

$$\begin{cases} \psi_{n+1} = \pi_n \psi_n + (1-\pi_n)[\xi \psi_n + (1-\xi)\Im_\omega \psi_n], \\ \phi_n = \dfrac{1}{n}\displaystyle\sum_{t=0}^{n-1} \psi_t, \quad n \geq 1, \end{cases} \tag{3.9}$$

Then $\{\phi_n\}$ converges weakly to $\wp \in \mathcal{F}(\Im)$, where $\wp = \lim_{n\to\infty} \mathcal{P}_{\mathcal{F}(\Im)}\psi_n$. In particular, for $\psi \in \Omega$, define

$$\eth_n \psi = \frac{1}{n}\sum_{t=0}^{n-1} \Im^t_{\xi,\eta}\psi, \quad n \geq 1, \tag{3.10}$$

where $\Im_{\xi,\eta} = \xi I + (1-\xi)\Im_\omega$. Then $\{\eth_n\psi\}$ converges weakly to $\phi \in \mathcal{F}(\Im)$, where $\phi = \lim_{n\to\infty} P_{\mathcal{F}(\Im)}\Im^n_{\xi,\eta}\psi$.

Proof Set $\Im_{\xi,\eta} = \xi\psi + (1-\xi)\Im_\omega\psi$. Then, for all $\psi, \phi \in \Omega$, we obtain the following estimates:

$$\begin{aligned}
&\|\Im_{\xi,\eta}\psi - \Im_{\xi,\eta}\phi\|^2 \\
&\quad = \left\| \xi(\psi-\phi) + (1-\xi)\left[\frac{\eta+\Im}{\eta+1}\psi - \frac{\eta+\Im}{\eta+1}\phi\right]\right\|^2 \\
&\quad = \left\| \xi(\psi-\phi) + (1-\xi)\left[\frac{\eta}{\eta+1}(\psi-\phi) + \frac{1}{\eta+1}(\Im\psi - \Im\phi)\right]\right\|^2 \\
&\quad = \xi\|\psi-\phi\|^2 + (1-\xi)\left\|\frac{\eta}{\eta+1}(\psi-\phi) + \frac{1}{\eta+1}(\Im\psi-\Im\phi)\right\|^2 \\
&\qquad - \xi(1-\xi)\left\|\psi-\phi-\left[\frac{\eta}{\eta+1}(\psi-\phi) + \frac{1}{\eta+1}(\Im\psi-\Im\phi)\right]\right\|^2 \\
&\quad = \xi\|\psi-\phi\|^2 + \frac{(1-\xi)}{(\eta+)^2}\left\|\eta(\psi-\phi) + \Im\psi - \Im\phi\right\|^2 \\
&\qquad - \frac{\xi(1-\xi)}{(\eta+1)^2}\left\|\eta(\psi-\phi) + [\psi - (\eta+\Im)\psi - (\phi-(\eta+\Im))\phi]\right\|^2 \\
&\quad \leq \xi\|\psi-\phi\|^2 + \frac{(1-\xi)}{(\eta+)^2}\left\|\eta(\psi-\phi) + \Im\psi - \Im\phi\right\|^2 \\
&\qquad - \frac{\xi(1-\xi)}{(\eta+1)^2}\left\|\psi - (\eta+\Im)\psi - (\phi - (\eta+\Im)\phi)\right\|^2 \\
&\quad \leq \xi\|\psi-\phi\|^2 + \frac{(1-\xi)}{(\eta+1)^2}[(\eta+1)^2\|\psi-\phi\|^2 \\
&\qquad + \beta\|\psi - \Im\psi - (\phi-\Im\phi)\|^2 + 2\langle\psi-\Im\psi, \phi-\Im\phi\rangle] \\
&\qquad - \frac{\xi(1-\xi)}{(\eta+1)^2}\|\psi-(\eta+\Im)\psi - (\phi-(\eta+\Im)\phi)\|^2 \\
&\quad = \xi\|\psi-\phi\|^2 + (1-\xi)\|\psi-\phi\|^2 \\
&\qquad + \frac{(1-\xi)}{(\eta+1)^2}\beta\|\psi-\Im\psi-(\phi-\Im\phi)\|^2 + 2\frac{(1-\xi)}{(\eta+1)^2}\langle\psi-\Im\psi,\phi-\Im\phi\rangle \\
&\qquad - \frac{\xi(1-\xi)}{(\eta+1)^2}\|\eta(\phi-\psi) + [\psi-\Im\psi-(\phi-\Im\phi)]\|^2 \\
&\quad \leq \xi\|\psi-\phi\|^2 + (1-\xi)\|\psi-\phi\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(1-\xi)}{(\eta+1)^2}\beta\|\psi-\Im\psi-(\phi-\Im\phi)\|^2+2\frac{(1-\xi)}{(\eta+1)^2}\langle\psi-\Im\psi,\phi-\Im\phi\rangle\\
&-\frac{\xi(1-\xi)}{(\eta+1)^2}\|\psi-\Im\psi-(\phi-\Im\phi)\|^2\\
&=\xi\|\psi-\phi\|^2+(1-\xi)\|\psi-\phi\|^2\\
&-\frac{(1-\xi)}{(\eta+1)^2}(\xi-\beta)\|\psi-\Im\psi-(\phi-\Im\phi)\|^2+2\frac{(1-\xi)}{(\eta+1)^2}\langle\psi-\Im\psi,\phi-\Im\phi\rangle\\
&\le\|\psi-\phi\|^2+2\frac{(1-\xi)}{(\eta+1)^2}\langle\psi-\Im\psi,\phi-\Im\phi\rangle.
\end{aligned}\tag{3.11}
$$

Observe that

$$
\begin{aligned}
&\langle\psi-\Im\psi,\phi-\Im\phi\rangle\\
&=\langle\eta\psi+\psi-(\eta+\Im)\psi,\eta\phi+\phi-\phi(\eta+\Im)\phi\rangle\\
&=\langle\eta\psi+\psi-(\eta+\Im)\psi,\eta\phi\rangle+\langle\eta\psi+\psi-(\eta+\Im)\psi,\eta\phi+\phi-\phi(\eta+\Im)\phi\rangle\\
&=\langle-[(\eta+\Im)\psi-(\eta+)\psi],\eta\phi\rangle+\langle\eta\psi,-[(\eta+\Im)\phi-(\eta+)\phi]\rangle\\
&\quad+\langle\psi-(\eta+\Im)\psi,\phi-\phi(\eta+\Im)\phi\rangle\\
&=-\|(\eta+\Im)\psi-(\eta+)\psi\|\|\eta\phi\|-\|\eta\psi\|\|(\eta+\Im)\phi-(\eta+)\phi\|\\
&\quad+\langle\psi-(\eta+\Im)\psi,\phi-\phi(\eta+\Im)\phi\rangle\\
&\le\langle\psi-(\eta+\Im)\psi,\phi-(\eta+\Im)\phi\rangle.
\end{aligned}\tag{3.12}
$$

From (3.11) and (3.12) it follows that

$$
\|\Im_{\xi,\eta}\psi-\Im_{\xi,\eta}\phi\|^2\le\|\psi-\phi\|^2+2\frac{(1-\xi)}{(\eta+1)^2}\langle\psi-(\eta+\Im)\psi,\phi-(\eta+\Im)\phi\rangle.\tag{3.13}
$$

Since $\Im_{\xi,\eta}\psi=\xi\psi+(1-\xi)\Im_\omega\psi$, it follows that

$$
\begin{aligned}
\Im_{\xi,\eta}\psi&=\xi\psi+(1-\xi)[(1-\omega)\psi+\omega\Im\psi]\\
&=\xi\psi+(1-\xi)\Big[\frac{\eta}{\eta+1}\psi+\frac{1}{\eta+1}\Im\psi\Big]\\
&=\xi\psi+\frac{(1-\xi)}{\eta+1}(\eta+\Im)\psi.
\end{aligned}
$$

The last identity implies that

$$
\frac{1}{1-\xi}(\psi-\Im_{\xi,\eta}\psi)=\psi-\frac{1}{\eta+1}(\eta+\Im)\psi.
$$

Thus, if we set $V=\frac{1}{(1-\xi)^2}\langle\psi-\Im_{\xi,\eta}\psi,\phi-\Im_{\xi,\eta}\phi\rangle$, then

$$
\begin{aligned}
V&=\Big\langle\frac{\psi}{\eta+1}-\frac{1}{\eta+1}(\eta+\Im)\psi+\Big(\psi-\frac{\psi}{\eta+1}\Big),\frac{\phi}{\eta+1}-\frac{1}{\eta+1}(\eta+\Im)\phi+\Big(\phi-\frac{\phi}{\eta+1}\Big)\Big\rangle\\
&=\frac{1}{(\eta+1)^2}\langle\psi-(\eta+\Im)\psi,\phi-(\eta+\Im)\phi\rangle
\end{aligned}
$$

$$+\left\langle \frac{\psi}{\eta+1} - \frac{1}{\eta+1}(\eta+\Im)\psi, \phi - \frac{\phi}{\eta+1} \right\rangle$$
$$+\left\langle \psi - \frac{\psi}{\eta+1}, \frac{\phi}{\eta+1} - \frac{1}{\eta+1}(\eta+\Im)\phi \right\rangle + \left\langle \psi - \frac{\psi}{\eta+1}, \phi - \frac{\phi}{\eta+1} \right\rangle,$$

so that

$$\begin{aligned}\frac{1}{(\eta+1)^2}\left\langle \psi - (\eta+\Im)\psi, \phi - (\eta+\Im)\phi \right\rangle = {} & \frac{1}{(1-\xi)^2}\langle \psi - \Im_{\xi,\eta}\psi, \phi - \Im_{\xi,\eta}\phi \rangle \\ & -\left\langle \frac{\psi}{\eta+1} - \frac{1}{\eta+1}(\eta+\Im)\psi, \phi - \frac{\phi}{\eta+1} \right\rangle \\ & -\left\langle \psi - \frac{\psi}{\eta+1}, \frac{\phi}{\eta+1} - \frac{1}{\eta+1}(\eta+\Im)\phi \right\rangle \\ & -\left\langle \psi - \frac{\psi}{\eta+1}, \phi - \frac{\phi}{\eta+1} \right\rangle. \end{aligned} \tag{3.14}$$

From (3.13) and (3.14) it follows that

$$\|\Im_{\xi,\eta}\psi - \Im_{\xi,\eta}\phi\|^2 \le \|\psi - \phi\|^2 + \frac{2}{(1-\xi)}\langle \psi - \Im_{\xi,\eta}\psi, \phi - \Im_{\xi,\eta}\phi \rangle. \tag{3.15}$$

From (3.15), for each $\vartheta \in \mathcal{F}(\Im)$,

$$\begin{aligned}\|\psi_{n+1} - \vartheta\| &= \|\pi_n(\psi_n - \vartheta) + (1-\pi_n)(\Im_{\xi,\eta}\psi_n - \vartheta)\| \\ &\le \pi_n\|\psi_n - \vartheta\| + (1-\pi_n)\|\Im_{\xi,\eta}\psi_n - \vartheta\| \\ &\le \|\psi_n - \vartheta\|. \end{aligned} \tag{3.16}$$

Therefore $\{\psi_n\}$ is bounded.

Using (3.15) and Lemma 2.1(i), we obtain, for all integer $t \ge 1$ and for all $\phi \in \Omega$, that

$$\begin{aligned}\|\psi_{t+1} - \Im_{\xi,\eta}\phi\|^2 &= \|\pi_t(\psi_t - \Im_{\xi,\eta}\phi) + (1-\pi_t)(\Im_{\xi,\eta}\psi_t - \Im_{\xi,\eta}\phi)\|^2 \\ &= \pi_t\|\psi_t - \Im_{\xi,\eta}\phi\| + (1-\pi_t)\|\Im_{\xi,\eta}\psi_t - \Im_{\xi,\eta}\phi\|^2 \\ &\quad -\pi_t(1-\pi_t)\|\psi_t - \Im_{\xi,\eta}\psi_t\|^2 \\ &\le \pi_t\|\psi_t - \Im_{\xi,\eta}\phi\| \\ &\quad +(1-\pi_t)\left[\|\psi_t - \phi\|^2 + \frac{2}{(1-\xi)}\langle \psi_t - \Im_{\xi,\eta}\psi_t, \phi - \Im_{\xi,\eta}\phi \rangle\right]. \end{aligned} \tag{3.17}$$

From

$$\begin{aligned}\|\psi_t - \phi\|^2 &= \|\psi_t - \Im_{\xi,\eta}\phi + \Im_{\xi,\eta}\phi - \phi\|^2 \\ &= \|\psi_t - \Im_{\xi,\eta}\phi\|^2 + \|\Im_{\xi,\eta}\phi - \phi\|^2 + 2\langle \psi_t - \Im_{\xi,\eta}\phi, \Im_{\xi,\eta}\phi - \phi \rangle \end{aligned}$$

and the last inequality we get

$$\begin{aligned}&\|\psi_{t+1} - \Im_{\xi,\eta}\phi\|^2 \\ &\quad \le \pi_t\|\psi_t - \Im_{\xi,\eta}\phi\| \end{aligned}$$

$$
\begin{aligned}
&+ (1-\pi_t)\|\psi_t - \Im_{\xi,\eta}\phi\|^2 + (1-\pi_t)\|\Im_{\xi,\eta}\phi - \phi\|^2 \\
&+ 2(1-\pi_t)\langle \psi_t - \Im_{\xi,\eta}\phi, \Im_{\xi,\eta}\phi - \phi\rangle + \frac{2(1-\pi_t)}{(1-\xi)}\langle \psi_t - \Im_{\xi,\eta}\psi_t, \phi - \Im_{\xi,\eta}\phi\rangle \\
= &\|\psi_t - \Im_{\xi,\eta}\phi\| + (1-\pi_t)\|\Im_{\xi,\eta}\phi - \phi\|^2 \\
&+ 2\langle \psi_t - \Im_{\xi,\eta}\phi, \Im_{\xi,\eta}\phi - \phi\rangle - \pi_t\langle \psi_t - \Im_{\xi,\eta}\phi, \Im_{\xi,\eta}\phi - \phi\rangle \\
&+ \frac{2}{(1-\xi)}\langle (1-\pi_t)(\psi_t - \Im_{\xi,\eta}\psi_t), \phi - \Im_{\xi,\eta}\phi\rangle.
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
\|\psi_{t+1} - \Im_{\xi,\eta}\phi\|^2 \le \|\psi_t - \Im_{\xi,\eta}\phi\| + \|\Im_{\xi,\eta}\phi - \phi\|^2 \\
+2\langle \psi_t - \Im_{\xi,\eta}\phi, \Im_{\xi,\eta}\phi - \phi\rangle - \pi_t\langle \psi_t - \Im_{\xi,\eta}\phi, \Im_{\xi,\eta}\phi - \phi\rangle \\
+\frac{2}{(1-\xi)}\langle (\psi_t - \psi_{t+1}, \phi - \Im_{\xi,\eta}\phi\rangle. \qquad (3.18)
\end{aligned}
$$

Summing (3.18) from $t = 1$ to $t = n$ and dividing the outcome by n, we get

$$
\begin{aligned}
\frac{1}{n}\|\psi_{n+1} - \Im_{\xi,\eta}\phi\|^2 \le \frac{1}{n}\|\psi_1 - \Im_{\xi,\eta}\phi\| + \frac{2}{(1-\xi)}\langle \frac{\psi_1}{n} - \frac{\psi_{n+1}}{n}, \phi - \Im_{\xi,\eta}\phi\rangle \\
+\|\Im_{\xi,\eta}\phi - \phi\|^2 + 2\langle \phi_n - \Im_{\xi,\eta}\phi, \Im_{\xi,\eta}\phi - \phi\rangle \\
-\frac{2}{n}\sum_{t=1}^{n}\pi_t\langle \psi_t - \Im_{\xi,\eta}\phi, \Im_{\xi,\eta}\phi - \phi\rangle. \qquad (3.19)
\end{aligned}
$$

From the boundedness of $\{\psi_n\}$ it obviously follows that $\{\phi_n\}$ is also bounded. Consequently, we can find a subsequence $\{\phi_{n_k}\}$ of $\{\phi_n\}$ such that $\phi_{n_k} \rightharpoonup \wp$ as $k \to \infty$. By using n_k for n in (3.19) we obtain

$$
\begin{aligned}
\frac{1}{n_k}\|\psi_{n_k+1} - \Im_{\xi,\eta}\phi\|^2 \le \frac{1}{n_k}\|\psi_1 - \Im_{\xi,\eta}\phi\| + \frac{2}{(1-\xi)}\langle \frac{\psi_1}{n_k} - \frac{\psi_{n_k+1}}{n_k}, \phi - \Im_{\xi,\eta}\phi\rangle \\
+\|\Im_{\xi,\eta}\phi - \phi\|^2 + 2\langle \phi_{n_k} - \Im_{\xi,\eta}\phi, \Im_{\xi,\eta}\phi - \phi\rangle \\
-\frac{2}{n_k}\sum_{t=1}^{n_k}\pi_t\langle \psi_t - \Im_{\xi,\eta}\phi, \Im_{\xi,\eta}\phi - \phi\rangle. \qquad (3.20)
\end{aligned}
$$

Letting $k \to \infty$ in (3.20), we deduce

$$
0 \le \|\Im_{\xi,\eta}\phi - \phi\|^2 + 2\langle \wp - \Im_{\xi,\eta}\phi, \Im_{\xi,\eta}\phi - \phi\rangle. \qquad (3.21)
$$

Since $\phi \in \mathfrak{H}$ was arbitrarily chosen, putting $\phi = \wp$ in (3.21) yields

$$
0 \le \|\Im_{\xi,\eta}\wp - \wp\|^2 - 2\|\Im_{\xi,\eta}\wp - \wp\|^2,
$$

so that $\wp \in \mathcal{F}(\Im_{\xi,\eta}) = \mathcal{F}(\Im_\xi) = \mathcal{F}(\Im)$.

Also, since $\Im$ is (η,β)-ESPN with $\mathcal{F}(\Im) \neq \emptyset$, it follows from Proposition 3.3 that $\mathcal{F}(\Im)$ is closed and convex. Consequently, we can define the projection $\mathcal{P}_{\mathcal{F}(\Im)} : \mathfrak{H} \longrightarrow \mathcal{F}(\Im)$. Since

$$
\|\psi_{n+1} - \vartheta\|^2 \le \|\psi_n - \vartheta\|^2 \quad \forall \vartheta \in \mathcal{F}(\Im),
$$

it follows from Lemma 2.2 that $\{\mathcal{P}_{\mathcal{F}(\Im)}\psi_n\}$ converges strongly.

Let $\lim_{n\to\infty} \mathcal{P}_{\mathcal{F}(\Im)}\psi_n = \hbar$. Then it suffices to show that $\hbar = \wp$. Since $\wp \in \mathcal{F}(\Im)$, by (2.1) we have

$$\begin{aligned}\langle \wp - \hbar, \psi_t - \mathcal{P}_{\mathcal{F}(\Im)}\psi_t\rangle &= \langle \wp - \mathcal{P}_{\mathcal{F}(\Im)}\psi_t, \psi_t - \mathcal{P}_{\mathcal{F}(\Im)}\psi_t\rangle + \langle \mathcal{P}_{\mathcal{F}(\Im)}\psi_t - \hbar, \psi_t - \mathcal{P}_{\mathcal{F}(\Im)}\psi_t\rangle \\ &\leq \langle \mathcal{P}_{\mathcal{F}(\Im)}\psi_t - \hbar, \psi_t - \mathcal{P}_{\mathcal{F}(\Im)}\psi_t\rangle \\ &\leq \|\mathcal{P}_{\mathcal{F}(\Im)}\psi_t - \hbar\|\|\psi_t - \mathcal{P}_{\mathcal{F}(\Im)}\psi_t\| \\ &\leq Q\|\mathcal{P}_{\mathcal{F}(\Im)}\psi_t - \hbar\|, \end{aligned} \tag{3.22}$$

where such Q exists since $\{\psi_t\}$ and $\{\mathcal{P}_{\mathcal{F}(\Im)}\psi_t\}$ are bounded, and thus $\|\psi_t - \mathcal{P}_{\mathcal{F}(\Im)}\psi_t\| \leq Q$, $t \geq 1$, for some $Q > 0$.

Summing (3.22) from $t = 1$ to $t = n_k$ and dividing the outcome by n_k, we get

$$\langle \wp - \hbar, \phi_{n_k} - \frac{1}{n_k}\sum_{t=1}^{n_k} \mathcal{P}_{\mathcal{F}(\Im)}\psi_t\rangle \leq \frac{Q}{n_k}\sum_{t=1}^{n_k} \|\mathcal{P}_{\mathcal{F}(\Im)}\psi_t - \hbar\|.$$

Since $\phi_{n_k} \rightharpoonup \wp$ as $k \to \infty$ and $\mathcal{P}_{\mathcal{F}(\Im)}\psi_n \to \hbar$ as $n \to \infty$, we get

$$\langle \wp - \hbar, \wp - \hbar\rangle = \|\wp - \hbar\|^2 \leq 0.$$

Therefore $\wp = \hbar$, as required.

Furthermore, by setting $\pi_n = 0$ in (3.9) we infer

$$\begin{aligned}\psi_{n+1} &= \Im_{\xi,\eta}\psi_n = \Im^n_{\xi,\eta}\psi_1, \\ \phi_n &= \frac{1}{n}\sum_{t=1}^{n} \psi_t = \frac{1}{n}\sum_{t=1}^{n} \Im^{t-1}_{\xi,\eta}\psi_1, \quad n \geq 1.\end{aligned}$$

Consequently, if for $\psi = \psi_1 \in \Omega$, we define

$$\eth_n\psi = \frac{1}{n}\sum_{t=1}^{n} \Im^{t-1}_{\xi,\eta}\psi_1 = \phi_n, \quad n \geq 1,$$

then $\{\eth_n\}$ converges weakly to $\phi \in \mathcal{F}(\Im)$, where $\phi = \lim_{n\to\infty} \mathcal{P}_{\mathcal{F}(\Im)}\Im^n_{\xi,\eta}\psi$, as required. □

Remark 3.2 Taking $\eta = 0$ in Theorem 3.4, we get Theorem 3.1 of [2] as a corollary.

Theorem 3.5 *Consider a real Hilbert space $\mathfrak{H}$, $\emptyset \neq \Omega \subset \mathfrak{H}$, and an (η, β)-ESPN map $\Im : \Omega \longrightarrow \Omega$ such that $\mathcal{F}(\Im) \neq \emptyset$. Let $\xi \in [\beta, 1)$ and $\{\pi_n\}$ be in $[0, 1)$ such that (a) $\lim_{n\to\infty} \pi_n = 0$ and (b) $\sum_{n=1}^{\infty} \pi_n = \infty$. Let $u \in \Omega$, and let $\{\psi_n\}$ and $\{\phi_n\}$ be sequences in Ω developed from arbitrary $\psi_1 \in \Omega$ by*

$$\begin{cases} \psi_{n+1} = \pi_n u + (1 - \pi_n)\phi_n, \\ \phi_n = \frac{1}{n}\sum_{t=0}^{n-1} \Im^t_{\xi,\eta}\psi_n, \quad n \geq 1. \end{cases} \tag{3.23}$$

Then $\{\psi_n\}$ and $\{\phi_n\}$ converge strongly to $\mathcal{P}_{\mathcal{F}(\Im)}u$, where $\mathcal{P}_{\mathcal{F}(\Im)} : \mathfrak{H} \longrightarrow \mathcal{F}(\Im)$ is the metric projection of $\mathfrak{H}$ onto $\mathcal{F}(\Im)$.

Proof Let $\vartheta \in \mathcal{F}(\Im)$. Then

$$\begin{aligned}\|\phi_n - \vartheta\| &= \left\| \frac{1}{n}\sum_{t=0}^{n-1} \Im_{\xi,\eta}^t \psi_n - \vartheta \right\| \leq \frac{1}{n}\sum_{t=0}^{n-1} \|\Im_{\xi,\eta}^t \psi_n - \vartheta\| \\ &\leq \frac{1}{n}\sum_{t=0}^{n-1} \|\psi_n - \vartheta\| = \|\psi_n - \vartheta\|. \end{aligned} \tag{3.24}$$

Consequently,

$$\begin{aligned}\|\psi_{n+1} - \vartheta\| &= \|\pi_n u + (1-\pi_n)\phi_n - \vartheta\| \\ &= \|\pi_n(u-\vartheta) + (1-\pi_n)(\phi_n - \vartheta)\| \\ &= \pi_n\|u-\vartheta\| + (1-\pi_n)\|\phi_n - \vartheta\|. \end{aligned}$$

Considering

$$\|\psi_1 - \vartheta\| \leq \max\{\|u-\vartheta\|, \|\psi_1 - \vartheta\|\}$$

and

$$\|\psi_n - \vartheta\| \leq \max\{\|u-\vartheta\|, \|\psi_1 - \vartheta\|\},$$

we obtain

$$\begin{aligned}\|\psi_{n+1} - \vartheta\| &\leq \pi_n\|u-\vartheta\| + (1-\pi_n)\max\{\|u-\vartheta\|, \|\psi_1 - \vartheta\|\} \\ &\leq \pi_n \max\{\|u-\vartheta\|, \|\psi_1 - \vartheta\|\} + (1-\pi_n)\max\{\|u-\vartheta\|, \|\psi_1 - \vartheta\|\} \\ &= \max\{\|u-\vartheta\|, \|\psi_1 - \vartheta\|\}. \end{aligned}$$

Hence

$$\|\psi_n - \vartheta\| \leq \max\{\|u-\vartheta\|, \|\psi_1 - \vartheta\|\}.$$

Therefore $\{\psi_n\}$ and $\{\phi_n\}$ are bounded. Also, since $\|\Im_{\xi,\eta}\psi - \vartheta\| \leq \|\psi - \vartheta\|$, it follows that $\{\Im_{\xi,\eta}\psi_n\}$ is bounded. Hence, for all $\wp \in \Omega$ and $t = 0, 1, 2, \dots, n-1$, we obtain

$$\begin{aligned}&\|\Im_{\xi,\eta}^{t+1}\psi_n - \Im_{\xi,\eta}\wp\| \\ &\quad = \|\Im_{\xi,\eta}(\Im_{\xi,\eta}^t)\psi_n - \Im_{\xi,\eta}\wp\|^2 \\ &\quad \leq \|\Im_{\xi,\eta}^t\psi_n - \wp\|^2 + \frac{2}{1-\xi}\langle \Im_{\xi,\eta}^t\psi_n - \Im_{\xi,\eta}^{t+1}\psi_n, \wp - \Im_{\xi.\eta}\wp\rangle \\ &\quad = \|\Im_{\xi,\eta}^t\psi_n - \Im_{\xi,\eta}\wp + \Im_{\xi,\eta}\wp - \wp\|^2 + \frac{2}{1-\xi}\langle \Im_{\xi,\eta}^t\psi_n - \Im_{\xi,\eta}^{t+1}\psi_n, \wp - \Im_{\xi.\eta}\wp\rangle \\ &\quad = \|\Im_{\xi,\eta}^t\psi_n - \Im_{\xi,\eta}\wp\|^2 + \|\Im_{\xi,\eta}\wp - \wp\|^2 + 2\langle \Im_{\xi,\eta}^t\psi_n - \Im_{\xi,\eta}\wp, \Im_{\xi,\eta}\wp - \wp\rangle \\ &\qquad + \frac{2}{1-\xi}\langle \Im_{\xi,\eta}^t\psi_n - \Im_{\xi,\eta}^{t+1}\psi_n, \wp - \Im_{\xi.\eta}\wp\rangle. \end{aligned} \tag{3.25}$$

Summing (3.25) from $t = 0$ to $t = n-1$ and dividing the outcome by n, we have

$$\frac{1}{n}\|\Im_{\xi,\eta}^{n}\psi_n - \Im_{\xi,\eta}\wp\| \leq \frac{1}{n}\|\psi_n - \Im_{\xi,\eta}\wp\|^2 + \|\Im_{\xi,\eta}\phi - \phi\|^2 + 2\langle\phi_n - \Im_{\xi,\eta}\wp, \Im_{\xi,\eta}\wp - \wp\rangle + \frac{2}{n(1-\xi)}\langle\psi_n - \Im_{\xi,\eta}^{n}\psi_n, \wp - \Im_{\xi.\eta}\wp\rangle. \tag{3.26}$$

From the boundedness of $\{\phi_n\}$ it follows that we can find a subsequence $\{\phi_{n_k}\}$ of $\{\phi_n\}$ that converges weakly to $q \in \Omega$. Substituting n_k for n in (3.26), we get

$$\frac{1}{n_k}\|\Im_{\xi,\eta}^{n_k}\psi_n - \Im_{\xi,\eta}\wp\| \leq \frac{1}{n_k}\|\psi_{n_k} - \Im_{\xi,\eta}\wp\|^2 + \|\Im_{\xi,\eta}\phi - \phi\|^2 + 2\langle\phi_{n_k} - \Im_{\xi,\eta}\wp, \Im_{\xi,\eta}\wp - \wp\rangle + \frac{2}{n_k(1-\xi)}\langle\psi_{n_k} - \Im_{\xi,\eta}^{n_k}\psi_{n_k}, \wp - \Im_{\xi.\eta}\wp\rangle. \tag{3.27}$$

Since $\{\psi_n\}$ and $\{\Im_{\xi,\eta}^{n}\psi_n\}$ are bounded, letting $k \to \infty$, we deduce from (3.27) that

$$0 \leq |\Im_{\xi,\eta}\wp - \wp\|^2 + 2\langle q - \Im_{\xi,\eta}\wp, \Im_{\xi,\eta}\wp - \wp\rangle. \tag{3.28}$$

Since $\wp \in \Omega$ is arbitrary, setting $\wp = q$ in (3.28) yields $q \in \mathcal{F}(\Im_{\xi,\eta}) = \mathcal{F}(\Im_\xi) = \mathcal{F}(\Im)$.

Again, since $\lim_{n\to\infty}\pi_n = 0$, it follows from (3.23) and the boundedness of $\{\phi_n\}$ that

$$\|\psi_{n+1} - \phi_n\| = \pi_n\|u - \phi_n\| \to 0 \text{ as } n \to \infty.$$

Assume, without loss of generality, that there is a subsequence $\{\psi_{n_k}\}$ of $\{\psi_n\}$ such that

$$\limsup_{n\to\infty}\langle u - \mathcal{P}_{\mathcal{F}(\Im)}u, \psi_{n+1} - \mathcal{P}_{\mathcal{F}(\Im)}u\rangle = \lim_{k\to\infty}\langle u - \mathcal{P}_{\mathcal{F}(\Im)}u, \psi_{n_k+1} - \mathcal{P}_{\mathcal{F}(\Im)}u\rangle$$

and $\psi_{n_k+1} \to \phi$ and $k \to \infty$. Since $\|\psi_{n+1} - \phi_n\| \to 0$ as $n \to \infty$, it follows that for arbitrary bounded linear functional g on $\mathfrak{H}$,

$$\begin{aligned}\|g(\phi_{n_k}) - g(\phi)\| &\leq |g(\phi_{n_k}) - g(\psi_{n_k+1})| + |g(\psi_{n_k+1}) - g(\phi)| \\ &\leq \|g\||\phi_{n_k} - \psi_{n_k+1}| + |g(\psi_{n_k+1}) - g(\phi)| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Consequently, $\phi_{n_k} \rightharpoonup \phi$ as $k \to \infty$, and hence $\phi \in \mathcal{F}(\Im)$. Since $\mathcal{P}_{\mathcal{F}(\Im)} : \mathfrak{H} \longrightarrow \mathcal{F}(\Im)$ is the projection map, it follows that

$$\lim_{k\to\infty}\langle u - \mathcal{P}_{\mathcal{F}(\Im)}u, \psi_{n_k+1} - \mathcal{P}_{\mathcal{F}(\Im)}u\rangle = \langle u - \mathcal{P}_{\mathcal{F}(\Im)}u, \phi - \mathcal{P}_{\mathcal{F}(\Im)}u\rangle \leq 0,$$

so that

$$\limsup_{n\to\infty}\langle u - \mathcal{P}_{\mathcal{F}(\Im)}u, \psi_{n+1} - \mathcal{P}_{\mathcal{F}(\Im)}u\rangle \leq 0.$$

From Lemma 2.1(ii) and (3.24) we get

$$\begin{aligned}\|\psi_{n+1} - \mathcal{P}_{\mathcal{F}(\Im)}u\|^2 &= \|\pi_n(u - \mathcal{P}_{\mathcal{F}(\Im)}u) + (1-\pi_n)(\phi_n - \mathcal{P}_{\mathcal{F}(\Im)}u)\|^2 \\ &\leq (1-\pi_n)^2\|\phi_n - \mathcal{P}_{\mathcal{F}(\Im)}u\|^2 + 2\pi_n\langle u - \mathcal{P}_{\mathcal{F}(\Im)}u, \psi_{n+1} - \mathcal{P}_{\mathcal{F}(\Im)}u\rangle \\ &\leq (1-\pi_n)\|\phi_n - \mathcal{P}_{\mathcal{F}(\Im)}u\|^2 + 2\pi_n\langle u - \mathcal{P}_{\mathcal{F}(\Im)}u, \psi_{n+1} - \mathcal{P}_{\mathcal{F}(\Im)}u\rangle.\end{aligned}$$

Since $\lim_{n\to\infty}\pi_n = 0$, $\sum_{n=1}^{\infty}\pi_n = \infty$, and $\limsup_{n\to\infty}\langle u - \mathcal{P}_{\mathcal{F}(\Im)}u, \psi_{n+1} - \mathcal{P}_{\mathcal{F}(\Im)}u\rangle \leq 0$, it follows by Lemma 2.3 that $\|\psi_n - \mathcal{P}_{\mathcal{F}(\Im)}u\| = 0$. In addition,

$$0 \leq \|\phi_n - \mathcal{P}_{\mathcal{F}(\Im)}u\| \leq \|\phi_n - \psi_{n+1}\| + \|\psi_{n+1} - \mathcal{P}_{\mathcal{F}(\Im)}u\| \to 0 \ \text{ as } n \to \infty.$$

Therefore $\lim_{n\to\infty}\|\phi_n - \mathcal{P}_{\mathcal{F}(\Im)}u\| = 0$. □

Remark 3.3 If we set $\eta = 0$ in Theorem 3.5, then Theorem 3.2 of [2] follows immediately.

Remark 3.4 The inclusion of an auxiliary map $\Im_{\xi,\eta}$ in our theorems also produces the following strong convergence result of Halpan type for the class of (η,β)-ESPN maps, which further gives an affirmative answer to the lingering open problem raised by Kurokawa and Takahashi in their last remark in [1] for the case in which the operator $\Im$ is averaged.

Theorem 3.6 *Consider a real Hilbert space $\mathfrak{H}$, $\emptyset \neq \Omega \subset \mathfrak{H}$, and an (η,β)-ESPN map $\Im : \Omega \longrightarrow \Omega$ such that $\mathcal{F}(\Im) \neq \emptyset$. Let $\xi \in [\beta,1)$, $\omega = \dfrac{1}{\eta+1}$ for $\eta \in [0,\infty)$, and $\Im_{\xi,\eta} = \xi I + (1-\xi)[(1-\omega)I + \omega\Im]$. Let $\{\pi_n\}$ be in $[0,1)$ such that (a) $\lim_{n\to\infty}\pi_n = 0$ and (b) $\sum_{n=1}^{\infty}\pi_n = \infty$. Let $u \in \Omega$ be fixed, and let $\{\psi_n\}$ be a sequence in Ω developed from arbitrary $\psi_1 \in \Omega$ by*

$$\psi_{n+1} = \pi_n u + (1-\pi_n)\Im_{\xi,\eta}\psi_n, \quad n \geq 1. \tag{3.29}$$

Then $\{\psi_n\}$ converges strongly to $\vartheta \in \mathcal{F}(\Im)$.

Proof Let $\mathcal{F}(\Im_{\xi,\eta}) = \mathcal{F}(\Im) \neq \emptyset$. Following the same technique as in the proof of Theorem 3.4, we obtain

$$\begin{aligned}
&\|\Im_{\xi,\eta}\psi - \Im_{\xi,\eta}\phi\|^2 \\
&\quad = \left\|\xi(\psi-\phi) + (1-\xi)\left[\frac{\eta+\Im}{\eta+1}\psi - \frac{\eta+\Im}{\eta+1}\phi\right]\right\|^2 \\
&\quad = \left\|\xi(\psi-\phi) + (1-\xi)\left[\frac{\eta}{\eta+1}(\psi-\phi) + \frac{1}{\eta+1}(\Im\psi - \Im\phi)\right]\right\|^2 \\
&\quad = \xi\|\psi-\phi\|^2 + (1-\xi)\left\|\frac{\eta}{\eta+1}(\psi-\phi) + \frac{1}{\eta+1}(\Im\psi-\Im\phi)\right\|^2 \\
&\qquad - \xi(1-\xi)\left\|\psi - \phi - \left[\frac{\eta}{\eta+1}(\psi-\phi) + \frac{1}{\eta+1}(\Im\psi-\Im\phi)\right]\right\|^2 \\
&\quad = \xi\|\psi-\phi\|^2 + \frac{(1-\xi)}{(\eta+)^2}\left\|\eta(\psi-\phi) + \Im\psi - \Im\phi\right\|^2 \\
&\qquad - \frac{\xi(1-\xi)}{(\eta+1)^2}\left\|\eta(\psi-\phi) + [\psi - (\eta+\Im)\psi - (\phi - (\eta+\Im))\phi]\right\|^2 \\
&\quad \leq \xi\|\psi-\phi\|^2 + \frac{(1-\xi)}{(\eta+)^2}\left\|\eta(\psi-\phi) + \Im\psi - \Im\phi\right\|^2 \\
&\qquad - \frac{\xi(1-\xi)}{(\eta+1)^2}\left\|\psi - (\eta+\Im)\psi - (\phi - (\eta+\Im)\phi)\right\|^2 \\
&\quad \leq \xi\|\psi-\phi\|^2 + \frac{(1-\xi)}{(\eta+1)^2}[(\eta+1)^2\|\psi-\phi\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + \beta\|\psi - \Im\psi - (\phi - \Im\phi)\|^2 + 2\langle\psi - \Im\psi, \phi - \Im\phi\rangle]\\
&\quad - \frac{\xi(1-\xi)}{(\eta+1)^2}\|\psi - (\eta+\Im)\psi - (\phi - (\eta+\Im)\phi)\|^2\\
&= \xi\|\psi - \phi\|^2 + (1-\xi)\|\psi - \phi\|^2\\
&\quad + \frac{(1-\xi)}{(\eta+1)^2}\beta\|\psi - \Im\psi - (\phi - \Im\phi)\|^2 + 2\frac{(1-\xi)}{(\eta+1)^2}\langle\psi - \Im\psi, \phi - \Im\phi\rangle\\
&\quad - \frac{\xi(1-\xi)}{(\eta+1)^2}\|\eta(\phi - \psi) + [\psi - \Im\psi - (\phi - \Im\phi)]\|^2\\
&\le \xi\|\psi - \phi\|^2 + (1-\xi)\|\psi - \phi\|^2\\
&\quad + \frac{(1-\xi)}{(\eta+1)^2}\beta\|\psi - \Im\psi - (\phi - \Im\phi)\|^2 + 2\frac{(1-\xi)}{(\eta+1)^2}\langle\psi - \Im\psi, \phi - \Im\phi\rangle\\
&\quad - \frac{\xi(1-\xi)}{(\eta+1)^2}\|\psi - \Im\psi - (\phi - \Im\phi)\|^2\\
&= \xi\|\psi - \phi\|^2 + (1-\xi)\|\psi - \phi\|^2 - \frac{(1-\xi)}{(\eta+1)^2}(\xi - \beta)\|\psi - \Im\psi - (\phi - \Im\phi)\|^2\\
&\quad + \frac{2}{(1-\xi)}\langle\psi - \Im_{\xi,\eta}\psi, \phi - \Im_{\xi,\eta}\phi\rangle. \qquad (3.30)
\end{aligned}
$$

Therefore, for all $\psi \in \Omega$ and $\vartheta \in \mathcal{F}(\Im_{\xi,\eta}) = \mathcal{F}(\Im)$, we get

$$
\|\Im_{\xi,\eta}\psi - \vartheta\|^2 \le \|\psi - \phi\|^2 - \frac{(1-\xi)}{(\eta+1)^2}(\xi - \beta)\|\psi - \Im\psi\|^2.
$$

Consequently, $\{\psi_n\}$ converges strongly to a point $\vartheta \in \mathcal{F}(\Im_{\xi,\eta}) = \mathcal{F}(\Im)$. □

Remark 3.5 If $\eta = 0$ in Theorem 3.6, then Theorem 3.3 of [2] follows immediately.

4 Conclusions

In this paper, weak and strong convergence theorems have been established for new classes of (η, β)-enriched strictly pseudononspreading maps in the setup of a real Hilbert space. Further, by means of a robust auxiliary map incorporated in our theorems we proved a strong convergence theorem of Halpern-type, thereby resolving in the affirmative the open problem raised by Kurokawa and Takahashi [1] in their concluding remark for the case in which the map $\Im$ is averaged. Also, we constructed some examples of the classes of maps studied to demonstrate their existence. The results obtained extend, improve, and generalize several well-known results in [2, 14, 15] and others.

Author contributions
I.K.A. made conceptualization, methodology and writing draft preparation. H.I. performed the formal analysis, writing-review and editing. D.I.I. made investigation, review and validation. All authors read and approved the final version.

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