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Relaxed inertial self-adaptive algorithm for the split feasibility problem with multiple output sets and fixed-point problem in the class of demicontractive mappings

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Abstract

The purpose of this paper is to study the split feasibility problem with multiple output sets and fixed point problem in the class of demicontractive mappings and propose relaxed inertial self-adaptive algorithms that do not use the least squares method. Under some appropriate assumptions, we establish a strong convergence result for the sequence generated by the proposed algorithm. Our result generalizes and extends several results existing in the literature. Finally, we illustrate the convergence of the proposed algorithm by using a numerical example.

Keywords: Split feasibility problems with multiple output sets; Fixed point; Generalized Fermat-Torricelli problem; Relaxed; Viscosity; Demicontractive

1 Introduction

Let C be a nonempty, closed, and convex subset of a real Hilbert space H, $S : C \to C$ be a mapping, and $\text{Fix}(S) := \{x \in C : Sx = x\}$. Then S is said to be

(a) nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|,$$

$\forall x, y \in C$.

(b) quasi-nonexpansive if Fix(S) $\neq \emptyset$ and

$$\|Sx - y\| \leq \|x - y\|,$$

$\forall x \in C$ and $y \in \text{Fix(S)}$.

(c) k-strictly pseudo-contractive if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2,$$

$\forall x, y \in C$.

(d) demicontractive if Fix(S) $\neq \emptyset$ and there exists a constant $k \in [0,1)$ such that

$$\|Sx - y\|^2 \leq \|x - y\|^2 + k\|(I - S)x\|^2,$$

$\forall x \in C$ and $y \in$ Fix(S).

Observe that the class of demicontractive mappings include various types of nonlinear mappings such as nonexpansive mappings, quasi-nonexpansive mappings, and strictly pseudo-contractive mappings.

A fixed point problem for a demicontractive mapping $S : C \to C$: Find $x \in C$ such that

$$Sx = x. \tag{1}$$

The split feasibility problem (SFP) is to find a point

$$x^* \in C \text{ such that } Ax^* \in Q, \tag{2}$$

where C and Q are nonempty, closed, and convex subsets of real Hilbert spaces H and H_1, respectively and $A : H \to H_1$ is a bounded linear operator. Let the solution set of the SFP (2) is given by $SFP(C,Q) := \{x \in C : Ax \in Q\}$.

The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [7] for modeling various inverse problems that have many real life applications such as medical image reconstruction and signal processing (see [5, 6]). The SFP attracts the attention of many authors due to its wide range of applications and several generalizations of the SFP have been studied by many authors, see, for instance, the multiple-sets SFP [8, 20, 21], the SFP with multiple output sets (SFPMOS) [12, 15, 17, 18, 23, 24], the split variational inequality problem [9, 10, 15], the multiple-sets split variational inequality problem [25], the split variational inequality problem with multiple output sets [1], and the multiple-sets split feasibility problem with multiple output sets [13, 22].

In 2024, Berinde [4] introduced an inertial self-adaptive viscosity algorithm for solving split feasibility and fixed point problems in the class of demicontractive mappings, which is shown below.

$$\begin{aligned} v_n &:= z_n + \mu_n(z_n - z_{n-1}) \\ w_n &:= p_C\left((1-\beta_n)(v_n - \zeta_n A^*(I - P_Q)Av_n) + \beta_n S_\lambda v_n\right), \\ z_{n+1} &:= \sigma_n g(z_n) + \theta_n v_n + \alpha_n w_n, \end{aligned}$$

with $S_\lambda := (1-\lambda)I + \lambda S, \lambda \in (0,1)$,

$$\mu_n = \begin{cases} \min\left\{\mu, \dfrac{\tau_n}{||z_n - z_{n-1}||}\right\} & \text{if } z_n \neq z_{n-1}, \\ \mu & \text{otherwise,} \end{cases} \tag{3}$$

$\mu \geq 0$ is a given number, $\zeta_n := \frac{\delta_n f(v_n)}{\|\nabla f(v_n)\|^2}$ where $f(v_n) := \frac{1}{2}\|(I - P_Q)Av_n\|^2$, $\delta_n \in (0,4)$ and $\{\sigma_n\}, \{\theta_n\}, \{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1)$ and τ_n is a positive sequence satisfying some suitable conditions, and $S : C \to C$ is a k-demicontractive mapping. He proved the strong convergence of the sequence generated by his algorithm to some $x^* \in$ Fix$(S) \cap SFP(C,Q)$.

The SFPMOS which was introduced by Kim et al. [13] in general Hilbert spaces is to find a point x^* such that

$$x^* \in C \cap \left(\cap_{j=1}^{p} T_j^{-1} \left(\cap_{k=1}^{r_j} Q_{jk} \right) \right) \neq \emptyset, \tag{4}$$

where C and $Q_{jk}, j = 1, 2, \ldots, p, k = 1, \ldots, r_j$ are nonempty, closed and convex subsets of real Hilbert spaces H and H_j, respectively, and $T_j : H \to H_j$ are bounded linear operators.

In order to approximate a solution to the SFP (2), many algorithms first transform this problem into an equivalent unconstrained convex minimization problem and obtain a minimizing element using the least squares method. In 2023, Reich and Tuyen [19] extended the Fermat-Torricelli problem and showed that the SFPMOS can be considered as a special case of this problem. Moreover, they provide a new approach for solving the SFPMOS in Hilbert spaces. The generalized Fermat-Torricelli problem is stated as follows:

$$\begin{aligned} &f(x) \to \min, \\ &\text{subject to } x \in C, \end{aligned}$$

where $f(x) = \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} f_{jk}(T_j x), \beta_{jk}, j = 1, 2, \cdots, p, k = 1, 2, \cdots, r_j$, are given positive real numbers, and $f_{jk}(y) = \|(I^{H_j} - P_{Q_{jk}}^{H_j})y\|$ for all $y \in H_j$ and $j = 1, 2, \cdots, p, k = 1, 2, \cdots, r_j$.

As a continuation of the aforementioned work, Reich and Tuyen [16] developed self-adaptive algorithms for solving the split feasibility problem with multiple output sets that do not use the least square method and proved strong and weak convergence theorems.

Motivated by the above works specially that of Berinde [4], Riech and Tuyen [16], and Kim et al. [13], we propose relaxed inertial self-adaptive algorithm for solving the SFPMOS and fixed-point problem in the class of demicontractive mappings. The main contributions of our paper are summarized as follows.

- The problem considered is a general problem as it combines the SFPMOS and fixed point problem.
- The proposed algorithm incorporates the relaxation method in order to speed up its convergence.
- The proposed method does not use the least squares method.
- Our result generalizes and extends several related results existing in the literature as demicontractive mappings include various types of nonlinear mappings.

This work is structured as follows. In Sect. 2, we state some notations, basic definitions, and lemmas that we will need in the proof of our main result. In Sect. 3, we give convergence analysis of our proposed algorithm. In Sect. 4, we provide a numerical experiment to validate our proposed algorithm. Finally, in Sect. 5, we give a concluding remark.

2 Preliminaries

The weak ω-limit set of $\{t_n\}$ is given by $\omega_\omega(t_n) = \left\{t \in H : \exists \{t_{n_k}\} \subseteq \{t_n\} \text{ such that } t_{n_k} \rightharpoonup t\right\}$.

It is well known that for every element $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that

$$\|x - P_C(x)\| = \min\{\|x - z\| : z \in C\}.$$

The operator $P_C : H \to C$ is called a metric projection of H onto C. It has got important characterization shown below:

$$\langle x - P_C x, z - P_C x\rangle \leq 0, \tag{5}$$

for all $x \in H$ and $z \in C$. We can deduce from (5) that the operator P_C is a nonexpansive mapping.

Lemma 1 (*see* [2]) *For all* $x, y \in H$ *and* $z \in C$, *the following inequalities hold.*

(i) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$;
(ii) $\langle x - y, (I - P_C)x - (I - P_C)y\rangle \geq \|(I - P_C)x - (I - P_C)y\|^2$;
(iii) $\|P_C x - z\|^2 \leq \|x - z\|^2 - \|P_C x - x\|^2$.

Definition 1 Let $f : H \to (-\infty, +\infty]$ be a given function. Then,

(1) The function f is proper if

$$\{x \in H : f(x) < +\infty\} \neq \emptyset.$$

(2) A proper function f is convex if for each $\sigma \in (0, 1)$,

$$f(\sigma x + (1 - \sigma)y) \leq \sigma f(x) + (1 - \sigma)f(y), \forall x, y \in H.$$

(3) f is σ-strongly convex, where $\sigma > 0$, if

$$f(\delta x + (1 - \delta)y) + \frac{\sigma}{2}\delta(1 - \delta)\|x - y\|^2 \leq \delta f(x) + (1 - \delta)f(y), \forall \delta \in (0, 1) \text{ and } \forall x, y \in H.$$

Moreover, f is σ-strongly convex if $f(x) - (\sigma/2)\|x\|^2$ is convex.

Definition 2 Let $f : H \to (-\infty, +\infty]$ be a proper function.

(1) A vector $\xi \in H$ is a subgradient of f at a point x if

$$f(y) \geq f(x) + \langle \xi, y - x\rangle, \forall y \in H.$$

(2) The set of all subgradients of f at $x \in H$, denoted by $\partial f(x)$, is called the subdifferential of f, and is defined by

$$\partial f(x) = \{\xi \in H : f(y) \geq f(x) + \langle \xi, y - x\rangle, \text{ for each } y \in H\}.$$

(3) If $\partial f(x) \neq \emptyset$, f is said to be subdifferentiable at x. If the function f is continuously differentiable then $\partial f(x) = \{\nabla f(x)\}$.

Definition 3 Let $f : H \to (-\infty, +\infty]$ be a proper function. Then, f is lower semi-continuous (lsc) at x if $x_n \to x$ implies

$$f(x) \leq \liminf_{k\to\infty} f(x_n).$$

Definition 4 Let C be a closed and convex subset of a real Hilbert space H and $S : C \to C$ is a mapping. If, for any sequence $\{x_k\}$ in C, such that $x_k \rightharpoonup x$, and $Sx_k \to 0$, we have $Sx = 0$, then S is said to be demiclosed at 0 in C.

Lemma 2 [3] *Let H be a real Hilbert space, $C \subset H$ be a closed and convex set. If $T : C \to C$ is k-demicontractive, then for any $\lambda \in (0, 1-k)$, $T_\lambda := (1-\lambda)I + \lambda T$ is a quasi-nonexpansive.*

Lemma 3 (*see* [11]) *Let $\{s_n\}$ be a non-negative real sequence, such that*

$$s_{n+1} \leq (1-\sigma_n)s_n + \sigma_n\mu_n,\ n \geq 1,$$
$$s_{n+1} \leq s_n - \phi_n + \varphi_n,\ n \geq 1,$$

where $\{\sigma_n\} \subset (0,1)$, $\{\phi_n\} \subset [0,\infty)$, and $\{\mu_n\}, \{\varphi_n\} \subset (-\infty,\infty)$. In addition, suppose the following conditions hold.
(i) $\sum_{n=1}^{\infty} \sigma_n = \infty$;
(ii) $\lim_{n\to\infty} \varphi_n = 0$;
(iii) $\lim_{k\to\infty} \phi_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \mu_{n_k} \leq 0$ *for every subsequence* $\{n_k\}$ *of* $\{n\}$.
Then, $\lim_{n\to\infty} s_n = 0$.

3 Main results

Let C and Q_{jk}, $j = 1,2,\dots,p$, $k = 1,\dots,r_j$ be nonempty, closed and convex subsets of real Hilbert spaces H and H_j, respectively, and $T_j : H \to H_j$ are bounded linear operators. Let $S : C \to C$ be a demicontractive mapping. Assuming that

$$x^* \in \Omega := C \cap \Big(\cap_{j=1}^{p} T_j^{-1} \Big(\cap_{k=1}^{r_j} Q_{jk} \Big)\Big) \cap \operatorname{Fix}(S) \neq \emptyset, \tag{6}$$

we consider the problem of finding an element $x^* \in \Omega$.

In this section, we state our algorithms and analyze their convergence.

For simplicity, let $\Gamma := C \cap \Big(\cap_{j=1}^{p} T_j^{-1} \Big(\cap_{k=1}^{r_j} Q_{jk} \Big)\Big)$, $J_1 := \{1,2,\dots,p\}$, and $J_2 := \{1,2,\dots,r_j\}$.

We take the following assumptions to undergo the analysis.

(C1) The nonempty level sets C and Q_{jk} in the Problem (6) are defined as follows

$$C = \{x \in H : c(x) \leq 0\} \quad \text{and} \quad Q_{jk} = \{y \in H_j : q_{jk}(y) \leq 0\}, \tag{7}$$

where $c : H \to (-\infty,+\infty]$ and $q_{jk} : H_j \to (-\infty,+\infty]$ for all $j \in J_1, k \in J_2$ are ϖ-strongly and ω_j-strongly convex subdifferentiable functions, respectively. Then c and q_{jk} are also lower semicontinuous (See, [2] Theorem 9.1)

The projections onto C and Q_{jk} are not easily implemented in general. To avoid this difficulty, we adopted a technique of projecting on to the half spaces and construct the relaxed sets (half-spaces) C^n and $Q_{jk}^n (j \in J_1, k \in J_2$ (see [14] for more).

(C2) Let c and q_{jk} be as defined in (7). Assume that at least one subgradient $\xi_n \in \partial c(x)$ and $\eta_{jk,n} \in \partial q_{jk}(y)$ can be computed for any $x \in H$ and $y \in H_j$. Moreover, both ∂c and $\partial q_{jk} (j \in J_1, k \in J_2)$ are bounded operators (bounded on bounded sets). The sets C^n and Q_{jk}^n $(j \in J_1, k \in J_2)$ are constructed as follows:

$$C^n = \Big\{ x \in H : c(x_n) + \langle \xi_n,\ x - x_n \rangle + \frac{\varpi}{2} \|x - x_n\|^2 \leq 0 \Big\}, \tag{8}$$

where $\xi_n \in \partial c(x_n)$ and

$$Q_{jk}^n = \left\{ y \in H_j : q_{jk}(T_j x_n) + \langle \eta_{jk,n},\ y - T_j x_n \rangle + \frac{\omega_j}{2} \|y - T_j x_n\|^2 \leq 0 \right\}, \tag{9}$$

where $\eta_{jk,n} \in \partial q_{jk}(T_j x_n)$. It is not difficult to show that $C \subset C^n$ and $Q_{jk} \subset Q_{jk}^n$.

(C3) Let the sequence $\{\rho_n\} \in (0,2)$, the sequences $\{\delta_n\}, \{\sigma_n\}, \{\gamma_n\}$, and $\{\alpha_n\}$ in $(0,1)$, and the sequence $\{\epsilon_n\} \in (0,\infty)$ satisfying the following conditions.
(i) $\liminf_{n\to\infty} \gamma_n > 0$;
(ii) $\lim\limits_{n\to\infty} \frac{\epsilon_n}{\sigma_n} = 0$;
(iii) $\lim_{n\to\infty} \sigma_n = 0$ and $\sum_{n=1}^{\infty} \sigma_n = \infty$;
(iv) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$;
(v) $\sigma_n + \alpha_n + \gamma_n = 1, \forall n \geq 1$.

Now, we delve into the details of the convergence analysis of Algorithm 1.

Proposition 1 In Algorithm 1, if $\sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n = 0$ and $w_n = y_n$, then y_n is a solution of Problem (6).

Proof Pick $t^* \in \Omega$. For each n, let $\Delta_n = \{(j,k) : d_{jk}^n \neq 0\}$.

By using an argument similar to the one used in the proof of Proposition 6 of [16], we get $y_n \in \Gamma$. Moreover, since $\sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n = 0$ and $w_n = y_n$, and using (14), we have $y_n \in \operatorname{Fix}(S)$. Consequently, $y_n \in \Omega = \Gamma \cap \operatorname{Fix}(S)$. □

Lemma 4 *Let $\Omega \neq \emptyset$ and $\{t_n\}$ be a sequence generated by Algorithm 1 such that Assumption $C(3)$ holds. Then the sequence $\{t_n\}$ is bounded.*

Proof Let $t^* \in \Omega$, then we have

$$\begin{aligned}
\|w_n - t^*\|^2 &= \left\| (1-\delta_n)\left(y_n - \tau_n \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \right) + \delta_n S_\lambda y_n - t^* \right\|^2 \\
&= \left\| (1-\delta_n)\left((y_n - t^*) - \tau_n \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \right) + \delta_n (S_\lambda y_n - t^*) \right\|^2 \\
&= (1-\delta_n) \left\| (y_n - t^*) - \tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} T_j^* d_{jk}^n \right\|^2 + \delta_n \|S_\lambda y_n - t^*\|^2 - \\
&\quad \delta_n(1-\delta_n) \left\| S_\lambda y_n - y_n + \tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} T_j^* d_{jk}^n \right\|^2 .
\end{aligned} \tag{17}$$

Estimating $\left\| (y_n - t^*) - \tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} T_j^* d_{jk}^n \right\|^2$, we get

$$\begin{aligned}
&\left\| (y_n - t^*) - \tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} T_j^* d_{jk}^n \right\|^2 \\
&= \|y_n - t^*\|^2 + \tau_n^2 \left\| \sum_{(j,k)\in\Delta_n} \beta_{jk} T_j^* d_{jk}^n \right\|^2 - 2\tau_n \left\langle \sum_{(j,k)\in\Delta_n} \beta_{jk} T_j^* d_{jk}^n, y_n - t^* \right\rangle
\end{aligned}$$

Algorithm 1 A strongly convergent algorithm for solving problem (6)

Step 0. Choose the sequences $\{\rho_n\}, \{\delta_n\}, \{\sigma_n\} \{\alpha_n\}, \{\epsilon_n\}$, and $\{\gamma_n\}$ in such away that they satisfy Assumption $(C3)$ and a positive constant τ.

Step 1. Select arbitrary points $t_0, t_1 \in C$ and choose $\theta \in (0,1)$ such that $0 \leq \theta_n \leq \hat{\theta}_n$ where

$$\tilde{\theta}_n = \begin{cases} \min\left\{\theta, \dfrac{\epsilon_n}{||t_n - t_{n-1}||}\right\} & \text{if } t_n \neq t_{n-1} \\ \theta & \text{otherwise} \end{cases}. \tag{10}$$

Step 2. Compute

$$y_n = t_n + \theta_n(t_n - t_{n-1}). \tag{11}$$

Step 3. Compute

$$d_{jk}^n = \begin{cases} \dfrac{\left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j y_n}{\left\|\left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j y_n\right\|} \text{ if } T_j y_n \notin Q_{jk}^n \\ 0 \text{ if } T_j y_n \in Q_{jk}^n \end{cases}, \tag{12}$$

for all $j \in J_1$ and $k \in J_2$.

Step 4. If $\sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n = 0$ and $w_n = y_n$, then stop. If not, compute z_n via

$$z_n = P_{C^n} w_n, \tag{13}$$

where

$$w_n := (1 - \delta_n)\left(y_n - \tau_n \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n\right) + \delta_n S_\lambda y_n, \tag{14}$$

$S_\lambda := (1 - \lambda)I + \lambda S$ where $\lambda \in (0,1)$, C^n, and Q_{jk}^n are defined as in (8) and (9), respectively, and

$$\tau_n := \frac{\rho_n \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} f_{jk}(T_j y_n)}{\Xi_n^2}, \tag{15}$$

where $f_{jk}(y) = \left\|\left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right)(y)\right\|$ for all $y \in \mathcal{H}_j$ and for all $j \in J_1, k \in J_2$ and $\Xi_n := \max\{\tau, \| \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \|\}$.

Step 5. Compute

$$t_{n+1} = \sigma_n \nu(t_n) + \alpha_n y_n + \gamma_n z_n, \tag{16}$$

where $\nu : \mathcal{H} \to C$ is a μ-contraction mapping such that $\mu \in (0,1)$.

Step 6. Set $n := n + 1$ and return to **Step 1.**

$$
\begin{aligned}
&= \|v_n - t^*\|^2 + \tau_n^2 \left\| \sum_{j=1}^{p}\sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \right\|^2 - 2\tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} \left\langle d_{jk}^n, T_j y_n - T_j t^* \right\rangle \\
&= \|y_n - t^*\|^2 + \tau_n^2 \left\| \sum_{j=1}^{p}\sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \right\|^2 \\
&\quad - 2\tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} \left\langle \frac{\left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j y_n}{\left\| \left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j y_n \right\|}, T_j y_n - T_j t^* \right\rangle \\
&= \|y_n - t^*\|^2 + \tau_n^2 \left\| \sum_{j=1}^{p}\sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \right\|^2 \\
&\quad - 2\tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} \left\langle \frac{\left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j y_n - \left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j t^*}{\left\| \left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j y_n \right\|}, T_j y_n - T_j t^* \right\rangle \\
&\le \|y_n - t^*\|^2 + \tau_n^2 \left\| \sum_{j=1}^{p}\sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \right\|^2 - 2\tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} \left\| \left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j y_n \right\|. \qquad (18)
\end{aligned}
$$

Substituting (15) into (18) and simplifying, we get

$$
\begin{aligned}
\left\| (y_n - t^*) - \tau_n \sum_{(j,k)\in\delta_n} \beta_{jk} T_j^* d_{jk}^n \right\|^2 &\le \|y_n - t^*\|^2 - \rho_n(2-\rho_n) g_{jk}(y_n) \qquad (19) \\
&\le \|y_n - t^*\|^2, \qquad (20)
\end{aligned}
$$

where

$$
g_{jk}(y_n) := \left(\frac{\sum_{(j,k)\in\Delta_n} \beta_{jk} f_{jk}(T_j y_n)}{\Xi_n} \right)^2. \qquad (21)
$$

Again, substituting (19) into (17) and simplifying, we get

$$
\begin{aligned}
\|w_n - t^*\|^2 \le \|y_n - t^*\|^2 - (1-\delta_n)\rho_n(2-\rho_n) g_{jk}(y_n) - \\
\delta_n(1-\delta_n) \left\| S_\lambda y_n - y_n + \tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} T_j^* d_{jk}^n \right\|^2. \qquad (22)
\end{aligned}
$$

Using the definition of z_n, we have

$$
\begin{aligned}
\|z_n - t^*\|^2 &= \left\| P_{C^n} w_n - t^* \right\|^2 \\
&\le \|w_n - t^*\|^2 - \|(I - P_{C^n}) w_n\|^2. \qquad (23)
\end{aligned}
$$

Substituting (22) into (23), we get

$$\begin{aligned}\|z_n - t^*\|^2 \le{}& \|y_n - t^*\|^2 - (1-\delta_n)\rho_n(2-\rho_n)g_{jk}(y_n) - \\ & \delta_n(1-\delta_n)\left\| S_\lambda y_n - y_n + \tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} T_j^* d_{jk}^n \right\|^2 - \\ & \|(I - P_{C^n})w_n\|^2 \qquad (24)\\ \le{}& \|y_n - t^*\|^2. \qquad (25)\end{aligned}$$

Now, denote

$$u_n := \frac{1}{1-\sigma_n}\left(\alpha_n y_n + \gamma_n z_n\right). \qquad (26)$$

It follows that

$$\begin{aligned}\|u_n - t^*\|^2 &= \left\| \frac{\alpha_n}{1-\sigma_n} y_n + \frac{\gamma_n}{1-\sigma_n} z_n - t^* \right\|^2 \\ &= \left\| \frac{\alpha_n}{1-\sigma_n}(y_n - t^*) + \frac{\gamma_n}{1-\sigma_n}(z_n - t^*) \right\|^2 \\ &\le \frac{\alpha_n}{1-\sigma_n}\|y_n - t^*\|^2 + \frac{\gamma_n}{1-\sigma_n}\|z_n - t^*\|^2. \qquad (27)\end{aligned}$$

Substituting (24) into (27) and simplifying, we get

$$\begin{aligned}\|u_n - t^*\|^2 \le{}& \|y_n - t^*\|^2 - (1-\delta_n)\rho_n(2-\rho_n)\frac{\gamma_n}{1-\sigma_n} g_{jk}(y_n) - \\ & \delta_n(1-\delta_n)\frac{\gamma_n}{1-\sigma_n}\left\| S_\lambda y_n - y_n + \tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} T_j^* d_{jk}^n \right\|^2 - \\ & \frac{\gamma_n}{1-\sigma_n}\|(I - P_{C^n})w_n\|^2 \qquad (28)\\ \le{}& \|y_n - t^*\|^2. \qquad (29)\end{aligned}$$

Using the definition of y_n, we have

$$\begin{aligned}\|y_n - t^*\| &= \|t_n + \theta_n(t_n - t_{n-1}) - t^*\| \\ &\le \|t_n - t^*\| + \theta_n\|t_n - t_{n-1}\| \\ &= \|t_n - t^*\| + \sigma_n\left[\frac{\theta_n}{\sigma_n}\|t_n - t_{n-1}\|\right] \qquad (30)\end{aligned}$$

Using the definition of t_n and (29), we obtain

$$\begin{aligned}\|t_{n+1} - t^*\| &= \|\sigma_n(v(t_n) - t^*) + (1-\sigma_n)(u_n - t^*)\| \\ &= \sigma_n\|v(t_n) - t^*\| + (1-\sigma_n)\|u_n - t^*\| \\ &\le \sigma_n\|v(t_n) - v(t^*)\| + \sigma_n\|v(t^*) - t^*\| + (1-\alpha_n)\|u_n - t^*\| \qquad (31)\end{aligned}$$

$$\leq \mu\sigma_n\|t_n - t^*\| + \sigma_n\|v(t^*) - t^*\| + (1-\sigma_n)\|u_n - t^*\|$$
$$\leq \mu\sigma_n\|t_n - t^*\| + \sigma_n\|v(t^*) - t^*\| + (1-\sigma_n)\|y_n - t^*\|$$

Now, combining (30) and (31), we have

$$\|t_{n+1} - t^*\| \leq [1-(1-\mu)\sigma_n]\|t_n - t^*\| + \sigma_n\left[\frac{\theta_n}{\sigma_n}\|t_n - t_{n-1}\| + \|v(t^*) - t^*\|\right]. \quad (32)$$

Using $C(3)(ii)$ and (10), we have $\lim_{n\to\infty}\frac{\theta_n}{\sigma_n}\|t_n - t_{n-1}\| = 0$. Hence, we can find a constant $M \geq 0$ such that

$$\frac{\theta_n}{\sigma_n}\|t_n - t_{n-1}\| \leq M.$$

Now, (32) becomes

$$\begin{aligned}\|t_{n+1} - t^*\| \leq &[1-(1-\eta)\sigma_n]\|t_n - t^*\| + \sigma_n\big[M + \|v(t) - t^*\|\big]\\ =&[1-(1-\eta)\sigma_n]\|t_n - t^*\| + \sigma_n(1-\eta)\left[\frac{M + \|v(t) - t^*\|}{1-\eta}\right].\end{aligned}$$

Proceeding inductively, we arrive at

$$\|t_{n+1} - t^*\| \leq \max\left\{\|t_1 - t^*\|, \frac{M + \|v(t^*) - t^*\|}{1-\eta}\right\},$$

for all $n \geq 1$, which proves that $\{t_n\}$ is bounded. □

Lemma 5 *Let $\Omega \neq \emptyset$, $S : C \to C$ be a demicontractive mapping such that $I - S$ is demiclosed at zero, $v : C \to C$ be a μ-contraction, and suppose that $\{\sigma_n\}, \{\rho_n\}, \{\alpha_n\}, \{\gamma_n\}, \{\delta_n\}$, and $\{\epsilon_n\}$ are sequences satisfying Assumption $C(3)$.*

Let $t^* \in \Omega$, $t^* = p_\Omega v(t^*)$, $\{t_n\}$ be the sequence generated by Algorithm 1, the function $g_{jk}(y_n)$ and the sequence u_n be given as in (21) and (26), respectively.

For $n \geq 1$, let us denote

$$\Theta_n := 2(1-\mu)\sigma_n;$$
$$\Phi_n := 2\sigma_n\langle t_{n+1} - t^*, v(t_n) - t^*\rangle;$$
$$\begin{aligned}\Delta_n := \frac{1}{2(1-\mu)}\bigg(&2((1-\sigma_n)^2 + \mu\sigma_n)\frac{\epsilon_n}{\sigma_n}\|y_n - t^*\| + \sigma_n\|v(t_n) - t^*\|^2 + \\ &2\sigma_n\|v(t_n) - t^*\|\|u_n - t^*\| + \sigma_n\|t_n - t^*\|^2 + 2\langle v(t^*) - t^*, u_n - t^*\rangle + \sigma_n\|t_n - t^*\|^2\bigg);\end{aligned}$$

and

$$\begin{aligned}\Psi_n :=\ &(1-\delta_n)\rho_n(4-\rho_n)\frac{\gamma_n}{1-\sigma_n}g_{jk}(y_n) + \\ &\delta_n(1-\delta_n)\frac{\gamma_n}{1-\sigma_n}\left\|S_\lambda y_n - y_n + \tau_n\sum_{(j,k)\in\Delta_n}\beta_{jk}T_j^* d_{jk}^n\right\|^2 +\end{aligned}$$

$$\frac{\gamma_n}{1-\sigma_n}\|(I-P_{C^n})w_n\|^2.$$

Then, for any subsequence $\{n_l\}$ *of* n, *we have*

$$\limsup_{l\to\infty}\Delta_{n_l}\leq 0, \tag{33}$$

whenever,

$$\lim_{l\to\infty}\Psi_{n_l}=0. \tag{34}$$

Proof Suppose (34) holds. It follows that

$$\lim_{l\to\infty}g_{jk}(y_{n_l})=0, \tag{35}$$

and based on the assumptions listed under $C(3)$, it follows that

$$\lim_{l\to\infty}\frac{\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}f_{jk}(T_jy_{n_l})}{\Xi_{n_l}}=0, \tag{36}$$

for all $(j,k)\in\Delta_{n_l}$.

Let $\Xi:=\max\{\tau,\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}\|T_j^*\|\}$. By using $\|d_{jk}^{n_l}\|=1$ for all $(j,k)\in\Delta_{n_l}$, we get

$$0\leq\frac{\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}f_{jk}(T_jy_{n_l})}{\Xi}\leq\frac{\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}f_{jk}(T_jy_{n_l})}{\Xi_{n_l}}. \tag{37}$$

Combining (36) and (37), we have

$$\lim_{l\to\infty}\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}f_{jk}(T_jy_{n_l})=0, \tag{38}$$

or equivalently

$$\lim_{l\to\infty}\|(I^{\mathcal{H}_j}-P^{\mathcal{H}_j}_{Q^{n_l}_{jk}})T_jy_{n_l}\|=0,$$

for all $(j,k)\in\Delta_{n_l}$.

Note that from the definition of Δ_{n_l} and $d_{jk}^{n_l}$, we have $T_jy_{n_l}\in Q_{jk}^{n_l}$ when $(j,k)\notin\Delta_{n_l}$ and hence $\|(I^{\mathcal{H}_j}-P^{\mathcal{H}_j}_{Q^{n_l}_{jk}})T_jy_{n_l}\|=0$. As a result, we get

$$\lim_{l\to\infty}\|(I^{\mathcal{H}_j}-P^{\mathcal{H}_j}_{Q^{n_l}_{jk}})T_jy_{n_l}\|=0, \tag{39}$$

for all $j\in J_1$ and $k\in J_2$.

By (34), we also get

$$\lim_{l\to\infty}\left\|S_\lambda y_{n_l}-y_{n_l}+\tau_{n_l}\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}T_j^*d_{jk}^{n_l}\right\|^2=0,$$

and due to (38), we obtain

$$\lim_{l\to\infty}\left\|S_\lambda y_{n_l}-y_{n_l}\right\|=0. \tag{40}$$

On the other hand, by (34) and using the definition of z_n, we also obtain

$$\begin{aligned}
&\lim_{l\to\infty}\left\|(I-P_{C^{n_l}})w_{n_l}\right\|\\
&=\lim_{l\to\infty}\left\|w_{n_l}-P_{C^n}w_{n_l}\right\|\\
&=\lim_{l\to\infty}\left\|\left((1-\delta_{n_l})\left(y_{n_l}-\tau_{n_l}\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}T_j^*d_{jk}^{n_l}\right)+\delta_{n_l}S_\lambda y_{n_l}\right)-z_{n_l}\right\|\\
&=\lim_{l\to\infty}\left\|(1-\delta_{n_l})y_{n_l}+\delta_{n_l}S_\lambda y_{n_l}-z_{n_l}-(1-\delta_{n_l})\tau_{n_l}\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}T_j^*d_{jk}^{n_l}\right\|\\
&=0.
\end{aligned} \tag{41}$$

Using (38) and (41), we get

$$\lim_{l\to\infty}\left\|(1-\delta_{n_l})y_{n_l}+\delta_{n_l}S_\lambda y_{n_l}-z_{n_{l l}}\right\|=\lim_{l\to\infty}\left\|y_{n_l}-z_{n_{l l}}+\delta_{n_l}\left(S_\lambda y_{n_l}-y_{n_l}\right)\right\|=0. \tag{42}$$

Similarly, using (40) and (42), we get

$$\lim_{l\to\infty}\|y_{n_l}-z_{n_l}\|=0. \tag{43}$$

By using the definition of u_n, we have

$$\begin{aligned}
\|u_{n_l}-y_{n_l}\| &= \left\|\frac{\alpha_{n_l}}{1-\sigma_{n_l}}y_{n_l}+\frac{\gamma_{n_l}}{1-\sigma_{n_l}}z_{n_l}-y_{n_l}\right\|\\
&\le \frac{\gamma_{n_l}}{1-\sigma_{n_l}}\left\|z_{n_l}-y_{n_l}\right\|,
\end{aligned}$$

which, by (43), yields

$$\lim_{l\to\infty}\|u_{n_l}-y_{n_l}\|=0. \tag{44}$$

Next, we need to show that $\omega_w(y_n)\subset\Omega$. Since $\{y_n\}$ is bounded, $\omega_w(y_n)\neq\emptyset$. Let $\bar{y}\in\omega_w(y_n)$. It follows that there exists a subsequence $\{y_{n_l}\}$ of $\{y_n\}$ such that $y_{n_l}\rightharpoonup\bar{y}$.

Now, due to the linearity and boundedness of T_j, we have $T_jy_{n_l}\rightharpoonup T_j\bar{y}$.

We claim that $\bar{y}\in\Omega$. To show this, it is suffices to show that $\bar{y}\in C^n$ and $T_j(\bar{y})\in Q_{jk}^n$ for all $j\in J_1,k\in J_2$.

From the assumption (C2), we can see that ∂q_{jk} is bounded on bounded sets for each $j\in J_1,k\in J_2$. It follows that we can find a constant $\eta>0$ such that $\|\eta_{jk,n_l}\|\le\eta$, where $\eta_{jk,n_l}\in\partial q_{jk}(T_jy_{n_l})$ for each $j\in J_1,k\in J_2$.

Now, using (9), (39), and the fact that $P_{Q_{jk}}^{n_l}\left(T_jy_{n_l}\right) \in Q_{jk}^{n_l}$, we get

$$\begin{aligned} q_{jk}\left(T_jy_{n_l}\right) &\le \left\langle \eta_{jk,n_l},\ T_jy_{n_l} - P_{Q_{jk}^{n_l}}\left(T_jy_{n_l}\right)\right\rangle - \frac{\omega_j}{2}\left\| T_jy_{n_l} - P_{Q_{jk}^{n_l}}\left(T_jy_{n_l}\right)\right\|^2 \\ &\le \left\langle \eta_{jk,n_l},\ T_jy_{n_l} - P_{Q_{jk}^{n_l}}\left(T_jy_{n_l}\right)\right\rangle \\ &\le \left\|\eta_{jk,n_l}\right\| \left\| \left(I - P_{Q_{jk}^{n_l}}\right)T_jy_{n_l}\right\| \\ &\le \eta \left\| \left(I - P_{Q_{jk}^{n_l}}\right)T_jy_{n_l}\right\| \to 0. \end{aligned} \tag{45}$$

Noting q_{jk} is weakly lower semi-continuous, it follows that

$$q_{jk}(T_j\bar{y}) \le \liminf_{l\to\infty} q_{jk}\left(T_jy_{n_l}\right) \le \lim_{l\to\infty} \eta \left\| \left(I - P_{Q_{jk}^{n_l}}\right)T_jy_{n_l}\right\| = 0,$$

for all $j \in J_1, k \in J_2$. It turns out that, $T_j\bar{y} \in Q_{jk}$ for all $j \in J_1, k \in J_2$.

Again, from the assumption (C2), we can see that ∂c is bounded on bounded sets. It follows that there is a constant $\xi > 0$ such that $\|\xi_{n_l}\| \le \xi$, where $\xi_{n_l} \in \partial c(y_{n_l})$.

By using (8) and (44), we have as $l \to \infty$ that

$$\begin{aligned} c(y_{n_l}) &\le \left\langle \xi_{n_l},\ u_{n_l} - y_{n_l}\right\rangle - \frac{\varpi}{2}\|u_{n_l} - y_{n_l}\|^2 \\ &\le \|\xi_{n_l}\| \|u_{n_l} - y_{n_l}\| \\ &\le \xi \|u_{n_l} - y_{n_l}\| \to 0. \end{aligned} \tag{46}$$

Noting c is weakly lower semi-continuous, it follows that

$$c(\bar{y}) \le \liminf_{l\to\infty} c(y_{n_l}) \le \lim_{l\to\infty} \xi \left\| y_{n_l} - u_{n_l}\right\| = 0.$$

Thus, $\bar{y} \in C^n$. Consequently, $\omega_\omega(y_{n_l}) \subset \Gamma$.

Since $S_\lambda = (I - \lambda)I + \lambda S$ and $I - S$ is demiclosed at zero, we see that $I - S_\lambda$ is demiclosed at zero. Now, taking $\{y_{n_l}\} \rightharpoonup \bar{y}$ and (40) into account, we deduce that $\omega_\omega(y_{n_l}) \subset \mathrm{Fix}(S)$. Putting the above results together, we see that $\omega_\omega(y_{n_l}) \subset \Omega = \mathrm{Fix}(S) \cap \Gamma$.

Since the mapping $P_\Omega \nu$ is a strict contraction on H, there exists a unique point $t^* \in H$ such that $t^* = P_\Omega \nu(t^*)$. It then follows from (5) that

$$\langle \nu(t^*) - t^*, z - t^*\rangle \le 0, \tag{47}$$

for all $z \in \Omega$.

Next, we choose a subsequence $\{y_{n_{l_m}}\}$ of $\{y_{n_l}\}$ such that

$$\limsup_{l\to\infty} \langle \nu(t^*) - t^*, y_{n_l} - t^*\rangle = \lim_{m\to\infty} \langle \nu(t^*) - t^*, y_{n_{l_m}} - t^*\rangle. \tag{48}$$

We may assume, without any loss of generality, that $y_{n_{l_m}} \rightharpoonup \bar{y}$ as $m \to \infty$.

Now, using (44), (47), and (48), we get

$$\limsup_{l\to\infty} \langle \nu(t^*) - t^*, u_{n_l} - t^*\rangle$$

$$
\begin{aligned}
&= \limsup_{l\to\infty} \langle \nu(t^*) - t^*, u_{n_l} - y_{n_l} + y_{n_l} - t^* \rangle \\
&= \limsup_{l\to\infty} \langle \nu(t^*) - t^*, u_{n_l} - y_{n_l} \rangle + \limsup_{l\to\infty} \langle \nu(t^*) - t^*, y_{n_l} - t^* \rangle \\
&\le \limsup_{l\to\infty} \|\nu(t^*) - t^*\| \|u_{n_l} - y_{n_l}\| + \limsup_{l\to\infty} \langle \nu(t^*) - t^*, y_{n_l} - t^* \rangle \\
&= \lim_{m\to\infty} \langle \nu(t^*) - t^*, y_{n_{l_m}} - t^* \rangle \\
&= \langle \nu(t^*) - t^*, \bar{y} - t^* \rangle \\
&\le 0, \qquad (49)
\end{aligned}
$$

which shows that (33) holds. □

Theorem 6 *Let $\Omega \neq \emptyset$, $S : C \to C$ be a demicontractive mappings such that $I - S$ is demiclosed at zero, $\nu : C \to C$ be a μ-contraction, and suppose that $\{\rho_n\}, \{\epsilon_n\}, \{\sigma_n\}, \{\alpha_n\}, \{\gamma_n\}$, and $\{\delta_n\}$ are sequences satisfying Assumption $C(3)$. Then, the sequence generated by Algorithm 1 converges to $t^* = P_\Omega \nu(t^*)$.*

Proof Using the definition of y_n, we have

$$
\begin{aligned}
\|y_n - t^*\|^2 &= \|t_n + \theta_n(t_n - t_{n-1}) - t^*\|^2 \\
&= \|(t_n - t^*) + \theta_n(t_n - t_{n-1})\|^2 \\
&\le \|t_n - t^*\|^2 + 2\theta_n \langle y_n - t^*, t_n - t_{n-1} \rangle \\
&\le \|t_n - t^*\|^2 + 2\theta_n \|t_n - t_{n-1}\| \|y_n - t^*\| \\
&\le \|t_n - t^*\|^2 + 2\epsilon_n \|y_n - t^*\|. \qquad (50)
\end{aligned}
$$

Using (29) and (50), we get

$$
\|u_n - t^*\|^2 \le \|t_n - t^*\|^2 + 2\epsilon_n \|y_n - t^*\|. \qquad (51)
$$

Using the definition of t_n, we have

$$
\begin{aligned}
&\|t_{n+1} - t^*\|^2 \\
&= \|\sigma_n(\nu(t_n) - t^*) + (1 - \sigma_n)(u_n - t^*)\|^2 \\
&= \sigma_n^2 \|\nu(t_n) - t^*\|^2 + (1 - \sigma_n)^2 \|u_n - t^*\|^2 + 2\sigma_n(1 - \sigma_n) \langle \nu(t_n) - t^*, u_n - t^* \rangle \\
&= \sigma_n^2 \|\nu(t_n) - t^*\|^2 + (1 - \sigma_n)^2 \|u_n - t^*\|^2 + 2\sigma_n \langle \nu(t_n) - t^*, u_n - t^* \rangle - \\
&\quad 2\sigma_n^2 \langle \nu(t^*) - t^*, u_n - t^* \rangle \\
&\le \sigma_n^2 \|\nu(t_n) - t^*\|^2 + (1 - \sigma_n)^2 \|u_n - t^*\|^2 + 2\sigma_n^2 \|\nu(t_n) - t^*\| \|u_n - t^*\| + \\
&\quad 2\sigma_n \langle \nu(t_n) - t^*, u_n - t^* \rangle \\
&= \sigma_n^2 \|\nu(t_n) - t^*\|^2 + (1 - \sigma_n)^2 \|u_n - t^*\|^2 + 2\sigma_n^2 \|\nu(t_n) - t^*\| \|u_n - t^*\| + \qquad (52) \\
&\quad 2\sigma_n \langle \nu(t_n) - \nu(t^*), u_n - t^* \rangle + 2\sigma_n \langle \nu(t^*) - t^*, u_n - t^* \rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sigma_n^2\|\nu(t_n)-t^*\|^2+(1-\sigma_n)^2\|u_n-t^*\|^2+2\sigma_n^2\|\nu(t_n)-t^*\|\|u_n-t^*\|+\\
&\quad 2\sigma_n\|\nu(t_n)-\nu(t^*)\|\|u_n-t^*\|+2\sigma_n\langle \nu(t^*)-t^*,u_n-t^*\rangle\\
&\leq \sigma_n^2\|\nu(t_n)-t^*\|^2+(1-\sigma_n)^2\|u_n-t^*\|^2+2\sigma_n^2\|\nu(t_n)-t^*\|\|u_n-t^*\|+\\
&\quad 2\mu\sigma_n\|t_n-t^*\|\|u_n-t^*\|+2\sigma_n\langle \nu(t^*)-t^*,u_n-t^*\rangle\\
&\leq \sigma_n^2\|\nu(t_n)-t^*\|^2+(1-\sigma_n)^2\|u_n-t^*\|^2+2\sigma_n^2\|\nu(t_n)-t^*\|\|u_n-t^*\|+\\
&\quad \mu\sigma_n(\|t_n-t^*\|^2+\|u_n-t^*\|^2)+2\sigma_n\langle \nu(t^*)-t^*,u_n-t^*\rangle.
\end{aligned}
$$

Substituting (51) into (52), we get

$$
\begin{aligned}
\|t_{n+1}-t^*\|^2 \leq&\sigma_n^2\|\nu(t_n)-t^*\|^2+[(1-\sigma_n)^2+2\mu\sigma_n]\|t_n-t^*\|^2+\\
&[2\epsilon_n(1-\sigma_n)^2+2\mu\sigma_n\epsilon_n]\|y_n-t^*\|+2\sigma_n^2\|\nu(t_n)-t^*\|\|u_n-t^*\|+\\
&2\sigma_n\langle \nu(t^*)-t^*,u_n-t^*\rangle\\
\leq&[1-2\sigma_n(1-\mu)]\|t_n-t^*\|^2+\sigma_n^2\|\nu(t_n)-t^*\|+\\
&2\mu\sigma_n\epsilon_n\|y_n-t^*\|+2\sigma_n^2\|\nu(t_n)-t^*\|\|u_n-t^*\|+\\
&2\sigma_n\langle \nu(t^*)-t^*,u_n-t^*\rangle+\sigma_n^2\|t_n-t^*\|^2\\
\leq&[1-2\sigma_n(1-\mu)]\|t_n-t^*\|^2+2\sigma_n(1-\mu)\frac{1}{2(1-\mu)}\Big[\sigma_n\|\nu(t_n)-t^*\|+\\
&2[(1-\sigma_n)^2+\mu\sigma_n]\frac{\epsilon_n}{\sigma_n}\|y_n-t^*\|+2\sigma_n\|\nu(t_n)-t^*\|\|u_n-t^*\|+\\
&2\langle \nu(t^*)-t^*,u_n-t^*\rangle+\sigma_n\|t_n-t^*\|^2\Big].
\end{aligned}
\tag{53}
$$

Again, using the definition of t_n and (28), we get

$$
\begin{aligned}
\|t_{n+1}-t^*\|^2 &= \|\sigma_n(\nu(t_n)-t^*)+(1-\sigma_n)(u_n-t^*)\|^2\\
&\leq \|u_n-t^*\|^2+\sigma_n\langle \nu(t_n)-t^*,t_{n+1}-t^*\rangle\\
&\leq \|y_n-t^*\|^2-(1-\delta_n)\rho_n(4-\rho_n)\frac{\gamma_n}{1-\sigma_n}g_{jk}(y_n)-\\
&\quad \delta_n(1-\delta_n)\frac{\gamma_n}{1-\sigma_n}\left\|S_\lambda y_n-y_n+\tau_n\sum_{(j,k)\in\Delta_n}\beta_{jk}T_j^*d_{jk}^n\right\|^2-\\
&\quad \frac{\gamma_n}{1-\sigma_n}\left\|(I-P_{C^n})y_n\right\|^2+2\sigma_n\langle t_{n+1}-t^*,u_n-t^*\rangle.
\end{aligned}
\tag{54}
$$

According to the notations, we introduced in Lemma 5, inequalities (53) and (54) can be briefly expressed as

$$
\begin{aligned}
&\|t_{n+1}-t^*\|^2\leq(1-\Theta_n)\|t_n-t^*\|^2+\Theta_n\Delta_n,\forall n\geq 1,\\
&\|t_{n+1}-t^*\|^2\leq\|t_n-t^*\|^2-\Psi_n+\Phi_n,\forall n\geq 1,
\end{aligned}
$$

respectively.

Applying the conditions listed under $C(3)$, we immediately obtain

$$\sum_{n=1}^{\infty}\Theta_n = \infty \text{ and } \lim_{n\to\infty}\Phi_n = 0.$$

All is now set to give the strong convergence of $\{t_n\}$. From the results we obtained above and in Lemma 5, we can see that all the hypotheses of Lemma 3 are satisfied. Hence

$$\lim_{n\to\infty}\|t_n - t^*\| = 0,$$

which shows that the sequence $\{t_n\}$ converges strongly to $t^* = P_\Omega \nu(t^*)$. □

4 Numerical experiment

In this section, we illustrate the convergence of Algorithm 1 using a numerical example.

Example 1 Let $H = \mathbb{R}^S$, $H_1 = \mathbb{R}^R$, $H_2 = \mathbb{R}^N$, $H_3 = \mathbb{R}^M$, $H_4 = \mathbb{R}^L$.

Let $C = \{x \in \mathbb{R}^S : \|x - \mathbf{o}\|^2 \leq \mathbf{r}^2\}$ where $\mathbf{o} \in \mathbb{R}^S$ and $\mathbf{r} \in \mathbb{R}$. Clearly C is a nonempty closed and convex subsets of H.

Let $Q_{11} = \{T_1x \in \mathbb{R}^R : \|T_1x - \mathbf{a}_1\|^2 \leq \varrho_1^2\}$, $Q_{21} = \{T_2x \in \mathbb{R}^N : \|T_2x - \mathbf{a}_2\|^2 \leq \varrho_2^2\}$, $Q_{31} = \{T_3x \in \mathbb{R}^M : \|T_3x - \mathbf{a}_3\|^2 \leq \varrho_3^2\}$, and $Q_{41} = \{T_4x \in \mathbb{R}^L : \|T_4x - \mathbf{a}_4\|^2 \leq \varrho_4^2\}$ where $\mathbf{a}_1 \in \mathbb{R}^R$, $\mathbf{a}_2 \in \mathbb{R}^N$, $\mathbf{a}_3 \in \mathbb{R}^M$, $\mathbf{a}_4 \in \mathbb{R}^L$ and $\varrho_1, \varrho_2, \varrho_3, \varrho_4 \in \mathbb{R}$.

Let $T_1 : \mathbb{R}^S \to \mathbb{R}^R$, $T_2 : \mathbb{R}^S \to \mathbb{R}^N$, $T_3 : \mathbb{R}^S \to \mathbb{R}^M$, $T_4 : \mathbb{R}^S \to \mathbb{R}^L$ where their entries are randomly generated in the closed interval $[-5, 5]$.

Now, we construct the balls C^n and Q_{j1}^n $(j = 1, 2, 3, 4)$ given in (8) and (9) of the sets C and Q_{j1}, respectively, as follows.

For any $x \in \mathbb{R}^S$, we have $c(x) = \|x - \mathbf{o}\|^2 - \mathbf{r}^2$ and $q_{j1}(T_jx) = \|T_jx - \mathbf{a}_j\|^2 - \varrho_j^2$ for $j = 1, 2, 3, 4$. In what follows, the subgradients ξ_n and $\eta_{j1,n}$ of respectively $c(y_n)$ and $q_{j1}(T_jy_n)$ can be calculated respectively at the points y_n and T_jy_n by $\xi_n(y_n) = 2(y_n - \mathbf{o})$ and $\eta_{j1,n}(T_jy_n) = 2T_j^*(T_jy_n - \mathbf{c}_j)$. The metric projections onto the balls C^n $(i = 1, 2, 3, 4)$ and Q_{j1}^n $(j = 1, 2, 3, 4)$, can be easily calculated.

We randomly generate the coordinates of $\mathbf{o}$ and $\mathbf{a}_j$ in $[-1, 1]$ and, $\mathbf{r}$ and ϱ_j in $[S, 2S]$, $[R, 2R]$, $[N, 2N]$, $[M, 2M]$, and $[L, 2L]$, respectively. We take the initial points as $t_0 = 100(1, 1, \dots, 1)^T \in \mathbb{R}^S$ and $t_1 = -10(1, 1, \dots, 1)^T \in \mathbb{R}^S$. We take a $\frac{1}{6}$-demicontractive mapping $S(x) = -\frac{7}{5}x, x \in \mathbb{R}^S$ and $\nu(x) = 0.95x, x \in \mathbb{R}^S$, respectively. Now, using Lemma 2, taking $\lambda = \frac{1}{3}$, we get $S_\lambda(x) = \frac{1}{5}x, x \in \mathbb{R}^S$ which is a quasi-nonexpansive mapping.

We take $\varpi = 0.5$. For $j = 1, 2, 3, 4$, we take $\beta_j = \frac{j}{10}$ and $\omega_j = 1.5$, $\theta = 0.3$, $\epsilon_n = \frac{1}{20n^{30}+1}$, $\delta_n = 0.4$, $\alpha_n = 0.5$, $\rho_n = \frac{n}{40n+1}$, $\sigma_n = \frac{1}{n+1}$, $\alpha_n = 0.6$, $\tau = 2$, and $\gamma_n = 1 - \alpha_n - \sigma_n$. We use $Error_n = \|t_{n+1} - t_n\|^2 < 10^{-8}$ as a stopping criterion in this example. The algorithms are coded in MATLAB 2023b on a personal computer (13th Gen Intel(R) Core(TM) i7-1355U 1.70 GHz, and a 16.0 GB RAM). All results are reported in Table 1 and Fig. 1.

5 Conclusion

In this paper, we study the split feasibility problem with multiple output sets and fixed point problem in the class of demicontractive mappings. We propose relaxed inertial self-adaptive algorithm and prove strong convergence result for the sequence generated by the proposed algorithm. The proposed method combines the SFPMOS and fixed point

Table 1 Numerical results of Algorithm 1 (when $k = 1$) for different choices of S, R, N, M, L

Dimensions	Iter. (n)	*CPU(s)*	$Error_n$
$S = 5, R = 10, N = 15, M = 20, L = 25$	16	0.000870	9.9308e-09
$S = 15, R = 30, N = 45, M = 60, L = 75$	118	0.003496	9.9658e-09
$S = 30, R = 60, N = 90, M = 120, L = 150$	178	0.009372	9.9704e-09
$S = 100, R = 200, N = 300, M = 400, L = 500$	323	0.202976	9.9841e-09

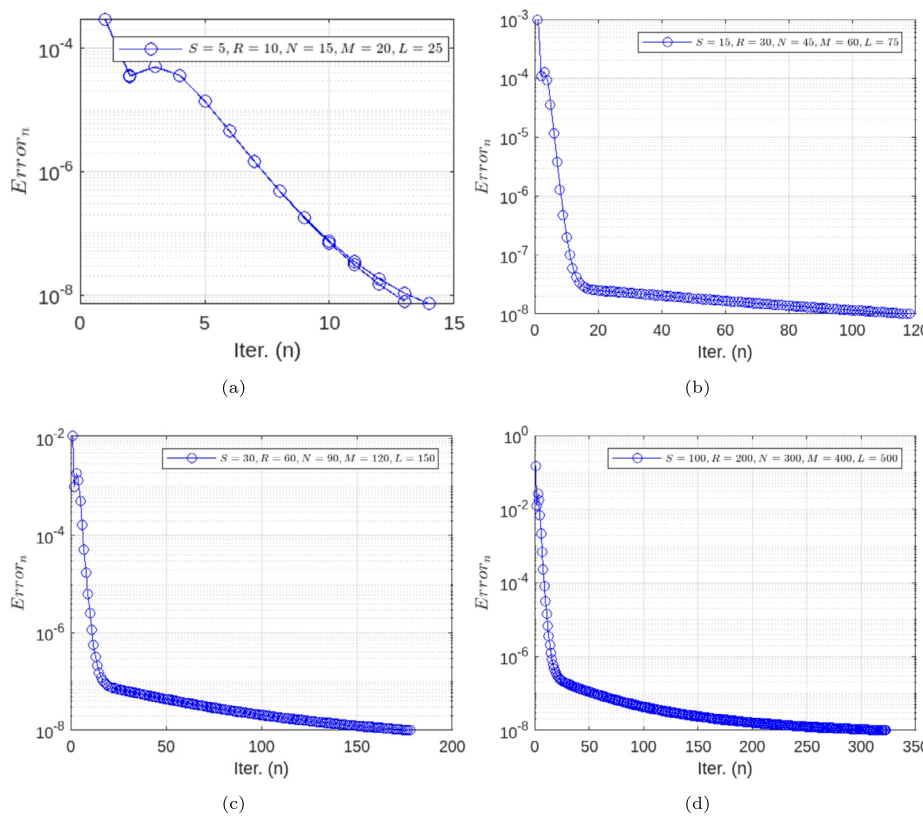

Figure 1 Iter. (n) vs $Error_n$, experimental results of Algorithm 1 (when $k = 1$) for different choices of S, R, N, M, L

problem of demicontractive mappings. So, it generalizes a number of related works as the two problems are larger classes of problems. The proposed algorithm also incorporates the relaxation method in order to speed up its convergence. We adopted a new approach for solving the SFPMOS which does not use the least squares method. Finally, we illustrate the convergence of the proposed algorithm using a numerical example.

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Declarations

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The authors declare no competing interests.

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