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Mathematical programming with multiple sets split monotone variational inclusion constraints

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Abstract

In this paper, we first study a hierarchical problem of Baillon's type, and we study a strong convergence theorem of this problem. For the special case of this convergence theorem, we obtain a strong convergence theorem for the ergodic theorem of Baillon's type. Our result of the ergodic theorem of Baillon's type improves and generalizes many existence theorems of this type of problem. Two numerical examples are given to demonstrate our results.

As applications of our convergence theorem of the hierarchical problem, we study the unique solution for the following problems: mathematical programming with multiply sets split variational inclusion and fixed point set constraints; mathematical programming with multiple sets split variational inequalities and fixed point set constraints; the variational inequality problem with a system of mixed type equilibria and fixed point set constraints; the variational inequality problem with multiple sets split system of mixed type equilibria and fixed point set constraints; mathematical programming with a system of mixed type equilibria and fixed point set constraints. We give iteration processes for these types of problems and establish the strong convergence for the unique solution of these problems. For our special case, our results can be reduced to the following problems: the unique minimal norm solution of the multiply sets split monotonic variational inclusion problems; the minimum norm solutions for the multiple sets split system of mixed type equilibria problem; the minimum norm solution of the system of mixed type equilibria problem. Our results will have many applications in diverse fields of science.

Keywords: hierarchical problems; split variational inclusion problems; fixed point problems; mathematical programming; minimum norm solution

1 Introduction

Let $C_1, C_2, \dots, C_m$ be nonempty closed convex subsets of a Hilbert space H_1. The well-known convex feasiblity problem (CFP) is to find $x^* \in H_1$ such that

$$x^* \in C_1 \cap C_2 \cap \cdots \cap C_m.$$

The split feasibility problem (SFP) is to find a point

$$x^* \in C \text{ such that } Ax^* \in Q,$$

where C is a nonempty closed convex subset of a Hilbert space H_1, Q is a nonempty closed convex subset of a Hilbert space H_2, and $A : H_1 \to H_2$ is an operation. The split feasibility problem (SFP) in the finite dimensional Hilbert spaces was first introduced by Censor *et al.* [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. Since then, the convex feasibility problem and the split feasibility problem (SFP) has received much attention due to its applications in signal processing, image reconstruction, approximation theory, control theory, biomedical engineering, communications, and geophysics. For example, one can refer to [1–5] and related literature.

Let $C_1, C_2, \dots, C_m$ be nonempty closed convex subsets of a Hilbert space H_1, let $Q_1, Q_2, \dots, Q_n$ be nonempty closed convex sets H_2 and let $A_1, A_2, \dots, A_m : H_1 \to H_2$ be linear operator operators. The well-known multiple sets split feasibility problem studied by Censor *et al.* [6]. Xu [7] and Lopez *et al.* [8] (MSSFP) is to find $x^* \in H_1$ such that

$$x^* \in C_i \text{ such that } A_i x^* \in Q_i \quad \text{for all } i = 1, 2, \dots, m.$$

In 2011, Moudafi [9] introduced and studied the following split monotone variational inclusion (SMVI):

$$\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in (B_1 + G_1)^{-1}0, \tag{1}$$

and

$$\bar{y} = A\bar{x} \in H_2 \text{ such that } \bar{y} \in (B + G)^{-1}0, \tag{2}$$

where H_1 and H_2 are real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator, $B_1 : H_1 \to H_1$ and $B : H_2 \to H_2$ are given operators, $G_1 : H_1 \multimap H_1$ and $G : H_2 \multimap H_2$ are given multivalued mappings.

Moudafi [9] proved a weakly convergence theorem for the solution of the split monotone variational inclusion (SMVI) with an iteration process.

In 2011, Maruyama *et al.* [10] proved the following ergodic theorem of Baillon's type [11].

Theorem 1.1 [10] *Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to C$ be a 2-generalized hybrid mapping with $\mathrm{Fix}(T) \neq \emptyset$ and let $P_{\mathrm{Fix}(T)}$ be the metric projection of H_1 onto $\mathrm{Fix}(T)$. Then, for any $x \in C$,*

$$S_n x := \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element p of $\mathrm{Fix}(T)$, where $p = \lim_{n\to\infty} P_{\mathrm{Fix}(T)} T^n x$.

In this paper, we first study a hierarchical problem of Baillon's type, and we study a strong convergence theorem of this problem. For the special case of this convergence theorem, we obtain a strong convergence theorem for the ergodic theorem of Baillon's type. Our result of the ergodic theorem of Baillon's type improves and generalizes many existence theorems of this type of problem. Two numerical examples are given to demonstrate our results.

As applications of our convergence theorem of the hierarchical problem, we study the unique solution for the following problems: mathematical programming with multiply sets split variational inclusion and a fixed point set constraints; mathematical programming with multiple sets split variational inequalities and fixed point set constraints; the variational inequality problem with a system of mixed type equilibria and fixed point set constraints; the variational inequality problem with multiple sets split system of mixed type equilibria and a fixed point set constraints; mathematical programming with system of mixed type equilibrium and a fixed point set constraints. We give iteration processes for these types of problems and establish the strong convergence for the unique solution of these problems. For the special case of our results, our results can be reduced to the following problems: the unique minimal norm solution of the multiply sets split monotonic variational inclusion problems; the minimum norm solutions for the multiple sets split system of mixed type equilibrium problem; the minimum norm solution of the system of the mixed type equilibria problem. Our results will have many applications in diverse fields of science.

2 Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers and let $\mathbb{R}$ be the set of real numbers, H_1 be a (real) Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively, and C be a nonempty closed convex subset of H_1. We denote the strongly convergence and the weak convergence of $\{x_n\}$ to $x \in H_1$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let $T : C \to H_1$ be a mapping, and let $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$ denote the set of fixed points of T. A mapping $T : C \to H_1$ is called

(i) a 2-generalized hybrid mapping [10] if there exist $\delta_1, \delta_2, \epsilon_1, \epsilon_2 \in \mathbb{R}$ such that

$$\begin{aligned}&\delta_1\left\|T^2x - Ty\right\|^2 + \delta_2\|Tx - Ty\|^2 + (1 - \delta_1 - \delta_2)\|x - Ty\|^2\\ &\quad\le \epsilon_1\left\|T^2x - y\right\|^2 + \epsilon_2\|Tx - y\|^2 + (1 - \epsilon_1 - \epsilon_2)\|x - y\|^2\end{aligned}$$

for all $x, y \in C$.

We know that the class of 2-generalized hybrid mapping contains the classes of nonexpansive mappings, nonspreading mappings, and a (α,β)-generalized hybrid [12] in a Hilbert space. We give an example for a 2-generalized hybrid mapping.

Example 2.1 [13] Let $T : [0,2] \to \mathbb{R}$ be defined as

$$Tx = \begin{cases} 0 & \text{if } x \in [0,2),\\ 1 & \text{if } x = 2.\end{cases}$$

Then T is a 2-generalized hybrid mapping and $\mathrm{Fix}(T) = \{0\}$.

Proof Let $\epsilon_1 = \epsilon_2 = \frac{1}{4}$, $\delta_1 = \delta_2 = \frac{1}{2}$.

Case 1: If $x \in [0,2)$, $y = 2$, then $Tx = T^2x = 0$, $Ty = 1$ and $\|x - Ty\| \le 1$. We know that

$$\begin{aligned}&\delta_1\left\|T^2x - Ty\right\|^2 + \delta_2\|Tx - Ty\|^2 + (1 - \delta_1 - \delta_2)\|x - Ty\|^2\\ &\quad= \delta_1 + \delta_2 + (1 - \delta_1 - \delta_2)\|x - Ty\|^2\\ &\quad\le \delta_1 + \delta_2 + (1 - \delta_1 - \delta_2) = 1\end{aligned}$$

and

$$\begin{aligned}
&\epsilon_1\left\|T^2x-y\right\|^2+\epsilon_2\|Tx-y\|^2+(1-\epsilon_1-\epsilon_2)\|x-y\|^2\\
&\quad=4\epsilon_1+4\epsilon_2+(1-\epsilon_1-\epsilon_2)\|x-2\|^2\\
&\quad\geq 4\epsilon_1+4\epsilon_2\geq 1+1=2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\delta_1\left\|T^2x-Ty\right\|^2+\delta_2\|Tx-Ty\|^2+(1-\delta_1-\delta_2)\|x-Ty\|^2\\
&\quad\leq\epsilon_1\left\|T^2x-y\right\|^2+\epsilon_2\|Tx-y\|^2+(1-\epsilon_1-\epsilon_2)\|x-y\|^2.
\end{aligned}$$

Case 2: If $x\in[0,2)$, $y\in[0,2)$, then $Tx=T^2x=0$, $Ty=T^2y=0$. We know that

$$\begin{aligned}
&\delta_1\left\|T^2x-Ty\right\|^2+\delta_2\|Tx-Ty\|^2+(1-\delta_1-\delta_2)\|x-Ty\|^2\\
&\quad=(1-\delta_1-\delta_2)x^2\\
&\quad=0\\
&\quad\leq\epsilon_1\left\|T^2x-y\right\|^2+\epsilon_2\|Tx-y\|^2+(1-\epsilon_1-\epsilon_2)\|x-y\|^2.
\end{aligned}$$

Case 3: If $x=y=2$, then $Tx=1$, $T^2x=0$, $Ty=1$, $T^2y=0$. We know that

$$\begin{aligned}
&\delta_1\left\|T^2x-Ty\right\|^2+\delta_2\|Tx-Ty\|^2+(1-\delta_1-\delta_2)\|x-Ty\|^2\\
&\quad=\delta_1+(1-\delta_1-\delta_2)\\
&\quad=(1-\delta_2)=\frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
&\epsilon_1\left\|T^2x-y\right\|^2+\epsilon_2\|Tx-y\|^2+(1-\epsilon_1-\epsilon_2)\|x-y\|^2\\
&\quad=4\epsilon_1+\epsilon_2\geq\frac{5}{4}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\delta_1\left\|T^2x-Ty\right\|^2+\delta_2\|Tx-Ty\|^2+(1-\delta_1-\delta_2)\|x-Ty\|^2\\
&\quad\leq\epsilon_1\left\|T^2x-y\right\|^2+\epsilon_2\|Tx-y\|^2+(1-\epsilon_1-\epsilon_2)\|x-y\|^2.
\end{aligned}$$

By the above case, we know that T is a 2-generalized hybrid. □

A mapping $V:H_1\to H_1$ is called

(i) strongly monotone if there exists $\bar{\gamma}>0$ such that $\langle x-y, Vx-Vy\rangle\geq\bar{\gamma}\|x-y\|^2$ for all $x,y\in H_1$;

(ii) α-inverse-strongly monotone if $\langle x-y, Vx-Vy\rangle\geq\alpha\|Vx-Vy\|^2$ for all $x,y\in H_1$ and $\alpha>0$.

We also know that if V is a α-inverse-strongly monotone mapping and $0 < \lambda \leq 2\alpha$, then $I - \lambda V : C \to H_1$ is nonexpansive.

Let $G : H_1 \multimap H_1$ be a multivalued mapping. The effective domain of G is denoted by $D(G)$, that is, $D(G) = \{x \in H_1 : Gx \neq \emptyset\}$.

Then $G : H_1 \multimap H_1$ is called

(i) a monotone operator on H_1 if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in D(G)$, $u \in Gx$, and $v \in Gy$;

(ii) a maximal monotone operator on H_1 if G is a monotone operator on H_1 and its graph is not properly contained in the graph of any other monotone operator on H_1.

Lemma 2.1 [14] *Let C be a nonempty closed convex subset of a real Hilbert space H_1. Let T be a nonexpansive mapping of C into itself, and let $\{x_n\}$ be a sequence in C. If $x_n \rightharpoonup w$ and* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, *then $Tw = w$.*

In 2012, Hojo *et al.* [15] also gave an example for a 2-generalized hybrid mapping which is not a generalized hybrid mapping with $\mathrm{Fix}(T) = \{(0,0)\}$. We shall prove that this example for a 2-generalized hybrid mapping does not satisfy the demiclosed property as in Lemma 2.1.

Example 2.2 Let $A = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ and $TA \to \mathbb{R}^2$ be defined as

$$Tx = \begin{cases} (0,0) & \text{if } x \in A, \\ \frac{x}{\|x\|} & \text{if } x \in \mathbb{R}^2/A. \end{cases}$$

Hojo *et al.* [15] showed that T is a 2-generalized hybrid mapping, but T is not a generalized hybrid mapping. Note that T does not have the demiclosed property. Indeed, there exists a sequence $\{x_n\} \in A$ such that $x_n \rightharpoonup w$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, but w in $\mathbb{R}^2/\mathrm{Fix}(T) = \mathbb{R}^2/\{(0,0)\}$.

Proof Let $r_n = 1 + \frac{1}{n}$, $x_n = (r_n \cos\theta, r_n \sin\theta)$ for all $n \in \mathbb{N}$, then $x_n \to (\cos\theta, \sin\theta)$ and $Tx_n = (\cos\theta, \sin\theta)$. We also have $\|Tx_n - x_n\| = \|((r_n - 1)\cos\theta, (r_n - 1)\sin\theta)\| = r_n - 1 \to 0$, but $(\cos\theta, \sin\theta) \neq (0,0)$. □

Lemma 2.2 [16] *Let $V : H_1 \to H_1$ be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Let $\theta \in H_1$, and $V_1 : H_1 \to H_1$ such that $V_1 x = Vx - \theta$. Then V_1 is a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous mapping. Furthermore, there is a unique fixed point z_0 in C satisfying $z_0 = P_C(z_0 - Vz_0 + \theta)$. This point $z_0 \in C$ is also a unique solution of the hierarchical variational inequality $\langle Vz_0 - \theta, q - z_0\rangle \geq 0$, for all $q \in C$.*

Let C be a nonempty subset of a real Hilbert space H_1. Then $T : C \to H_1$ is a firmly nonexpansive mapping if $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$ for every $x, y \in C$, that is, $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle$ for every $x, y \in C$.

Lemma 2.3 [17] *Let G_1 be a maximal monotone mapping on H_1. Let $J_r^{G_1}$ be the resolvent of G_1 defined by $J_r^{G_1} = (I + rG_1)^{-1}$ for each $r > 0$. Then the following holds*:

$$\frac{|s - t|}{s}\langle J_s^{G_1}x - J_t^{G_1}x, J_s^{G_1}x - x\rangle \geq \|J_s^{G_1}x - J_t^{G_1}x\|^2$$

for all $s,t>0$ and $x\in H_1$. In particular,

$$\|J_s^{G_1}x-J_t^{G_1}x\|\leq\frac{|s-t|}{s}\|J_s^{G_1}x-x\|$$

for all $s,t>0$ and $x\in H_1$.

A mapping $T:H_1\to H_1$ is said to be averaged if $T=(1-\alpha)I+\alpha S$, where $\alpha\in(0,1)$ and $S:H_1\to H_1$ is nonexpansive. In this case, we also say that T is α-averaged. A firmly nonexpansive mapping is $\frac{1}{2}$-averaged.

Lemma 2.4 ([7, 18]) *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T:C\to C$ be a mapping. Then the following are satisfied*:

(i) *T is nonexpansive if and only if the complement $(I-T)$ is $1/2$-ism.*

(ii) *If S is υ-ism, then for $\gamma>0$, γS is υ/γ-ism.*

(iii) *S is averaged if and only if the complement $I-S$ is υ-ism for some $\upsilon>1/2$.*

(iv) *If S and T are both averaged, then the product (composite) ST is averaged.*

(v) *If the mappings $\{T_i\}_{i=1}^n$ are averaged and have a common fixed point, then $\bigcap_{i=1}^n \mathrm{Fix}(T_i)=\mathrm{Fix}(T_1\cdots T_n)$.*

Lemma 2.5 [19] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i}<a_{n_i+1}$ for all $i\in\mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}\subseteq\mathbb{N}$ such that $m_k\to\infty$ and the following properties are satisfied by all (sufficiently large) numbers $k\in\mathbb{N}$:*

$$a_{m_k}\leq a_{m_k+1}\quad\textit{and}\quad a_k\leq a_{m_k+1}.$$

In fact, $m_k=\max\{j\leq k:a_j<a_{j+1}\}$.

Lemma 2.6 [20] *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty}u_n<\infty$, $\{t_n\}$ a sequence of real numbers with $\limsup t_n\leq 0$. Suppose that $a_{n+1}\leq(1-\alpha_n)a_n+\alpha_n t_n+u_n$ for each $n\in\mathbb{N}$. Then $\lim_{n\to\infty}a_n=0$.*

3 Convergence theorems of hierarchical problems

Let H_1 be a real Hilbert space and let C be a nonempty closed convex subset of H_1. For each $i=1,2$, and $\kappa_i>0$, let F_i be a κ_i-inverse-strongly monotone mapping of C into H_1. For each $i=1,2$, let G_i be a maximal monotone mapping on H_1 such that the domain of G_i is included in C and define the set $G_i^{-1}0$ as $G_i^{-1}0=\{x\in H_1:0\in G_ix\}$. Let $J_{\lambda_n}^{G_1}=(I+\lambda_nG_1)^{-1}$ and $J_{r_n}^{G_2}=(I+r_nG_2)^{-1}$ for each $n\in\mathbb{N}$, $\lambda_n>0$ and $r_n>0$. Let $\{\theta_n\}\subset H_1$ be a sequence. Let V be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma}>0$ and $L>0$. Throughout this paper, we use these notations and assumptions unless specified otherwise.

The following strong convergence theorem for hierarchical problems is one of our main results in this paper.

Theorem 3.1 *Let $T : C \to H_1$ be a 2-generalized hybrid mapping with* $\mathrm{Fix}(T) \cap (F_1 + G_1)^{-1}0 \cap (F_2 + G_2)^{-1}0 \neq \emptyset$. *Take $\mu \in \mathbb{R}$ as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H_1$ be defined by

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = J_{\lambda_n}^{G_1}(I - \lambda_n F_1) J_{r_n}^{G_2}(I - r_n F_2) x_n, \\ s_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k y_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n \theta_n + (I - \beta_n V) s_n) \end{cases} \tag{3.1}$$

for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, and $\{r_n\} \subset (0,\infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $0 < a_1 \le \lambda_n \le b_1 < 2\kappa_1$, *and* $0 < a_2 \le r_n \le b_2 < 2\kappa_2$;
(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H_1$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\mathrm{Fix}(T)\cap(F_1+G_1)^{-1}0\cap(F_2+G_2)^{-1}0}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \ge 0, \quad \forall q \in \mathrm{Fix}(T) \cap (F_1 + G_1)^{-1}0 \cap (F_2 + G_2)^{-1}0.$$

Proof Take any $\bar{x} \in \mathrm{Fix}(T) \cap (F_1 + G_1)^{-1}0 \cap (F_2 + G_2)^{-1}0$ and let $\bar{x}$ be fixed. Then we have $\bar{x} = J_{\lambda_n}^{G_1}(I - \lambda_n F_1)\bar{x}$ and $\bar{x} = J_{r_n}^{G_2}(I - r_n F_2)\bar{x}$. Let $u_n = J_{r_n}^{G_2}(I - r_n F_2)x_n$. For each $n \in \mathbb{N}$, by the same argument as the proof of Theorem 3.1 [16], we have

$$\|u_n - \bar{x}\|^2 \le \|x_n - \bar{x}\|^2 - r_n(2\kappa_2 - r_n)\|F_2 x_n - F_2\bar{x}\|^2 \le \|x_n - \bar{x}\|^2 \tag{3}$$

and

$$\|y_n - \bar{x}\|^2 \le \|u_n - \bar{x}\|^2 - \lambda_n(2\kappa_1 - \lambda_n)\|F_1 u_n - F_1\bar{x}\|^2 \le \|x_n - \bar{x}\|^2. \tag{4}$$

By equations (3) and (4), we have

$$\|y_n - \bar{x}\| \le \|u_n - \bar{x}\| \le \|x_n - \bar{x}\|.$$

Since T is a 2-generalized hybrid mapping with $\mathrm{Fix}(T) \neq \emptyset$, we know that T is a quasi-nonexpansive, and

$$\begin{aligned} \|s_n - \bar{x}\| &= \left\| \frac{1}{n}\sum_{k=0}^{n-1} T^k y_n - \bar{x} \right\| \le \frac{1}{n}\sum_{k=0}^{n-1} \|T^k y_n - \bar{x}\| \\ &\le \|y_n - \bar{x}\| \le \|u_n - \bar{x}\| \le \|x_n - \bar{x}\|. \end{aligned} \tag{5}$$

By the same argument as in the proof of Theorem 3.1 [16], we find that the sequence $\{x_n\}$ is bounded. Furthermore, $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, and $\{s_n\}$ are bounded. We also have

$$\|x_{n+1} - x_n\|^2 \le (1 - \alpha_n)^2\big[\beta_n^2\|\theta_n - Vs_n\|^2 + \|s_n - x_n\|^2 + 2\beta_n\|\theta_n - Vs_n\|\|s_n - x_n\|\big] \tag{6}$$

and

$$\begin{aligned}
&\|x_{n+1}-\bar{x}\|^2-\|x_n-\bar{x}\|^2+(1-\alpha_n)\alpha_n\|s_n-x_n\|^2\\
&\quad\le 2(1-\alpha_n)\beta_n\langle\theta_n,x_n-\bar{x}\rangle-2(1-\alpha_n)\beta_n\langle Vs_n,x_n-\bar{x}\rangle\\
&\qquad+(1-\alpha_n)^2\bigl[\beta_n^2\|\theta_n-Vs_n\|^2+2\beta_n\|\theta_n-Vs_n\|\|s_n-x_n\|\bigr].
\end{aligned}\tag{7}$$

We will divide the proof into two cases.

Case 1: there exists a natural number N such that $\|x_{n+1}-\bar{x}\|\le\|x_n-\bar{x}\|$ for each $n\ge N$. Therefore, $\lim_{n\to\infty}\|x_n-\bar{x}\|$ exists. Hence, it follows from equation (7), (i), and (ii) that

$$\lim_{n\to\infty}\|s_n-x_n\|=0.\tag{8}$$

By equations (6), (8), (i), and (ii), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{9}$$

We also have

$$\|z_n-s_n\|\le\bigl\|\beta_n\theta_n+(1-\beta_n V)s_n-s_n\bigr\|\le\beta_n\|\theta_n-Vs_n\|.\tag{10}$$

By equation (10), (iv), and (ii) we have

$$\lim_{n\to\infty}\|z_n-s_n\|=0.\tag{11}$$

By equations (8) and (11),

$$\lim_{n\to\infty}\|z_n-x_n\|=0.\tag{12}$$

By the same argument as in the proof of Theorem 3.1 [16], we have

$$\lim_{n\to\infty}\|u_n-x_n\|=0\tag{13}$$

and

$$\lim_{n\to\infty}\|y_n-u_n\|=0.\tag{14}$$

Since $\mathrm{Fix}(T)\cap(F_1+G_1)^{-1}0\cap(F_2+G_2)^{-1}0$ is a nonempty closed convex subset of H_1, by Lemma 2.2, we can take $\bar{x}_0\in\mathrm{Fix}(T)\cap(F_1+G_1)^{-1}0\cap(F_2+G_2)^{-1}0$ such that

$$\bar{x}_0=P_{\mathrm{Fix}(T)\cap(F_1+G_1)^{-1}0\cap(F_2+G_2)^{-1}0}(\bar{x}_0-V\bar{x}_0+\theta).$$

This point $\bar{x}_0$ is also a unique solution of the hierarchical variational inequality:

$$\langle V\bar{x}_0-\theta,q-\bar{x}_0\rangle\ge 0,\quad\forall q\in\mathrm{Fix}(T)\cap(F_1+G_1)^{-1}0\cap(F_2+G_2)^{-1}0.\tag{15}$$

We shall show that

$$\limsup_{n\to\infty}\langle V\bar{x}_0-\theta, z_n-\bar{x}_0\rangle \geq 0.$$

Without loss of generality, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$z_{n_k} \rightharpoonup w \tag{16}$$

for some $w \in H_1$ and

$$\limsup_{n\to\infty}\langle V\bar{x}_0-\theta, z_n-\bar{x}_0\rangle = \lim_{k\to\infty}\langle V\bar{x}_0-\theta, z_{n_k}-\bar{x}_0\rangle. \tag{17}$$

By equations (12) and (13), we have

$$\lim_{n\to\infty}\|u_n - z_n\| = 0$$

and $u_{n_k} \rightharpoonup w$. On the other hand, since $0 < a_1 \leq \lambda_n \leq b_1 < 2\kappa_1$, there exists a subsequence $\{\lambda_{n_{k_j}}\}$ of $\{\lambda_{n_k}\}$ such that $\{\lambda_{n_{k_j}}\}$ converges to a number $\bar{\lambda} \in [a_1, b_1]$. By equation (14) and Lemma 2.3, we have

$$\begin{aligned}
&\bigl\|u_{n_{k_j}} - J_{\bar{\lambda}}^{G_1}(I-\bar{\lambda}F_1)u_{n_{k_j}}\bigr\| \\
&\quad\leq \bigl\|u_{n_{k_j}} - J_{\lambda_{n_{k_j}}}^{G_1}(I-\lambda_{n_{k_j}}F_1)u_{n_{k_j}}\bigr\| + \bigl\|J_{\lambda_{n_{k_j}}}^{G_1}(I-\bar{\lambda}F_1)u_{n_{k_j}} - J_{\bar{\lambda}}^{G_1}(I-\bar{\lambda}F_1)u_{n_{k_j}}\bigr\| \\
&\qquad + \bigl\|J_{\lambda_{n_{k_j}}}^{G_1}(I-\lambda_{n_{k_j}}F_1)u_{n_{k_j}} - J_{\lambda_{n_{k_j}}}^{G_1}(I-\bar{\lambda}F_1)u_{n_{k_j}}\bigr\| \\
&\quad\leq \|u_{n_{k_j}} - y_{n_{k_j}}\| + |\lambda_{n_{k_j}} - \bar{\lambda}|\|F_1 u_{n_{k_j}}\| \\
&\qquad + \frac{|\lambda_{n_{k_j}} - \bar{\lambda}|}{\bar{\lambda}}\bigl\|J_{\bar{\lambda}}^{G_1}(I-\bar{\lambda}F_1)u_{n_{k_j}} - (I-\bar{\lambda}F_1)u_{n_{k_j}}\bigr\| \to 0.
\end{aligned} \tag{18}$$

By equation (18), $u_{n_{k_j}} \rightharpoonup w$, and Lemma 2.1, $w \in \operatorname{Fix}(J_{\bar{\lambda}}^{G_1}(I-\bar{\lambda}F)) = (F+B)^{-1}0$.

Since $0 < a_2 \leq r_n \leq b_2 < 2\kappa_2$, there exists a subsequence $\{r_{n_{k_j}}\}$ of $\{r_{n_k}\}$ such that $\{r_{n_{k_j}}\}$ converges to a number $\bar{r} \in [a_2, b_2]$. By the same argument as for equation (18), we have

$$\bigl\|x_{n_{k_j}} - J_{\bar{r}}^{G_2}(I-\bar{r}F_2)x_{n_{k_j}}\bigr\| \to 0. \tag{19}$$

By equation (13) and $u_{n_k} \rightharpoonup w$, we have $x_{n_k} \rightharpoonup w$.

By equation (19), $x_{n_k} \rightharpoonup w$ and Lemma 2.1, we have $w \in \operatorname{Fix}(J_{\bar{r}}^{G_2}(I-\bar{r}F_2)) = (F_2+G_2)^{-1}0$.

Since T is a 2-generalized hybrid mapping, there exist $\delta_1, \delta_2, \epsilon_1, \epsilon_2 \in \mathbb{R}$ such that

$$\begin{aligned}
&\delta_1\bigl\|T^2x - Ty\bigr\|^2 + \delta_2\|Tx-Ty\|^2 + (1-\delta_1-\delta_2)\|x-Ty\|^2 \\
&\quad\leq \epsilon_1\bigl\|T^2x - y\bigr\|^2 + \epsilon_2\|Tx-y\|^2 + (1-\epsilon_1-\epsilon_2)\|x-y\|^2
\end{aligned} \tag{20}$$

for all $x, y \in C$. Replacing x by $T^k y_n$ in equation (20), we have, for all $y \in C$ and $k = 1, 2, \dots, n$,

$$\begin{aligned}
&\delta_1 \|T^{k+2}y_n - Ty\|^2 + \delta_2 \|T^{k+1}y_n - Ty\|^2 + (1 - \delta_1 - \delta_2)\|T^k y_n - Ty\|^2 \\
&\quad \le \epsilon_1 \|T^{k+2}y_n - y\|^2 + \epsilon_2 \|T^{k+1}y_n - y\|^2 + (1 - \epsilon_1 - \epsilon_2)\|T^k y_n - y\|^2 \\
&\quad \le \epsilon_1 \big[\|T^{k+2}y_n - Ty\|^2 + \|Ty - y\|^2 + 2\langle T^{k+2}y_n - Ty, Ty - y\rangle\big] \\
&\qquad + \epsilon_2 \big[\|T^{k+1}y_n - Ty\|^2 + \|Ty - y\|^2 + 2\langle T^{k+1}y_n - Ty, Ty - y\rangle\big] \\
&\qquad + (1 - \epsilon_1 - \epsilon_2)\big[\|T^k y_n - Ty\|^2 + \|Ty - y\|^2 + 2\langle T^k y_n - Ty, Ty - y\rangle\big].
\end{aligned}$$

This implies that

$$\begin{aligned}
0 &\le (\epsilon_1 - \delta_1)\|T^{k+2}y_n - Ty\|^2 + \|Ty - y\|^2 + 2\epsilon_1\langle T^{k+2}y_n - Ty, Ty - y\rangle \\
&\quad + (\epsilon_2 - \delta_2)\|T^{k+1}y_n - Ty\|^2 + 2\epsilon_2\langle T^{k+1}y_n - Ty, Ty - y\rangle \\
&\quad + (\delta_1 - \epsilon_1 + \delta_2 - \epsilon_2)\|T^k y_n - Ty\|^2 \\
&\quad + 2(1 - \epsilon_1 - \epsilon_2)\langle T^k y_n - Ty, Ty - y\rangle \\
&\le (\epsilon_1 - \delta_1)\big[\|T^{k+2}y_n - Ty\|^2 - \|T^k y_n - Ty\|^2\big] \\
&\quad + (\epsilon_2 - \delta_2)\big[\|T^{k+1}y_n - Ty\|^2 - \|T^k y_n - Ty\|^2\big] + \|Ty - y\|^2 \\
&\quad + 2\langle T^k y_n - Ty + \epsilon_1(T^{k+2}y_n - T^k y_n) + \epsilon_2(T^{k+1}y_n - T^k y_n), Ty - y\rangle.
\end{aligned} \tag{21}$$

Summing up these inequalities (21) with respect to $k = 0$ to $k = n - 1$ and dividing by n, we have

$$\begin{aligned}
0 &\le \frac{(\epsilon_1 - \delta_1)}{n}\big[\|T^{n+1}y_n - Ty\|^2 + \|T^n y_n - Ty\|^2 - \|Ty_n - Ty\|^2 - \|y_n - Ty\|^2\big] \\
&\quad + \frac{(\epsilon_2 - \delta_2)}{n}\big[\|T^n y_n - Ty\|^2 - \|y_n - Ty\|^2\big] \\
&\quad + \|Ty - y\|^2 + 2\langle s_n - Ty, Ty - y\rangle \\
&\quad + \frac{2}{n}\langle \epsilon_1(T^{n+1}y_n + T^n y_n - Ty_n - y_n) + \epsilon_2(T^n y_n - y_n), Ty - y\rangle.
\end{aligned} \tag{22}$$

Replacing n by n_{k_j} and let $n_{k_j} \to \infty$. Then from equation (11), (16), and (22), we have $s_{n_{k_j}} \rightharpoonup w$, and

$$0 \le \|Ty - y\|^2 + 2\langle w - Ty, Ty - y\rangle.$$

Taking $y = w$ in the above inequality, we have

$$0 \le \|Tw - w\|^2 + 2\langle w - Tw, Tw - w\rangle = \|Tw - w\|^2 - 2\|Tw - w\|^2 = -\|Tw - w\|^2.$$

This implies that $w \in \mathrm{Fix}(T)$. Hence, $w \in \mathrm{Fix}(T) \cap (F_1 + G_1)^{-1}0 \cap (F_2 + G_2)^{-1}0$. Therefore, we have from equations (15) and (17)

$$\limsup_{n\to\infty}\langle V\bar{x}_0 - \theta, z_n - \bar{x}_0\rangle = \lim_{k\to\infty}\langle V\bar{x}_0 - \theta, z_{n_k} - \bar{x}_0\rangle = \langle V\bar{x}_0 - \theta, w - \bar{x}_0\rangle \ge 0. \tag{23}$$

By the same argument as the proof of Theorem 3.1 [16], we have

$$\begin{aligned}
&\|x_{n+1}-\bar{x}_0\|^2 \\
&\quad\leq \bigl[1-2(1-\alpha_n)\beta_n\tau\bigr]\|x_n-\bar{x}_0\|^2+2(1-\alpha_n)\beta_n\tau\biggl(\frac{\beta_n\tau\|x_n-\bar{x}_0\|^2}{2} \\
&\qquad+\frac{\langle\theta_n-\theta,z_n-\bar{x}_0\rangle}{\tau}+\frac{\langle\theta-V\bar{x}_0,z_n-\bar{x}_0\rangle}{\tau}\biggr).
\end{aligned}\tag{24}$$

By equations (23), (24), assumptions, and Lemma 2.6, we know that $\lim_{n\to\infty}x_n=\bar{x}_0$, where

$$\bar{x}_0=P_{\mathrm{Fix}(T)\cap(F_1+G_1)^{-1}0\cap(F_2+G_2)^{-1}0}(\bar{x}_0-V\bar{x}_0+\theta).$$

Case 2: Suppose that there exists $\{n_i\}$ of $\{n\}$ such that $\|x_{n_i}-\bar{x}\|\leq\|x_{n_i+1}-\bar{x}\|$ for all $i\in\mathbb{N}$. By Lemma 2.5, there exists a nondecreasing sequence $\{m_j\}$ in $\mathbb{N}$ such that $m_j\to\infty$ and

$$\|x_{m_j}-\bar{x}\|\leq\|x_{m_j+1}-\bar{x}\|\quad\text{and}\quad\|x_j-\bar{x}\|\leq\|x_{m_j+1}-\bar{x}\|.\tag{25}$$

Hence, it follows from equations (7) and (25) that

$$\begin{aligned}
&(1-\alpha_{m_j})\alpha_{m_j}\|s_{m_j}-x_{m_j}\|^2 \\
&\quad\leq 2(1-\alpha_{m_j})\beta_{m_j}\langle\theta_{m_j},x_{m_j}-\bar{x}\rangle-2(1-\alpha_{m_j})\beta_{m_j}\langle Vs_{m_j},x_{m_j}-\bar{x}\rangle \\
&\qquad+(1-\alpha_{m_j})^2\bigl[\beta_{m_j}^2\|\theta_{m_j}-Vs_{m_j}\|^2+2\beta_{m_j}\|\theta_{m_j}-Vs_{m_j}\|\|s_{m_j}-x_{m_j}\|\bigr]
\end{aligned}\tag{26}$$

for each $j\in\mathbb{N}$. Hence, it follows from equation (26), (i), and (ii) that

$$\lim_{j\to\infty}\|s_{m_j}-x_{m_j}\|=0.\tag{27}$$

We show that

$$\limsup_{j\to\infty}\langle V\bar{x}_0-\theta,z_{m_j}-\bar{x}_0\rangle\geq 0.$$

Without loss of generality, there exists a subsequence $\{z_{m_{j_k}}\}$ of $\{z_{m_j}\}$ such that $z_{m_{j_k}}\rightharpoonup w$ for some $w\in H$ and

$$\limsup_{j\to\infty}\langle V\bar{x}_0-\theta,z_{m_j}-\bar{x}_0\rangle=\lim_{k\to\infty}\langle V\bar{x}_0-\theta,z_{m_{j_k}}-\bar{x}_0\rangle.\tag{28}$$

By a similar argument as in the proof of Case 1, we have $w\in\mathrm{Fix}(T)\cap(F_1+G_1)^{-1}0\cap(F_2+G_2)^{-1}0$. Therefore, we have from equations (28) and (15)

$$\limsup_{j\to\infty}\langle V\bar{x}_0-\theta,z_{m_j}-\bar{x}_0\rangle=\lim_{k\to\infty}\langle V\bar{x}_0-\theta,z_{m_{j_k}}-\bar{x}_0\rangle=\langle V\bar{x}_0-\theta,w-\bar{x}_0\rangle\geq 0.\tag{29}$$

Following a similar argument as in the proof of Case 1, we have

$$\begin{aligned}
&\|x_{m_j+1} - \bar{x}_0\|^2 \\
&\quad \le \bigl[1 - 2(1-\alpha_{m_j})\beta_{m_j}\tau\bigr]\|x_{m_j} - \bar{x}_0\|^2 + (1-\alpha_{m_j})(\beta_{m_j}\tau)^2\|x_{m_j} - \bar{x}_0\|^2 \\
&\qquad + 2\beta_{m_j}(1-\alpha_{m_j})\langle \theta_n - \theta, z_{m_j} - \bar{x}_0\rangle + 2\beta_{m_j}(1-\alpha_{m_j})\langle \theta - V\bar{x}_0, z_{m_j} - \bar{x}_0\rangle. \qquad (30)
\end{aligned}$$

From $\|x_{m_j} - \bar{x}\| \le \|x_{m_j+1} - \bar{x}\|$, we have

$$\begin{aligned}
&2(1-\alpha_{m_j})\beta_{m_j}\tau\|x_{m_j} - \bar{x}_0\|^2 \\
&\quad \le (1-\alpha_{m_j})(\beta_{m_j}\tau)^2\|x_{m_j} - \bar{x}_0\|^2 + 2\beta_{m_j}(1-\alpha_{m_j})\langle \theta_n - \theta, z_{m_j} - \bar{x}_0\rangle \\
&\qquad + 2\beta_{m_j}(1-\alpha_{m_j})\langle \theta - V\bar{x}_0, z_{m_j} - \bar{x}_0\rangle. \qquad (31)
\end{aligned}$$

Since $(1-\alpha_{m_j})\beta_{m_j} > 0$, we have

$$2\tau\|x_{m_j} - \bar{x}_0\|^2 \le \beta_{m_j}\tau\|x_{m_j} - \bar{x}_0\|^2 + 2\langle \theta_n - \theta, z_{m_j} - \bar{x}_0\rangle + 2\langle \theta - V\bar{x}_0, z_{m_j} - \bar{x}_0\rangle. \qquad (32)$$

By equations (29), (32), and the assumptions, we know that

$$\lim_{j\to\infty}\|x_{m_j} - \bar{x}_0\| = 0.$$

By (6), (27), and the assumptions, we know that

$$\lim_{j\to\infty}\|x_{m_{j+1}} - x_{m_j}\| = 0.$$

Thus, we have

$$\lim_{j\to\infty}\|x_{m_{j+1}} - \bar{x}_0\| = 0. \qquad (33)$$

By equations (25) and (33),

$$\lim_{j\to\infty}\|x_j - \bar{x}_0\| \le \lim_{j\to\infty}\|x_{m_{j+1}} - \bar{x}_0\| = 0.$$

Thus, the proof is completed. □

Remark 3.1

(i) The assumptions, method, conclusion, and applications of Theorem 3.1 are different from Theorem 3.1 in [21] and [22]. In Theorem 3.1, Lemma 2.5 is used to prove the result, but in [21] and [22] we did not use this lemma.

(ii) The assumptions, method, and conclusion of Theorem 3.1 are different from Theorem 3.1 [16]. In Theorem 3.1 [16], T is a quasi-nonexpansive with the demiclosed property, but in Theorem 3.1, T is a 2-generalized hybrid mapping, and by Example 2.2, we know that T does not satisfy the demiclosed property. Therefore Theorem 3.1 [16] cannot apply for a 2-generalized hybrid mapping.

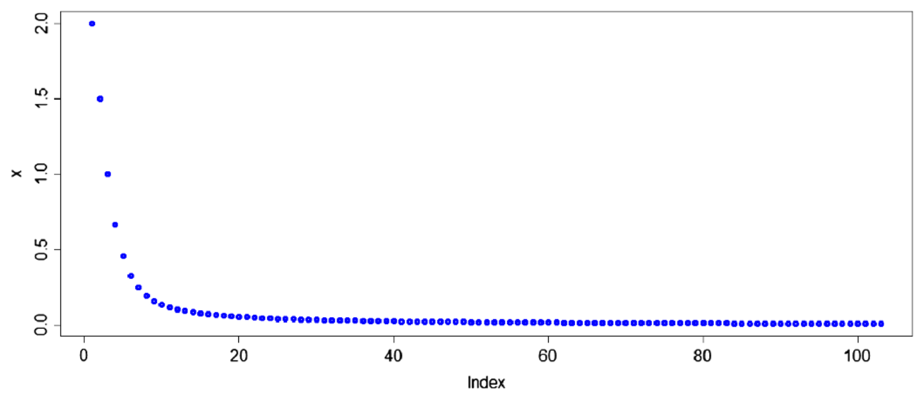

Figure 1 *x*-axis: index; *y*-axis: x_n.

Example 3.1 Let T be the same as Example 2.1. Let $\alpha_n = 1/2$, $\beta_n = 1/n$, $\theta_n = 1$, $Vx = x$, $G_1x = F_1x = 2x$, $G_2x = F_2x = 3x$, $r_n = 1/2$, $\lambda_n = 1/2$. Then V, G_1, G_2, F_1, F_2 satisfy all conditions of Theorem 3.1 and $\mathrm{Fix}(T) \cap (F_1 + G_1)^{-1}0 \cap (F_2 + G_2)^{-1}0 = \{0\}$, and if we let $x_1 = 2.000000000$, we see the following numerical results and graph (see Figure 1) demonstrating Theorem 3.1:

$n = 1-5$	2.000000000	1.500000000	1.000000000	0.666666667	0.458333333
$n = 6-10$	0.329166667	0.247916667	0.195386905	0.160193452	0.135652282
$n = 11-15$	0.117826141	0.104367616	0.093850475	0.085386776	0.078407674

...

Besides, we know the following.

If $|x_n - x_{n-1}| < 10^{-3}$, then $n = 35$; if $|x_n - x_{n-1}| < 10^{-4}$, then $n = 103$; if $|x_n - x_{n-1}| < 10^{-5}$, then $n = 319$; if $|x_n - x_{n-1}| < 10^{-6}$, then $n = 1{,}003$.

For $i = 1, 2$, let $F_i = 0$, $G_i = \partial i_C$, and $\lambda_n = r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.1. Furthermore, put $\theta_n = \theta$, and $V(x) = x$ for all $x \in H_1$; we obtain the following theorem which generalizes Theorem 4.1 in [10].

Theorem 3.2 *Let $T : C \to H_1$ be a 2-generalized hybrid mapping such that $F(T) \neq \emptyset$. Let $\theta \in C$, and $\{x_n\} \subset C$ be defined by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ s_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k x_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n \theta + (1-\beta_n)s_n) \end{cases} \tag{3.2}$$

for each $n \in \mathbb{N}$, $\{\alpha_n\} \subset (0,1)$, and $\{\beta_n\} \subset (0,1)$. Assume that $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ and $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\mathrm{Fix}(T)}\theta$.

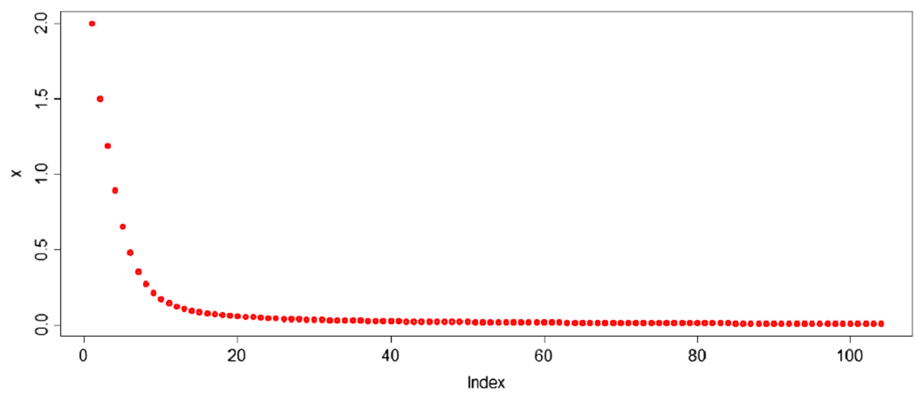

Figure 2 *x*-axis: index; *y*-axis: x_n.

Example 3.2 Let T be the same as Example 2.1. Let $\alpha_n = 1/2$, $\beta_n = 1/n$, $\theta = 1$. Then the following numerical results and graph (see Figure 2) demonstrate Theorem 3.2:

$n = 1 - 5$	2.000000000	1.500000000	1.187500000	0.892361111	0.654839410
$n = 6 - 10$	0.479806858	0.356556683	0.271536914	0.213118132	0.172638974
$n = 11 - 15$	0.144088241	0.123452725	0.108108238	0.096353820	0.087086603
$n = 16 - 20$	0.079585996	0.073374619	0.068130205	0.063630247	0.059717263

Besides, we know the following.

If $|x_n - x_{n-1}| < 10^{-3}$, then $n = 36$; if $|x_n - x_{n-1}| < 10^{-4}$, then $n = 104$; if $|x_n - x_{n-1}| < 10^{-5}$, then $n = 320$; if $|x_n - x_{n-1}| < 10^{-6}$, then $n = 1{,}004$.

4 Mathematical programming with multiple sets split feasibility constraints

Let H_1 be a Hilbert space, let f be a proper lower semicontinuous convex function of H_1 into $(-\infty, \infty)$. The subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H_1 : f(x) + \langle z, y - x\rangle \le f(y), \forall y \in H_1\}$$

for all $x \in H_1$. From Rockafellar [23], we know that ∂f is a maximal monotone operator. Let C be a nonempty closed convex subset of a real Hilbert space H_1, and i_C be the indicator function of C, *i.e.*

$$i_C x = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{if } x \notin C. \end{cases}$$

Furthermore, we also define the normal cone $N_C u$ of C at u as follows:

$$N_C u = \{z \in H_1 : \langle z, v - u\rangle \le 0, \forall v \in C\}.$$

Then i_C is a proper lower semicontinuous convex function on H, and the subdifferential ∂i_C of i_C is a maximal monotone operator. Thus, we can define the resolvent $J_\lambda^{\partial i_C}$ of ∂i_C for $\lambda > 0$, *i.e.*

$$J_\lambda^{\partial i_C} x = (I + \lambda \partial i_C)^{-1} x$$

for all $x \in H$. Since

$$\begin{aligned}\partial i_C x &= \{z \in H_1 : i_C x + \langle z, y - x\rangle \le i_C y, \forall y \in H_1\} \\ &= \{z \in H_1 : \langle z, y - x\rangle \le 0, \forall y \in C\} \\ &= N_C x\end{aligned}$$

for all $x \in C$, we have

$$\begin{aligned} u = J_\lambda^{\partial i_C} x \quad &\Leftrightarrow \quad x \in u + \lambda \partial i_C u \quad \Leftrightarrow \quad x - u \in \lambda N_C u \\ &\Leftrightarrow \quad \langle x - u, y - u\rangle \le 0, \quad \forall y \in C \\ &\Leftrightarrow \quad u = P_C x. \end{aligned} \tag{34}$$

The equilibrium problem is to find $z \in C$ such that

$$g(z, y) \ge 0 \quad \text{for each } y \in C. \tag{EP}$$

The solutions set of the equilibrium problem (EP) is denoted by $EP(g)$. For solving the equilibrium problem, let us assume that the bifunction $g : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $g(x, x) = 0$ for each $x \in C$;
(A2) g is monotone, *i.e.*, $g(x, y) + g(y, x) \le 0$ for any $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} g(tz + (1 - t)x, y) \le g(x, y)$;
(A4) for each $x \in C$, the scalar function $y \to g(x, y)$ is convex and lower semicontinuous.

Lemma 4.1 [24, 25] *Let $g : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions* (A1)-(A4). *Let $r > 0$ and $x \in C$. Then there exists $z \in C$ such that*

$$g(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 \quad \textit{for all } y \in C.$$

Furthermore, if

$$T_r^g(x) := \left\{z \in C : g(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 \textit{ for all } y \in C\right\},$$

then we have:

(i) T_r^g *is single-valued*;
(ii) T_r^g *is a firmly nonexpansive mapping*;
(iii) $EP(g)$ *is a closed convex subset of* C;
(iv) $EP(g) = \mathrm{Fix}(T_r^g)$.

We call such T_r^g the resolvent of g for $r > 0$. Throughout these section, we use these notations and assumptions unless specified otherwise.

Takahashi *et al.* [26] gave the following lemma.

Lemma 4.2 [26] *Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4). *Define A_g as follows*:

$$A_g x = \begin{cases} \{z \in H_1 : g(x,y) \geq \langle y - x, z\rangle, \forall y \in C\} & \text{if } x \in C; \\ \emptyset & \text{if } x \notin C. \end{cases} \tag{L4.2}$$

Then, $EP(g) = A_g^{-1}0$ and A_g is a maximal monotone operator with the domain of $A_g \subset C$. Furthermore, for any $x \in H_1$ and $r > 0$, the resolvent T_r^g of g coincides with the resolvent of A_g, i.e., $T_r^g x = (I + rA_g)^{-1}x$.

Let C, Q, and Q' be nonempty closed convex subsets of real Hilbert spaces H_1, H_2, and H_3, respectively, let G_i be a maximal monotone mapping on H_1 such that the domains of G_i is included in C for each $i = 1,2$. Let $J_\lambda^{G_1} = (I + \lambda G_1)^{-1}$ and $J_r^{G_2} = (I + rG_2)^{-1}$ for each $\lambda > 0$ and $r > 0$, let L_1 be a κ_1-inverse-strongly monotone mapping of C into H_1, let L_2 be a κ_2-inverse-strongly monotone mapping of C into H_1, let B be a ν-inverse-strongly monotone mapping of Q into H_2 and let B' be a ν'-inverse-strongly monotone mapping of Q' into H_3. Let G be a maximal monotone mappings on H_2 such that the domain of G is included in Q and let G' be maximal a monotone mappings on H_3 such that the domain of G' is included in Q'. Let $J_{\lambda'}^G = (I + \lambda' G)^{-1}$ and $J_{r'}^{G'} = (I + r'G')^{-1}$ for each $\lambda' > 0$ and $r' > 0$. Let $A, A_1 : H_1 \to H_2$ be bounded linear operators, A_1^* and A^* the adjoints of A_1 and A respectively, $A_2 : H_1 \to H_3$ a bounded linear operator, and A_2^* the adjoint of A_2. Let R_i be the spectral radius of the operator $A_i^*A_i$ for $i = 1,2$, respectively, and R the spectral radius of the operator A^*A. Let I, I_2, I_3 be the identity mappings of H_1, H_2, H_3, respectively. We use these notations throughout this section unless specified otherwise.

In order to study the convergence theorems for the solutions set of the multiple sets split monotone variational inclusion problem, we study the following essential problem (SFP-1):

$$\text{Find } \bar{x} \in H_1 \text{ such that } A\bar{x} \in (G + B)^{-1}(0).$$

Theorem 4.1 *Given any $\bar{x} \in H_1$ we have the following.*

(i) *If $\bar{x}$ is a solution of* (SFP-1), *then $(I - \lambda A^*(I_2 - U)A)\bar{x} = \bar{x}$, where $\lambda > 0$, $U = J_\sigma^G(I_2 - \sigma B)$ and $\sigma > 0$.*

(ii) *Suppose that $U = J_\sigma^G(I_2 - \sigma B)$, $0 < \lambda < \frac{1}{R}$, $0 < \sigma < 2\nu$. Then $A^*(I_2 - U)A$ is a $\frac{\kappa}{R}$-ism mapping, $J_\sigma^G(I_2 - \sigma B)$, and $I - \lambda A^*(I_2 - U)A$ are averaged for some $\kappa > \frac{1}{2}$. Suppose further that solution set of* (SFP-1) *is nonempty and $(I - \lambda A^*(I_2 - U)A)\bar{x} = \bar{x}$. Then $\bar{x}$ is a solution of* (SFP-1).

Proof (i) Suppose that $\bar{x} \in H_1$ is a solution of (SFP-1). Then $\bar{x} \in \mathcal{H}_1$, $A\bar{x} \in \mathrm{Fix}(U)$. It is easy to see that $(I - \lambda A^*(I_2 - U)A)\bar{x} = \bar{x}$.

(ii) Since the solutions set of (SFP-1) is nonempty, there exists $\bar{w} \in H_1$ such that $A\bar{w} \in F(U)$. Since B is a ν-inverse-strongly monotone mapping of Q into H_2, it follows from Lemma 2.4(iii) and (iv) that

$$J_\sigma^G(I_2 - \sigma B) \text{ is averaged.} \tag{35}$$

By Lemma 2.4(iii), for some $\kappa > \frac{1}{2}$, we know that

$$I_2 - U = I_2 - J_\sigma^G(I_2 - \sigma B) \text{ is } \kappa\text{-ism.} \tag{36}$$

In Theorem 3.1 [9], Moudafi showed that

$$A_1^*(I_2 - U)A \text{ is } \frac{\kappa}{R}\text{-ism.} \tag{37}$$

By Lemma 2.4(iii) and $0 < \lambda < \frac{1}{R}$, we know that

$$I - \lambda A^*(I_2 - U)A \text{ is averaged} \tag{38}$$

for some $\kappa > \frac{1}{2}$. Since

$$\bar{x} = \bigl(I - \lambda A^*(I_2 - U)A\bigr)\bar{x}. \tag{39}$$

This implies

$$A^*(I_2 - U)A\bar{x} = 0. \tag{40}$$

We know that $U(A\bar{x}) = A\bar{x} + w_1$, with $A^*w_1 = 0$, which combined with the fact that $U(A\bar{w}) = A\bar{w}$ yields

$$\bigl\|U(A\bar{x}) - U(A\bar{w})\bigr\|^2 = \|A\bar{x} + w_1 - A\bar{w}\|^2 = \|A\bar{x} - A\bar{w}\|^2 + \|w_1\|^2. \tag{41}$$

Since $U = J_\sigma^G(I_2 - \sigma B)$ is a nonexpansive mapping and we have equation (41), we have $w_1 = 0$.

This implies that

$$A\bar{x} = \operatorname{Fix}(U) = \operatorname{Fix}\bigl(J_\sigma^G(I_2 - \sigma B)\bigr). \tag{42}$$

This shows that $\bar{x}$ is a solution of (SFP-1). □

In the following theorem, we consider the multiple set split monotonic variational inclusion problem (MSSMVIP-1):

$$\begin{aligned}&\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0),\ A_1\bar{x} \in (B + G)^{-1}(0),\\ &\text{and } A_2\bar{x} \in \bigl(B' + G'\bigr)^{-1}(0).\end{aligned}$$

That is,

$$\begin{aligned}&\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in \operatorname{Fix}\bigl(J_\lambda^{G_1}\bigr) \cap \operatorname{Fix}\bigl(J_r^{G_2}\bigr),\ A_1\bar{x} \in \operatorname{Fix}(U_1) \text{ and } A_2\bar{x} \in \operatorname{Fix}(U_2)\\ &\text{where } U_1 = J_\sigma^G(I_2 - \sigma B),\ U_2 = J_{\sigma'}^{G'}\bigl(I_3 - \sigma' B'\bigr).\end{aligned}$$

Let Ω_1 be the solutions set of (MSSMVIP-1).

Theorem 4.2 *Let $T : C \to H_1$ be a 2-generalized hybrid mapping. Suppose that Ω_1 is the solutions set of* (MSSMVIP-1) *with* $\mathrm{Fix}(T) \cap \Omega_1 \neq \emptyset$. *Take $\mu \in \mathbb{R}$ as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H_1$ be defined by

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = J_\lambda^{G_1}(I - \lambda A_1^*(I_2 - U_1)A_1)J_r^{G_2}(I - rA_2^*(I_3 - U_2)A_2)x_n, \\ s_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k y_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (1-\beta_n V)s_n), \end{cases} \tag{4.2}$$

where $U_1 = J_\sigma^G(I_2 - \sigma B)$, $U_2 = J_{\sigma'}^{G'}(I_3 - \sigma' B')$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, $r \in (0,\infty)$ and $\lambda \in (0,\infty)$. We have

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $0 < \lambda < \frac{1}{R_1}$, $0 < r < \frac{1}{R_2}$, $0 < \sigma < 2\nu$ *and* $0 < \sigma' < 2\nu'$;
(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H_1$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega_1}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \mathrm{Fix}(T) \cap \Omega_1.$$

Proof Let $F_1 = A_1^*(I_2 - U_1)A_1$, $F_2 = A_2^*(I_3 - U_2)A_2$, $\lambda_n = \lambda$ and $r_n = r$ for all $n \in \mathbb{N}$ in Theorem 3.1. It follow from Theorem 4.1(ii) that F_i is $\frac{\mu_i}{R_i}$-ivm for some $\mu_i > \frac{1}{2}$ and each $i = 1, 2$. Then algorithm (3.1) in Theorem 3.1 follows immediately from algorithm (4.2) in Theorem 4.2.

Since $\mathrm{Fix}(T) \cap \Omega_1$ is nonempty, there exists $\bar{w} \in C$ such that

$$\begin{aligned} \bar{w} \in\ & \mathrm{Fix}(T) \cap \mathrm{Fix}\big(J_\lambda^{G_1}\big) \cap \mathrm{Fix}\big(I - \lambda A_1^*(I_2 - U_1)A_1\big) \\ & \cap \mathrm{Fix}\big(J_r^{G_2}\big) \cap \mathrm{Fix}\big(I - rA_2^*(I_3 - U_2)A_2\big). \end{aligned} \tag{43}$$

This implies that

$$\begin{aligned} \bar{w} \in\ & \mathrm{Fix}(T) \cap \mathrm{Fix}\big(J_\lambda^{G_1}\big(I - \lambda A_1^*(I_2 - U_1)A_1\big)\big) \\ & \cap \mathrm{Fix}\big(J_r^{G_2}\big(I - rA_2^*(I_3 - U_2)A_2\big)\big). \end{aligned} \tag{44}$$

That is,

$$\bar{w} \in \mathrm{Fix}(T) \cap \mathrm{Fix}\big(J_\lambda^{G_1}(I - \lambda F_1)\big) \cap \mathrm{Fix}\big(J_r^{G_2}(I - rF_2)\big). \tag{45}$$

Hence,

$$\bar{w} \in \mathrm{Fix}(T) \cap (G_1 + F_1)^{-1}0 \cap (G_2 + F_2)^{-1}0 \neq \emptyset. \tag{46}$$

It follows from Theorem 3.1 that $\lim_{n\to\infty} x_n = \bar{x}$, where

$$\bar{x} = P_{\mathrm{Fix}(T)\cap(F_1+G_1)^{-1}0\cap(F_2+G_2)^{-1}0}(\bar{x} - V\bar{x} + \theta).$$

This point $\bar{x}$ is also a unique solution of the hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \ge 0, \quad \forall q \in \mathrm{Fix}(T) \cap (F_1 + G_1)^{-1}0 \cap (F_2 + G_2)^{-1}0.$$

If

$$w \in \mathrm{Fix}(T) \cap (G_1 + F_1)^{-1}0 \cap (G_2 + F_2)^{-1}0. \tag{47}$$

By equations (44), (45), and (46), we know that

$$w = Tw, \qquad w = J_\lambda^{G_1}\bigl(I - \lambda A_1^*(I_2 - U_1)A_1\bigr)w, \qquad w = J_r^{G_2}\bigl(I - rA_2^*(I_3 - U_2)A_2\bigr)w. \tag{48}$$

By $\Omega_1 \neq \emptyset$, equation (48), and Lemma 2.4(v), we have

$$\begin{aligned} &w = Tw, \qquad w = J_\lambda^{G_1} w, \qquad w = \bigl(I - \lambda A_1^*(I_2 - U_1)A_1\bigr)w, \qquad w = J_r^{G_2} w, \\ &w = \bigl(I - rA_2^*(I_3 - U_2)A_2\bigr)w. \end{aligned} \tag{49}$$

It follows from Theorem 4.1(ii) that w is a solution of (MSSMVIP-1). Therefore, $w \in \mathrm{Fix}(T) \cap \Omega_1$ and

$$\mathrm{Fix}(T) \cap (G_1 + F_1)^{-1}0 \cap (G_2 + F_2)^{-1}0 \subseteq \mathrm{Fix}(T) \cap \Omega_1.$$

Conversely, if $w \in \mathrm{Fix}(T) \cap \Omega_1$, by equations (43), (44), (45), and (46), we know that $w \in \mathrm{Fix}(T) \cap (G_1 + F_1)^{-1}0 \cap (G_2 + F_2)^{-1}0$ and

$$\mathrm{Fix}(T) \cap \Omega_1 \subseteq \mathrm{Fix}(T) \cap (G_1 + F_1)^{-1}0 \cap (G_2 + F_2)^{-1}0.$$

Therefore, $\mathrm{Fix}(T) \cap \Omega_1 = \mathrm{Fix}(T) \cap (G_1 + F_1)^{-1}0 \cap (G_2 + F_2)^{-1}0$ and the proof is completed. □

Remark 4.1 Moudafi [9] studied a weak convergence theorem for the split monotone variational inclusion problem, while Theorem 4.2 is a strong convergence theorem for the multiply sets split monotone variational inclusion problem.

By Theorem 4.2, we study the mathematical programming problem with (MSSMVIP-1) and fixed point set constraints.

Theorem 4.3 *In Theorem* 4.2, *let* $h : C \to \mathbb{R}$ *be a convex Gâteaux differential function with Gâteaux derivative* V, *and the assumption* (iv) *is replaced by* $\lim_{n\to\infty} \theta_n = 0$. *Then* $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega_1}(\bar{x} - V\bar{x})$. *This point* $\bar{x}$ *is also a unique solution of the mathematical programming problem with* (MSSMVIP-1) *and fixed point constraints*: $\min_{x\in\mathrm{Fix}(T)\cap\Omega_1} h(x)$.

Proof Let $\theta = 0$ in Theorem 4.2, by Theorem 4.2, we see that

$$\langle V\bar{x}, q - \bar{x}\rangle \geq 0, \quad \forall q \in F(T) \cap \Omega_1. \tag{50}$$

Since $h : C \to \mathbb{R}$ is a convex Gâteaux differential function with Gâteaux dirivitive V, we obtain

$$\begin{aligned}
\langle V\bar{x}, y - \bar{x}\rangle &= \lim_{t\to 0} \frac{h(\bar{x} + t(y - \bar{x})) - h(\bar{x})}{t} \\
&= \lim_{t\to 0} \frac{h((1-t)\bar{x} + ty) - h(\bar{x})}{t} \\
&\leq \lim_{t\to 0} \frac{(1-t)h(\bar{x}) + th(y) - h(\bar{x})}{t} \\
&= h(y) - h(\bar{x})
\end{aligned} \tag{51}$$

for all $y \in C$. By equations (50) and (51), it is easy to see that $h(\bar{x}) \leq h(q)$ for all $q \in \operatorname{Fix}(T) \cap \Omega_1$. □

If we put $h(x) = \frac{1}{2}\|x\|^2$ in Theorem 4.3, then $V = I$, and we have the following minimum norm of common solutions for (MSSMVIP-1) and $\operatorname{Fix}(T)$.

Theorem 4.4 *In Theorem* 4.3, *let the iteration process* $\{x_{n+1}\}$ *be replaced by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\bigl(\beta_n\theta_n + (1 - \beta_n)s_n\bigr), \quad n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\operatorname{Fix}(T)\cap\Omega_1}(0)$. *This point* $\bar{x}$ *is also a unique minimum solution of* $\operatorname{Fix}(T) \cap \Omega_1$: $\min_{x\in\operatorname{Fix}(T)\cap\Omega_1} \|x\|$.

The multiple sets split variational inequality problem (MSSMVIP-2) is defined as follows:

$$\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0), \tag{52}$$

and

$$\bigl\langle B(A_1\bar{x}), y - A_1\bar{x}\bigr\rangle \geq 0 \quad \text{for all } y \in Q, \tag{53}$$

$$\bigl\langle B'(A_2\bar{x}), y - A_2\bar{x}\bigr\rangle \geq 0 \quad \text{for all } y \in Q'. \tag{54}$$

That is,

$$\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in \operatorname{Fix}\bigl(J_\lambda^{G_1}\bigr) \cap \operatorname{Fix}\bigl(J_r^{G_2}\bigr) \tag{55}$$

and

$$A_1\bar{x} \in \operatorname{Fix}\bigl(P_Q(I - \sigma B)\bigr) \quad \text{and} \quad A_2\bar{x} \in \operatorname{Fix}\bigl(P_{Q'}\bigl(I - \sigma' B'\bigr)\bigr). \tag{56}$$

By Theorem 4.2, we can study a variational inequality problem with the split variational inequality (MSSMVIP-2) and fixed point set constraints.

Theorem 4.5 *In Theorem* 4.2, *let* $U_1 = J_\sigma^G(I-\sigma B)$, $U_2 = J_{\sigma'}^{G'}(I-\sigma' B')$ *be replaced by* $U_1 = P_Q(I-\sigma B)$, $U_2 = P_{Q'}(I-\sigma' B')$, *respectively. Suppose that the set of solutions for* (MSSMVIP-2) *is* Ω_2 *and* $\mathrm{Fix}(T) \cap \Omega_2 \neq \emptyset$. *Then* $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega_2}(\bar{x} - V\bar{x} + \theta)$. *This point* $\bar{x}$ *is also a unique solution of the hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \mathrm{Fix}(T) \cap \Omega_2.$$

Proof Let $G = \partial i_Q$ and $G' = \partial i_{Q'}$ in Theorem 4.2, then, by equation (34), we have $J_\sigma^G(I - \sigma B) = P_Q(I-\sigma B)$, $J_{\sigma'}^{G'}(I-\sigma' B') = P_{Q'}(I-\sigma' B')$. Since $\mathrm{Fix}(T) \cap \Omega_2 \neq \emptyset$, there exists $\bar{w} \in C$ such that we can find

$$\bar{w} \in H_1 \text{ such that } \bar{w} \in \mathrm{Fix}\bigl(J_\lambda^{G_1}\bigr) \cap \mathrm{Fix}\bigl(J_r^{G_2}\bigr) \tag{57}$$

and

$$A_1\bar{w} \in \mathrm{Fix}\bigl(P_Q(I-\sigma B)\bigr) \quad \text{and} \quad A_2\bar{w} \in \mathrm{Fix}\bigl(P_{Q'}\bigl(I-\sigma' B'\bigr)\bigr). \tag{58}$$

This implies that

$$A_1\bar{w} \in \mathrm{Fix}\bigl(J_\sigma^G(I-\sigma B)\bigr) \quad \text{and} \quad A_2\bar{w} \in \mathrm{Fix}\bigl(J_{\sigma'}^{G'}\bigl(I-\sigma' B'\bigr)\bigr). \tag{59}$$

Therefore, $\bar{w} \in \mathrm{Fix}(T) \cap \Omega_1 \neq \emptyset$. It follows from Theorem 4.2 that $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega_1}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \mathrm{Fix}(T) \cap \Omega_1.$$

By equations (57), (58), and (59), $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega_2}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \mathrm{Fix}(T) \cap \Omega_2. \qquad \square$$

Remark 4.2 Censor *et al.* [27] studied a weak convergence theorem for the split variational inequalities problem with the additional assumption, while Theorem 4.5 studies a strong convergence theorem for multiply sets split variational inequalities problem without this additional assumption.

By Theorem 4.5, we study a mathematical programming problem with (MSSMVIP-2) and fixed point set constraints.

Theorem 4.6 *In Theorem* 4.5, *let* $h : C \to \mathbb{R}$ *be a convex Gâteaux differential function with Gâteaux derivative* V, *and the assumption* (iv) *is replaced by* $\lim_{n\to\infty} \theta_n = 0$. *Then* $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega_2}(\bar{x} - V\bar{x})$. *This point* $\bar{x}$ *is also a unique solution of the following mathematical programming problem with* (MSSMVIP-2) *and fixed point constraints*:

$$\min_{x \in \mathrm{Fix}(T)\cap\Omega_2} h(x).$$

Proof By Theorem 4.5 and following the same argument as in the proof of Theorem 4.3, we see that the proof is complete. □

In the following theorem, we consider the following split monotonic variational inclusion problem (MSSMVIP-3):

$$\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in G_2^{-1}(0),\ \bar{x} \in (G_1 + F_1)^{-1}0,\ \text{and } A_2\bar{x} \in \bigl(B' + G'\bigr)^{-1}(0).$$

That is,

$$\begin{aligned}&\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in \operatorname{Fix}\bigl(J_\lambda^{G_1}(I - \lambda F_1)\bigr) \cap \operatorname{Fix}\bigl(J_r^{G_2}\bigr),\ \text{and } A_2\bar{x} \in \operatorname{Fix}(U_2)\\ &\text{where } U_2 = J_{\sigma'}^{G'}\bigl(I_3 - \sigma' B'\bigr).\end{aligned}$$

Let Ω_3 be the solutions set of (MSSMVIP-3).

Theorem 4.7 *Let $T : C \to H_1$ be a 2-generalized hybrid mapping. Suppose that Ω_3 is the solutions set of* (MSSMVIP-3) *with* $\operatorname{Fix}(T) \cap \Omega_3 \neq \emptyset$. *Take $\mu \in \mathbb{R}$ as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H_1$ be defined by

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily},\\ y_n = J_\lambda^{G_1}(I - \lambda F_1)J_r^{G_2}(I - rA_2^*(I_3 - U_2)A_2)x_n,\\ s_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k y_n,\\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n\theta_n + (1 - \beta_n V)s_n), \end{cases} \tag{4.7}$$

where $U_2 = J_{\sigma'}^{G'}(I_3 - \sigma' B')$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, $r \in (0,\infty)$ *and* $\lambda \in (0,\infty)$.

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $0 < \lambda < 2\kappa_1$, $0 < r < \frac{1}{R_2}$ *and* $0 < \sigma' < 2\nu'$;
(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H_1$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\operatorname{Fix}(T)\cap\Omega_3}(\bar{x} - V\bar{x} + \theta)$. *This point $\bar{x}$ is also a unique solution of the hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \operatorname{Fix}(T) \cap \Omega_3.$$

Proof Let $F_2 = A_2^*(I - U_2)A_2$, $\lambda_n = \lambda$ and $r_n = r$ for all $n \in \mathbb{N}$ in Theorem 3.1. It follows from Theorem 4.1(ii) that F_2 is $\frac{\mu_i}{R_i}$-ivm for some $\mu_i > \frac{1}{2}$. Then algorithm (3.1) in Theorem 3.1 follows immediately from algorithm (4.7) in Theorem 4.7.

Since $\operatorname{Fix}(T) \cap \Omega_3$ is nonempty, there exists $\bar{w} \in C$ such that

$$\bar{w} \in \operatorname{Fix}(T) \cap \operatorname{Fix}\bigl(J_\lambda^{G_1}(I - \lambda F_1)\bigr) \cap \operatorname{Fix}\bigl(J_r^{G_2}\bigr) \cap \operatorname{Fix}\bigl(I - rA_2^*(I - U_2)A_2\bigr). \tag{60}$$

This implies that

$$\bar{w} \in \operatorname{Fix}(T) \cap \operatorname{Fix}\bigl(J_\lambda^{G_1}(I - \lambda F_1)\bigr) \cap \operatorname{Fix}\bigl(J_r^{G_2}\bigl(I - rA_2^*(I - U_2)A_2\bigr)\bigr). \tag{61}$$

That is,

$$\bar{w} \in \mathrm{Fix}(T) \cap \mathrm{Fix}\big(J_\lambda^{G_1}(I - \lambda F_1)\big) \cap \mathrm{Fix}\big(J_r^{G_2}(I - rF_2)\big). \tag{62}$$

Hence,

$$\bar{w} \in \mathrm{Fix}(T) \cap (G_1 + F_1)^{-1}0 \cap (G_2 + F_2)^{-1}0 \neq \emptyset. \tag{63}$$

It follows from Theorem 3.1 that $\lim_{n\to\infty} x_n = \bar{x}$, where

$$\bar{x} = P_{\mathrm{Fix}(T)\cap(F_1+G_1)^{-1}0\cap(F_2+G_2)^{-1}0}(\bar{x} - V\bar{x} + \theta).$$

This point $\bar{x}$ is also a unique solution of the hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \mathrm{Fix}(T) \cap (F_1 + G_1)^{-1}0 \cap (F_2 + G_2)^{-1}0.$$

If

$$w \in \mathrm{Fix}(T) \cap (G_1 + F_1)^{-1}0 \cap (G_2 + F_2)^{-1}0. \tag{64}$$

That is,

$$w = Tw, \qquad w = J_\lambda^{G_1}(I - F_1)w \quad \text{and} \quad w = J_r^{G_2}\big(I - rA_2^*(I - U_2)A_2\big)w. \tag{65}$$

By $\Omega_3 \neq \emptyset$, equation (65) and Lemma 2.4(v), we have

$$w = Tw, \qquad w = J_\lambda^{G_1}(I - \lambda F_1)w, \qquad w = J_r^{G_2}w, \qquad w = \big(I - rA_2^*(I_3 - U_2)A_2\big)w. \tag{66}$$

By $\Omega_3 \neq \emptyset$, equation (66), and Theorem 4.1(ii), we see that w is a solution of (MSSMVIP-3). Therefore, $w \in \mathrm{Fix}(T) \cap \Omega_3$ and $\mathrm{Fix}(T) \cap (G_1 + F_1)^{-1}0 \cap (G_2 + F_2)^{-1}0 \subseteq \mathrm{Fix}(T) \cap \Omega_3$. Conversely, if $w \in \mathrm{Fix}(T) \cap \Omega_3$, by equations (60), (61), (62), and (63), we know that $w \in \mathrm{Fix}(T) \cap (G_1 + F_1)^{-1}0 \cap (G_2 + F_2)^{-1}0$ and

$$\mathrm{Fix}(T) \cap \Omega_3 \subseteq \mathrm{Fix}(T) \cap (G_1 + F_1)^{-1}0 \cap (G_2 + F_2)^{-1}0.$$

Therefore, $\mathrm{Fix}(T) \cap \Omega_3 = \mathrm{Fix}(T) \cap (G_1 + F_1)^{-1}0 \cap (G_2 + F_2)^{-1}0$ and the proof is completed. □

Remark 4.3 Theorem 4.7 also improve Theorem 3.1 [9].

For each $i = 1, 2$, let $f_i : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A4). The system of mixed type equilibria problem (MSSMVIP-4) is defined as follows.

$$\text{Find } \bar{x} \in C \text{ such that } f_1(\bar{x}, x) + \langle x - \bar{x}, F_1\bar{x}\rangle \geq 0 \text{ and } f_2(\bar{x}, x) + \langle x - \bar{x}, F_2\bar{x}\rangle \geq 0$$

for all $x \in C$.

By Theorem 3.1 and Lemma 4.2, we study a variational inequality problem with (MSSMVIP-4) and fixed point set constraints.

Theorem 4.8 *Let $T : C \to H_1$ be a 2-generalized hybrid mapping. For each $i = 1,2$, let $f_i : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4), *and $J_\lambda^{A_{f_1}}$, $J_r^{A_{f_2}}$, defined as Lemma* 4.2. *Suppose that Ω_4 is the solutions set of* (MSSMVIP-4) *with $(A_{f_1} + F_1)^{-1}0 \cap (A_{f_2} + F_2)^{-1}0 \cap \mathrm{Fix}(T) \neq \emptyset$. Let $\{x_n\} \subset H$ be defined by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = J_\lambda^{A_{f_1}}(I - \lambda F_1) J_r^{A_{f_2}}(I - rF_2)x_n, \\ s_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k y_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n \theta_n + (1-\beta_n V)s_n), \end{cases} \tag{4.8}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, $r \in (0,\infty)$ and $\lambda \in (0,\infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $0 < a_1 \leq \lambda \leq b_1 < 2\kappa_1$, $0 < a_2 \leq r \leq b_2 < \kappa_2$;

(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H_1$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega_4}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \mathrm{Fix}(T) \cap \Omega_4.$$

Proof For each $i = 1,2$, let A_{f_i} be as in Lemma 4.2. By Lemma 4.2, we see that A_{f_i} is a maximal monotone operator with the domain of $A_{f_i} \subset C$. Furthermore, for any $x \in H_1$ and $r > 0$, the resolvent $T_\lambda^{f_i}$ of f_i coincides with the resolvent of A_{f_i}, *i.e.*,

$$T_\lambda^{f_i} x = (I + \lambda A_{f_i})^{-1} x. \tag{67}$$

For $i = 1,2$, let $G_i = A_{f_i}$ in Theorem 3.1. By equation (67), we have

$$T_\lambda^{f_i} x = (I + \lambda A_{f_1})^{-1} x = J_\lambda^{G_1} x, \qquad T_r^{f_2} x = (I + rA_{f_2})^{-1} x = J_r^{G_2} x. \tag{68}$$

Then algorithm (3.1) in Theorem 3.1 follows immediately from algorithm (4.8) in Theorem 4.8.

By equation (68), we have $\mathrm{Fix}(T_\lambda^{f_1}) = A_{f_1}^{-1}0 = \mathrm{Fix}(J_\lambda^{A_{f_1}})$ and $\mathrm{Fix}(T_r^{f_2}) = A_{f_2}^{-1}0 = \mathrm{Fix}(J_r^{A_{f_2}})$.

It follows from Theorem 3.1 that $\lim_{n\to\infty} x_n = \bar{x}$, where

$$\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega_1}(\bar{x} - V\bar{x} + \theta).$$

This point $\bar{x}$ is also a unique solution of the hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \mathrm{Fix}(T) \cap \Omega_1.$$

Here $w \in (G_1 + F_1)^{-1}0 \cap (G_2 + F_2)^{-1}0$. That is, $w \in (A_{f_1} + F_1)^{-1}0 \cap (A_{f_2} + F_2)^{-1}0$. That is, $w \in \mathrm{Fix}(J_\lambda^{A_{f_1}}(I - \lambda F_1)) \cap \mathrm{Fix}(J_r^{A_{f_2}}(I - rF_2))$. That is,

$$f_1(w,x) + \langle x - w, F_1 w\rangle \geq 0$$

and

$$f_2(w,x) + \langle x - w, F_2 w\rangle \geq 0$$

for all $x \in C$. Therefore, $\Omega_1 = \Omega_4$ and the proof is complete. □

By Theorem 4.2, we study a mathematical programming problem with (MSSMVIP-4) and fixed point set constraints.

Theorem 4.9 *In Theorem* 4.8, *let* $h : C \to \mathbb{R}$ *be a convex Gâteaux differential function with Gâteaux derivative* V, *and let the assumption* (iv) *be replaced by* $\lim_{n\to\infty} \theta_n = 0$. *Then* $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega_4}(\bar{x} - V\bar{x})$. *This point* $\bar{x}$ *is also a unique solution of the mathematical programming problem with* (MSSMVIP-2) *constraints*:

$$\min_{x\in\mathrm{Fix}(T)\cap\Omega_4} h(x).$$

For each $i = 1,2$, let $f_i : C \times C \to \mathbb{R}$ and $g_1 : Q \times Q \to \mathbb{R}$, $g_2 : Q' \times Q' \to \mathbb{R}$ be bifunctions satisfying conditions (A1)-(A4). The multiple sets split system of mixed type equilibrium problems (MSSMVIP-5) is defined as follows.

$$\text{Find } \bar{x} \in C \text{ such that } \bar{x} \in EP(f_1) \cap EP(f_2),\ g_1(A_1\bar{x}, y) + \langle y - A_1\bar{x}, BA_1\bar{x}\rangle \geq 0 \text{ and}$$
$$g_2\bigl(A_2\bar{x}, y'\bigr) + \bigl\langle y' - A_2\bar{x}, B'A_2\bar{x}\bigr\rangle \geq 0$$

for all $y \in Q$, $y' \in Q'$.

By Theorem 4.2, we can study a variational inequality problem with (MSSMVIP-5) and fixed point set constraints.

Theorem 4.10 *For each* $i = 1,2$, *let* $f_i : C \times C \to \mathbb{R}$ *and* $g_1 : Q \times Q \to \mathbb{R}$, $g_2 : Q' \times Q' \to \mathbb{R}$ *be bifunctions satisfying the conditions* (A1)-(A4). *Let* $T : C \to H_1$ *be a* 2-*generalized hybrid mapping, and let* $J_\lambda^{A_{f_1}}$, $J_r^{A_{f_2}}$, $J_\sigma^{A_{g_1}}$, $J_\sigma^{A_{g_2}}$ *be defined as in Lemma* 4.2. *Suppose that* Ω_5 *is the solutions set of* (MSSMVIP-4) *with* $\Omega_5 \cap \mathrm{Fix}(T) \neq \emptyset$. *Let* $\{x_n\} \subset H$ *be defined by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = J_\lambda^{A_{f_1}}(I - \lambda A_1^*(I_2 - U_1)A_1)J_\lambda^{A_{f_2}}(I - rA_2^*(I_3 - U_2)A_2)x_n, \\ s_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k y_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n\theta_n + (1-\beta_n V)s_n), \end{cases} \tag{4.10}$$

where $U_1 = J_\sigma^{A_{g_1}}(I_2 - \sigma B)$, $U_2 = J_{\sigma'}^{A_{g_2}}(I_3 - \sigma' B')$. *Then* $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega_5}(\bar{x} - V\bar{x} + \theta)$. *This point* $\bar{x}$ *is also a unique solution of the hierarchical variational inequality*:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \mathrm{Fix}(T) \cap \Omega_5.$$

Proof For $i = 1,2$, let $G_i = A_{f_i}$, $G = A_{g_1}$, and $G' = A_{g_2}$ in Theorem 4.2. By Theorem 4.2 and following the same argument as in the proof of Theorem 4.8, we prove Theorem 4.10. □

For each $i = 1, 2$, let $f_i : C \times C \to \mathbb{R}$ and $g_i : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions (A1)-(A4). The split mixed type equilibrium problem (MSSMVIP-6) is defined as follows.

$$\text{Find } \bar{x} \in C \text{ such that } \bar{x} \in EP(f_2),\ f_1(\bar{x}, x) + \langle x - \bar{x}, F_1\bar{x}\rangle \geq 0 \text{ and}$$
$$g_2(A_2\bar{x}, y) + \langle y - A_2\bar{x}, B'A_2\bar{x}\rangle \geq 0$$

for all $x \in C$, $y \in Q$.

Applying Theorem 4.7 and following a similar argument as in Theorem 4.10, we can study a variational inequality problem with (MSSMVIP-6) and with fixed point set constraints.

Theorem 4.11 *For each $i = 1, 2$, let $f_i : C \times C \to \mathbb{R}$ and $g_i : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions* (A1)-(A4). *Let $T : C \to H_1$ be a* 2*-generalized hybrid mapping, and $J_\lambda^{A_{f_1}}$, $J_r^{A_{f_2}}$, $J_\sigma^{A_{g_1}}$, $J_\sigma^{A_{g_2}}$ defined as in Lemma* 4.2. *Suppose that Ω_6 is the solutions set of* (MSSMVIP-6) *with $\Omega_6 \cap \mathrm{Fix}(T) \neq \emptyset$. Let $\{x_n\} \subset H$ be defined by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = J_\lambda^{A_{f_1}}(I - \lambda F_1)J_\lambda^{A_{f_2}}(I - rA_2^*(I_3 - U_2)A_2)x_n, \\ s_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k y_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n\theta_n + (1 - \beta_n V)s_n), \end{cases} \tag{4.11}$$

where $U_2 = J_{\sigma'}^{A_{g_2}}(I_3 - \sigma' B')$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, $r \in (0, \infty)$ and $\lambda \in (0, \infty)$. Assume further that

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $0 < \lambda < 2\kappa_1$, $0 < r < \frac{1}{R_2}$ *and* $0 < \sigma' < 2\nu'$;
(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H_1$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega_6}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \mathrm{Fix}(T) \cap \Omega_6.$$

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Authors' contributions
Z-TY carry out the project, draft and revise the manuscript. L-JL design this research project, coordination, revise the paper. C-SC coordinate and the project and revise the paper, and give the numerical results. All authors read and approved the final manuscript.

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