

## Research Article

# Fixed Point Results for Generalized Contractive Multimaps in Metric Spaces

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The concept of generalized contractive multimaps in the setting of metric spaces is introduced, and the existence of fixed points for such maps is guaranteed under certain conditions. Consequently, our results either generalize or improve a number of fixed point results including the corresponding recent fixed point results of Ćirić (2008), Latif-Albar (2008), Klim-Wardowski (2007), and Feng-Liu (2006). Examples are also given.

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## 1. Introduction and Preliminaries

Let  $(X, d)$  be a metric space,  $2^X$  a collection of nonempty subsets of  $X$ ,  $CB(X)$  a collection of nonempty closed bounded subsets of  $X$ ,  $Cl(X)$  a collection of nonempty closed subsets of  $X$ ,  $K(X)$  a collection of nonempty compact subsets of  $X$  and  $H$  the Hausdorff metric induced by  $d$ . Then for any  $A, B \in CB(X)$ ,

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}, \quad (1.1)$$

where  $d(x, B) = \inf_{y \in B} d(x, y)$ .

An element  $x \in X$  is called a *fixed point* of a multivalued map  $T : X \rightarrow 2^X$  if  $x \in T(x)$ . We denote  $\text{Fix}(T) = \{x \in X : x \in T(x)\}$ . A sequence  $\{x_n\}$  in  $X$  is called an *orbit* of  $T$  at  $x_0 \in X$  if  $x_n \in T(x_{n-1})$  for all  $n \geq 1$ .

A map  $f : X \rightarrow \mathbb{R}$  is called *lower semicontinuous* if for any sequence  $\{x_n\} \subset X$  with  $x_n \rightarrow x \in X$  it implies that  $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$ .

Using the concept of Hausdorff metric, Nadler [1] established the following fixed point result for multivalued contraction maps, known as Nadler's contraction principle which in turn is a generalization of the well-known Banach contraction principle.

**Theorem 1.1** (see [1]). *Let  $(X, d)$  be a complete metric space and let  $T : X \rightarrow CB(X)$  be a contraction map. Then  $\text{Fix}(T) \neq \emptyset$ .*

Using the concept of the Hausdorff metric, many authors have generalized Nadler's contraction principle in many directions. But, in fact for most cases the existence part of the results can be proved without using the concept of Hausdorff metric. Recently, Feng and Liu [2] extended Nadler's fixed point theorem without using the concept of Hausdorff metric. They proved the following result.

**Theorem 1.2.** *Let  $(X, d)$  be a complete metric space and let  $T : X \rightarrow Cl(X)$  be a map such that for any fixed constants  $h, b \in (0, 1)$ ,  $h < b$ , and for each  $x \in X$  there is  $y \in T(x)$  satisfying the following conditions:*

$$\begin{aligned} bd(x, y) &\leq d(x, T(x)), \\ d(y, T(y)) &\leq hd(x, y). \end{aligned} \tag{1.2}$$

*Then  $\text{Fix}(T) \neq \emptyset$  provided a real-valued function  $g$  on  $X$ ,  $g(x) = d(x, T(x))$  is lower semicontinuous.*

Recently, Klim and Wardowski [3] generalized Theorem 1.2 and proved the following two results.

**Theorem 1.3.** *Let  $(X, d)$  be a complete metric space and let  $T : X \rightarrow Cl(X)$ . Assume that the following conditions hold:*

- (i) *there exist a number  $b \in (0, 1)$  and a function  $k : [0, \infty) \rightarrow [0, b)$  such that for each  $t \in [0, \infty)$ ,*

$$\limsup_{r \rightarrow t^+} k(r) < b, \tag{1.3}$$

- (ii) *for any  $x \in X$  there is  $y \in T(x)$  satisfying*

$$\begin{aligned} bd(x, y) &\leq d(x, T(x)), \\ d(y, T(y)) &\leq k(d(x, y))d(x, y). \end{aligned} \tag{1.4}$$

*Then  $\text{Fix}(T) \neq \emptyset$  provided a real-valued function  $g$  on  $X$ ,  $g(x) = d(x, T(x))$  is lower semicontinuous.*

**Theorem 1.4.** *Let  $(X, d)$  be a complete metric space and let  $T : X \rightarrow K(X)$ . Assume that the following conditions hold:*

- (i) *there exists a function  $k : [0, \infty) \rightarrow [0, 1)$  such that for each  $t \in [0, \infty)$ ,*

$$\limsup_{r \rightarrow t^+} k(r) < 1, \tag{1.5}$$

(ii) for any  $x \in X$  there is  $y \in T(x)$  satisfying

$$\begin{aligned} d(x, y) &= d(x, T(x)), \\ d(y, T(y)) &\leq k(d(x, y))d(x, y). \end{aligned} \tag{1.6}$$

Then  $\text{Fix}(T) \neq \emptyset$  provided a real-valued function  $g$  on  $X$ ,  $g(x) = d(x, T(x))$  is lower semicontinuous.

Note that Theorem 1.3 generalizes Nadler's contraction principle and Theorem 1.2. Most recently, Ćirić [4] obtained some interesting fixed point results which extend and generalize the cited results. Namely, [4, Theorem 5] generalizes [5, Theorem 5], [4, Theorem 6] generalizes [4, Theorems 1.2, 1.3], and [3, theorem 7] generalizes Theorem 1.4.

In [6], Kada et al. introduced the concept of  $w$ -distance on a metric space as follows:

A function  $\omega : X \times X \rightarrow [0, \infty)$  is called  $w$ -distance on  $X$  if it satisfies the following for each  $x, y, z \in X$ :

- (w<sub>1</sub>)  $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$ ;
- (w<sub>2</sub>) a map  $\omega(x, \cdot) : X \rightarrow [0, \infty)$  is lower semicontinuous; that is, if a sequence  $\{y_n\}$  in  $X$  with  $y_n \rightarrow y \in X$ , then  $\omega(x, y) \leq \liminf_{n \rightarrow \infty} \omega(x, y_n)$ ;
- (w<sub>3</sub>) for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $\omega(z, x) \leq \delta$  and  $\omega(z, y) \leq \delta$  imply  $d(x, y) \leq \epsilon$ .

Note that, in general for  $x, y \in X$ ,  $\omega(x, y) \neq \omega(y, x)$  and not either of the implications  $\omega(x, y) = 0 \Leftrightarrow x = y$  necessarily hold. Clearly, the metric  $d$  is a  $w$ -distance on  $X$ . Let  $(Y, \|\cdot\|)$  be a normed space. Then the functions  $\omega_1, \omega_2 : Y \times Y \rightarrow [0, \infty)$  defined by  $\omega_1(x, y) = \|y\|$  and  $\omega_2(x, y) = \|x\| + \|y\|$  for all  $x, y \in Y$  are  $w$ -distances [6]. Many other examples and properties of the  $w$ -distance can be found in [6, 7].

The following lemma is crucial for the proofs of our results.

**Lemma 1.5** (see [8]). *Let  $K$  be a closed subset of  $X$  and  $\omega$  be a  $w$ -distance on  $X$ . Suppose that there exists  $u \in X$  such that  $\omega(u, u) = 0$ . Then  $\omega(u, K) = 0 \Leftrightarrow u \in K$ , where  $\omega(u, K) = \inf_{y \in K} \omega(u, y)$ .*

Most recently, the authors of this paper generalized Latif and Albar [9, Theorem 1.3] as follows.

**Theorem 1.6** (see [10]). *Let  $(X, d)$  be a complete metric space with a  $w$ -distance  $\omega$ . Let  $T : X \rightarrow Cl(X)$  be a multivalued map satisfying that for any constant  $b \in (0, 1)$  and for each  $x \in X$  there is  $y \in J_b^x$  such that*

$$\omega(y, T(y)) \leq k(\omega(x, y))\omega(x, y), \tag{1.7}$$

where  $J_b^x = \{y \in T(x) : b\omega(x, y) \leq \omega(x, T(x))\}$  and  $k$  is a function from  $[0, \infty)$  to  $[0, b)$  with  $\limsup_{r \rightarrow t^+} k(r) < b$ , for every  $t \in [0, \infty)$ . Suppose that a real-valued function  $g$  on  $X$  defined by  $g(x) = \omega(x, T(x))$  is lower semicontinuous. Then there exists  $v_0 \in X$  such that  $g(v_0) = 0$ . Further, if  $\omega(v_0, v_0) = 0$ , then  $v_0 \in \text{Fix}(T)$ .

The aim of this paper is to present some more general results on the existence of fixed points for multivalued maps satisfying certain conditions. Our results unify and generalize

the corresponding results of Mizoguchi and Takahashi [5], Klim and Wardowski [3], Latif and Abdou [10], Ćirić [4], Feng and Liu [2], Latif and Albar [9] and several others.

## 2. The Results

First we prove a theorem which is a generalization of Ćirić [4, Theorem 5] and due to Klim and Wardowski [3, Theorem 1.4].

**Theorem 2.1.** *Let  $(X, d)$  be a complete metric space with a  $\omega$ -distance  $\omega$ . Let  $T : X \rightarrow Cl(X)$  be a multivalued map. Assume that the following conditions hold:*

(i) *there exists a function  $\varphi : [0, \infty) \rightarrow [0, 1)$  such that for each  $t \in [0, \infty)$*

$$\limsup_{r \rightarrow t^+} \varphi(r) < 1 \quad (2.1)$$

(ii) *for any  $x \in X$ , there exists  $y \in T(x)$  satisfying*

$$\begin{aligned} \omega(x, y) &\leq (2 - \varphi(\omega(x, y)))\omega(x, T(x)), \\ \omega(y, T(y)) &\leq \varphi(\omega(x, y))\omega(x, y) \end{aligned} \quad (2.2)$$

(iii) *the map  $f : X \rightarrow \mathbb{R}$ , defined by  $f(x) = \omega(x, T(x))$  is lower semicontinuous.*

*Then there exists  $v_0 \in X$  such that  $f(v_0) = 0$ . Further if  $\omega(v_0, v_0) = 0$ , then  $v_0 \in T(v_0)$ .*

*Proof.* let  $x_0 \in X$  be any initial point. Then there exists  $x_1 \in T(x_0)$  such that

$$\begin{aligned} \omega(x_0, x_1) &\leq (2 - \varphi(\omega(x_0, x_1)))\omega(x_0, T(x_0)), \\ \omega(x_1, T(x_1)) &\leq \varphi(\omega(x_0, x_1))\omega(x_0, x_1). \end{aligned} \quad (2.3)$$

From (2.3) we get

$$\omega(x_1, T(x_1)) \leq \varphi(\omega(x_0, x_1))(2 - \varphi(\omega(x_0, x_1)))\omega(x_0, T(x_0)). \quad (2.4)$$

Define a function  $\psi : [0, \infty) \rightarrow [0, \infty)$  by

$$\psi(t) = \varphi(t)(2 - \varphi(t)) = 1 - (1 - \varphi(t))^2. \quad (2.5)$$

Using the facts that for each  $t \in [0, \infty)$ ,  $\varphi(t) < 1$  and  $\lim_{r \rightarrow t^+} \sup \varphi(r) < 1$ , we have

$$\psi(t) < 1, \quad (2.6)$$

$$\limsup_{r \rightarrow t^+} \psi(r) < 1 \quad \forall t \in [0, \infty) \quad (2.7)$$

From (2.4) and (2.5), we have

$$\omega(x_1, T(x_1)) \leq \psi(\omega(x_0, x_1))\omega(x_0, T(x_0)). \quad (2.8)$$

Similarly, for  $x_1 \in X$ , there exists  $x_2 \in T(x_1)$  such that

$$\begin{aligned} \omega(x_1, x_2) &\leq (2 - \varphi(\omega(x_1, x_2)))\omega(x_1, T(x_1)), \\ \omega(x_2, T(x_2)) &\leq \varphi(\omega(x_1, x_2))\omega(x_1, x_2). \end{aligned} \quad (2.9)$$

Thus

$$\omega(x_2, T(x_2)) \leq \varphi(\omega(x_1, x_2))\omega(x_1, T(x_1)). \quad (2.10)$$

Continuing this process we can get an orbit  $\{x_n\}$  of  $T$  in  $X$  satisfying the following:

$$\omega(x_n, x_{n+1}) \leq (2 - \varphi(\omega(x_n, x_{n+1})))\omega(x_n, T(x_n)), \quad (2.11)$$

$$\omega(x_{n+1}, T(x_{n+1})) \leq \varphi(\omega(x_n, x_{n+1}))\omega(x_n, T(x_n)), \quad (2.12)$$

for each integer  $n \geq 0$ . Since  $\varphi(t) < 1$  for each  $t \in [0, \infty)$  and from (2.12), we have for all  $n \geq 0$

$$\omega(x_{n+1}, T(x_{n+1})) < \omega(x_n, T(x_n)). \quad (2.13)$$

Thus the sequence of nonnegative real numbers  $\{\omega(x_n, T(x_n))\}$  is decreasing and bounded below, thus convergent. Therefore, there is some  $\delta \geq 0$  such that

$$\lim_{n \rightarrow \infty} \omega(x_n, T(x_n)) = \delta. \quad (2.14)$$

From (2.11), as  $\varphi(t) < 1$  for all  $t \geq 0$ , we get

$$\omega(x_n, T(x_n)) \leq \omega(x_n, x_{n+1}) < 2\omega(x_n, T(x_n)), \quad (2.15)$$

Thus, we conclude that the sequence of nonnegative reals  $\{\omega(x_n, x_{n+1})\}$  is bounded. Therefore, there is some  $\theta \geq 0$  such that

$$\liminf_{n \rightarrow \infty} \omega(x_n, x_{n+1}) = \theta. \quad (2.16)$$

Note that  $\omega(x_n, x_{n+1}) \geq \omega(x_n, T(x_n))$  for each  $n \geq 0$ , so we have  $\theta \geq \delta$ . Now we will show that  $\theta = \delta$ . Suppose that  $\delta = 0$ . Then we get

$$\lim_{n \rightarrow \infty} \omega(x_n, x_{n+1}) = 0. \quad (2.17)$$

Now consider  $\delta > 0$ . Suppose to the contrary, that  $\theta > \delta$ . Then  $\theta - \delta > 0$  and so from (2.14) and (2.16) there is a positive integer  $n_0$  such that

$$\omega(x_n, T(x_n)) < \delta + \frac{\theta - \delta}{4} \quad \forall n \geq n_0, \quad (2.18)$$

$$\theta - \frac{\theta - \delta}{4} < \omega(x_n, x_{n+1}) \quad \forall n \geq n_0. \quad (2.19)$$

Then from (2.19), (2.11) and (2.18), we get

$$\begin{aligned} \theta - \frac{\theta - \delta}{4} &< \omega(x_n, x_{n+1}) \\ &\leq (2 - \varphi(\omega(x_n, x_{n+1})))\omega(x_n, T(x_n)) \\ &< (2 - \varphi(\omega(x_n, x_{n+1}))) \left[ \delta + \frac{\theta - \delta}{4} \right]. \end{aligned} \quad (2.20)$$

Thus for all  $n \geq n_0$ ,

$$(2 - \varphi(\omega(x_n, x_{n+1}))) > \frac{3\theta + \delta}{3\delta + \theta}, \quad (2.21)$$

that is,

$$1 + (1 - \varphi(\omega(x_n, x_{n+1}))) > 1 + \frac{2(\theta - \delta)}{3\delta + \theta}, \quad (2.22)$$

and we get

$$-(1 - \varphi(\omega(x_n, x_{n+1})))^2 < -\left[ \frac{2(\theta - \delta)}{3\delta + \theta} \right]^2. \quad (2.23)$$

Thus for all  $n \geq n_0$ ,

$$\begin{aligned} \varphi(\omega(x_n, x_{n+1})) &= 1 - (1 - \varphi(\omega(x_n, x_{n+1})))^2 \\ &< 1 - \left[ \frac{2(\theta - \delta)}{3\delta + \theta} \right]^2. \end{aligned} \quad (2.24)$$

Thus, from (2.12) and (2.24), we get

$$\omega(x_{n+1}, T(x_{n+1})) \leq h\omega(x_n, T(x_n)) \quad \forall n \geq n_0, \quad (2.25)$$

where  $h = 1 - [2(\theta - \delta)/(3\delta + \theta)]^2$ . Clearly  $h < 1$  as  $\theta > \delta$ . From (2.18) and (2.25), we have for any  $k \geq 1$

$$\omega(x_{n_0+k}, T(x_{n_0+k})) \leq h^k \omega(x_{n_0}, T(x_{n_0})). \quad (2.26)$$

Since  $\delta > 0$  and  $h < 1$ , there is a positive integer  $k_0$  such that  $h^{k_0} \omega(x_{n_0}, T(x_{n_0})) < \delta$ . Now, since  $\delta \leq \omega(x_n, T(x_n))$  for each  $n \geq 0$ , by (2.26) we have

$$\delta \leq \omega(x_{n_0+k_0}, T(x_{n_0+k_0})) \leq h^{k_0} \omega(x_{n_0}, T(x_{n_0})) < \delta. \quad (2.27)$$

a contradiction. Hence, our assumption  $\theta > \delta$  is wrong. Thus  $\delta = \theta$ . Now we will show that  $\theta = 0$ . Since  $\theta = \delta \leq \omega(x_n, T(x_n)) \leq \omega(x_n, x_{n+1})$ , then from (2.16) we can read as

$$\liminf_{n \rightarrow \infty} \omega(x_n, x_{n+1}) = \theta+, \quad (2.28)$$

so, there exists a subsequence  $\{\omega(x_{n_k}, x_{n_k+1})\}$  of  $\{\omega(x_n, x_{n+1})\}$  such that

$$\lim_{k \rightarrow \infty} \omega(x_{n_k}, x_{n_k+1}) = \theta+. \quad (2.29)$$

Now from (2.7) we have

$$\limsup_{\omega(x_{n_k}, x_{n_k+1}) \rightarrow \theta+} \psi(\omega(x_{n_k}, x_{n_k+1})) < 1, \quad (2.30)$$

and from (2.12), we have

$$\omega(x_{n_k}, T(x_{n_k+1})) \leq \psi(\omega(x_{n_k}, x_{n_k+1})) \omega(x_{n_k}, T(x_{n_k})) \quad (2.31)$$

Taking the limit as  $k \rightarrow \infty$  and using (2.14), we get

$$\begin{aligned} \delta &= \limsup_{k \rightarrow \infty} \omega(x_{n_{k+1}}, T(x_{n_{k+1}})) \\ &\leq \left( \limsup_{k \rightarrow \infty} \psi(\omega(x_{n_{k+1}}, x_{n_{k+1}})) \right) \left( \limsup_{k \rightarrow \infty} \omega(x_{n_k}, T(x_{n_k})) \right) \\ &= \left( \limsup_{\omega(x_{n_k}, x_{n_k+1}) \rightarrow \theta+} \psi(\omega(x_{n_k}, x_{n_k+1})) \right) \delta. \end{aligned} \quad (2.32)$$

If we suppose that  $\delta > 0$ , then from last inequality, we have

$$\limsup_{\omega(x_{n_k}, x_{n_k+1}) \rightarrow \theta+} \psi(\omega(x_{n_k}, x_{n_k+1})) \geq 1, \quad (2.33)$$

which contradicts with (2.30). Thus  $\delta = 0$ . Then from (2.14) and (2.15), we have

$$\lim_{n \rightarrow \infty} \omega(x_n, T(x_n)) = 0+, \quad (2.34)$$

and thus

$$\lim_{n \rightarrow \infty} \omega(x_n, x_{n+1}) = 0+. \quad (2.35)$$

Now, let

$$\alpha = \lim_{\omega(x_{n_k}, x_{n_{k+1}}) \rightarrow 0^+} \sup \psi(\omega(x_{n_k}, x_{n_{k+1}})). \quad (2.36)$$

Then by (2.7),  $\alpha < 1$ . Let  $q$  be such that  $\alpha < q < 1$ . Then there is some  $n_0 \in \mathbb{N}$  such that

$$\psi(\omega(x_n, x_{n+1})) < q \quad \forall n \geq n_0. \quad (2.37)$$

Thus it follows from (2.12),

$$\omega(x_{n+1}, T(x_{n+1})) \leq q\omega(x_n, T(x_n)) \quad \forall n \geq n_0. \quad (2.38)$$

By induction we get

$$\omega(x_{n+1}, T(x_{n+1})) \leq q^{n+1-n_0} \omega(x_{n_0}, T(x_{n_0})) \quad \forall n \geq n_0. \quad (2.39)$$

Now, using (2.15) and (2.39), we have

$$\omega(x_n, x_{n+1}) \leq 2q^{n-n_0} \omega(x_{n_0}, T(x_{n_0})) \quad \forall n \geq n_0. \quad (2.40)$$

Now, we show that  $\{x_n\}$  is a Cauchy sequence, for all  $m > n \geq n_0$ , we get

$$\begin{aligned} \omega(x_n, x_m) &\leq \sum_{k=n}^{m-1} \omega(x_k, x_{k+1}) \\ &\leq 2 \sum_{k=n}^{m-1} q^{k-n_0} \omega(x_{n_0}, T(x_{n_0})) \\ &\leq 2 \left( \frac{q^{n-n_0}}{1-q} \right) \omega(x_{n_0}, T(x_{n_0})). \end{aligned} \quad (2.41)$$

Hence we conclude, as  $q < 1$ , that  $\{x_n\}$  is Cauchy sequence. Due to the completeness of  $X$ , there exists some  $v_0 \in X$  such that  $\lim_{n \rightarrow \infty} x_n = v_0$ . Since  $f$  is lower semicontinuous and from (2.34), we have

$$0 \leq f(v_0) \leq \liminf_{n \rightarrow \infty} f(x_n) = \omega(x_n, T(x_n)) = 0, \quad (2.42)$$



and thus,  $f(v_0) = \omega(v_0, T(v_0)) = 0$ . Since  $\omega(v_0, v_0) = 0$ , and  $T(v_0)$  is closed, it follows from Lemma 1.5 that  $v_0 \in T(v_0)$ .  $\square$

We also have the following interesting result by replacing the hypothesis (iii) of Theorem 2.1 with another natural condition.

**Theorem 2.2.** *Suppose that all the hypotheses of Theorem 2.1 except (iii) hold. Assume that*

$$\inf\{\omega(x, v) + \omega(x, T(x)) : x \in X\} > 0, \quad (2.43)$$

for every  $v \in X$  with  $v \notin T(v)$ . Then  $\text{Fix}(T) \neq \emptyset$ .

*Proof.* Following the proof of Theorem 2.1, there exists a Cauchy sequence  $\{x_n\}$  with  $x_n \in T(x_{n-1})$ . Due to the completeness of  $X$ , there exists  $v_0 \in X$  such that  $\lim_{n \rightarrow \infty} x_n = v_0$ . Since  $\omega(x, \cdot)$  is lower semicontinuous and  $x_m \rightarrow v_0 \in X$ , it follows for all  $n \geq n_0$

$$\begin{aligned} \omega(x_n, v_0) &\leq \liminf_{m \rightarrow \infty} \omega(x_n, x_m) \leq \left(\frac{2q^{n-n_0}}{1-q}\right) \omega(x_{n_0}, T(x_{n_0})), \\ \omega(x_n, T(x_n)) &\leq \omega(x_n, x_{n+1}) \leq 2q^{n-n_0} \omega(x_{n_0}, T(x_{n_0})). \end{aligned} \quad (2.44)$$

Assume that  $v_0 \notin T(v_0)$ . Then, we have

$$\begin{aligned} 0 &< \inf\{\omega(x, v_0) + \omega(x, T(x)) : x \in X\} \\ &\leq \inf\{\omega(x_n, v_0) + \omega(x_n, T(x_n)) : n \geq n_0\} \\ &\leq \inf\left\{\left(\frac{2q^{n-n_0}}{1-q}\right) \omega(x_{n_0}, T(x_{n_0})) + 2q^{n-n_0} \omega(x_{n_0}, T(x_{n_0})) : n \geq n_0\right\} \\ &= \frac{2(2-q)}{(1-q)q^{n_0}} \omega(x_{n_0}, T(x_{n_0})) \inf\{q^n : n \geq n_0\} = 0, \end{aligned} \quad (2.45)$$

which is impossible and hence  $v_0 \in \text{Fix}(T)$ .  $\square$

Now, we present an improved version of Ciric [4, Theorem 6] and which also generalizes due to Latif and Abdou [10, Theorem 1.6] and due to Klim and Wardowski [3, Theorem 1.3].

**Theorem 2.3.** *Let  $(X, d)$  be a complete metric space with a  $\omega$ -distance  $\omega$ . Let  $T : X \rightarrow Cl(X)$ , be a multivalued map. Assume that the following condition hold:*

- (i) *there exist functions  $\varphi : [0, \infty) \rightarrow (0, 1)$  and  $\mu : [0, \infty) \rightarrow [b, 1)$ , with  $b > 0$ ,  $\mu$  nondecreasing such that*

$$\varphi(t) < \mu(t), \quad \limsup_{r \rightarrow t^+} \varphi(r) < \limsup_{r \rightarrow t^+} \mu(r), \quad (2.46)$$

(ii) for any  $x \in X$ , there exists  $y \in T(x)$  satisfying the following conditions:

$$\begin{aligned}\mu(\omega(x, y))\omega(x, y) &\leq \omega(x, T(x)), \\ \omega(y, T(y)) &\leq \varphi(\omega(x, y))\omega(x, y),\end{aligned}\tag{2.47}$$

(iii) the map  $f : X \rightarrow \mathbb{R}$ , defined by  $f(x) = \omega(x, T(x))$  is lower semicontinuous.

Then there exists  $v_0 \in X$  such that  $f(v_0) = 0$ . Further if  $\omega(v_0, v_0) = 0$ , then  $v_0 \in T(v_0)$ .

*Proof.* Let  $x_0$  be an arbitrary, then there exists  $x_1 \in T(x_0)$  such that

$$\begin{aligned}\mu(\omega(x_0, x_1))\omega(x_0, x_1) &\leq \omega(x_0, T(x_0)), \\ \omega(x_1, T(x_1)) &\leq \varphi(\omega(x_0, x_1))\omega(x_0, x_1).\end{aligned}\tag{2.48}$$

From (2.48) we have

$$\omega(x_1, T(x_1)) \leq \frac{\varphi(\omega(x_0, x_1))}{\mu(\omega(x_0, x_1))}\omega(x_0, T(x_0)).\tag{2.49}$$

Define a function  $\psi : [0, \infty) \rightarrow [0, \infty)$  by

$$\psi(t) = \frac{\varphi(t)}{\mu(t)} \quad \forall t \in [0, \infty).\tag{2.50}$$

Since  $\varphi(t) < \mu(t)$ , we have

$$\psi(t) < 1,\tag{2.51}$$

$$\limsup_{r \rightarrow t^+} \psi(r) < 1 \quad \forall t \in [0, \infty).\tag{2.52}$$

Thus from (2.49)

$$\omega(x_1, T(x_1)) \leq \psi(\omega(x_0, x_1))\omega(x_0, T(x_0)).\tag{2.53}$$

Similarly, there exists  $x_2 \in T(x_1)$  such that

$$\begin{aligned}\mu(\omega(x_1, x_2))\omega(x_1, x_2) &\leq \omega(x_1, T(x_1)), \\ \omega(x_2, T(x_2)) &\leq \varphi(\omega(x_1, x_2))\omega(x_1, x_2).\end{aligned}\tag{2.54}$$

Then by definition of  $\psi$ , we get

$$\omega(x_2, T(x_2)) \leq \psi(\omega(x_1, x_2))\omega(x_1, T(x_1)).\tag{2.55}$$

Continuing this process, we get an orbit  $\{x_n\}$  of  $T$  at  $x_0$  such that

$$\mu(\omega(x_n, x_{n+1}))\omega(x_n, x_{n+1}) \leq \omega(x_n, T(x_n)), \quad (2.56)$$

$$\omega(x_{n+1}, T(x_{n+1})) \leq \varphi(\omega(x_n, x_{n+1}))\omega(x_n, x_{n+1}). \quad (2.57)$$

Thus

$$\omega(x_{n+1}, T(x_{n+1})) \leq \varphi(\omega(x_n, x_{n+1}))\omega(x_n, T(x_n)). \quad (2.58)$$

Since  $\varphi(t) < 1$  for all  $t \in [0, \infty)$ , we get

$$\omega(x_{n+1}, T(x_{n+1})) < \omega(x_n, T(x_n)). \quad (2.59)$$

Thus the sequence of nonnegative real numbers  $\{\omega(x_n, T(x_n))\}$  is decreasing and bounded below, thus convergent. Now, we want to show that the sequence  $\{\omega(x_n, x_{n+1})\}$  is also decreasing. Suppose to the contrary, that  $\omega(x_n, x_{n+1}) \leq \omega(x_{n+1}, x_{n+2})$ , then as  $\mu(t)$  is nondecreasing, we have

$$\mu(\omega(x_n, x_{n+1})) \leq \mu(\omega(x_{n+1}, x_{n+2})), \quad (2.60)$$

Now using (2.56), (2.57) and (2.60) with  $n = n + 1$ , we get

$$\begin{aligned} \omega(x_{n+1}, x_{n+2}) &\leq \frac{\varphi(\omega(x_n, x_{n+1}))}{\mu(\omega(x_{n+1}, x_{n+2}))}\omega(x_n, x_{n+1}) \\ &\leq \frac{\varphi(\omega(x_n, x_{n+1}))}{\mu(\omega(x_n, x_{n+1}))}\omega(x_n, x_{n+1}) \\ &< \varphi(\omega(x_n, x_{n+1}))\omega(x_n, x_{n+1}) \\ &< \omega(x_n, x_{n+1}), \end{aligned} \quad (2.61)$$

a contradiction. Thus the sequences  $\{\omega(x_n, x_{n+1})\}$  is decreasing. Now let

$$\limsup_{n \rightarrow \infty} \varphi(\omega(x_n, x_{n+1})) = \alpha. \quad (2.62)$$

Thus by (2.52),  $\alpha < 1$ . Then for any  $q \in (\alpha, 1)$ , there exists  $n_0 \in \mathbb{N}$  such that

$$\varphi(\omega(x_n, x_{n+1})) < q \quad \forall n \geq n_0. \quad (2.63)$$

So, from (2.58), for all  $n \geq n_0$ , we get

$$\omega(x_{n+1}, T(x_{n+1})) < q\omega(x_n, T(x_n)). \quad (2.64)$$

Thus by induction we get for all  $n \geq n_0$

$$\omega(x_{n+1}, T(x_{n+1})) \leq q^{n+1-n_0} \omega(x_{n_0}, T(x_{n_0})). \quad (2.65)$$

Since  $\mu(t) \geq b$ , from (2.56) and (2.65), we have

$$\omega(x_n, x_{n+1}) \leq \frac{1}{b} \omega(x_n, T(x_n)) \leq \frac{1}{b} q^{n-n_0} \omega(x_{n_0}, T(x_{n_0})), \quad (2.66)$$

for all  $n \geq n_0$ . Note that  $\omega(x_n, T(x_n)) \rightarrow 0$ . Now, we show that  $\{x_n\}$  is a Cauchy sequence. For all  $m > n \geq n_0$ , we have

$$\begin{aligned} \omega(x_n, x_m) &\leq \sum_{k=n}^{m-1} \omega(x_k, x_{k+1}) \\ &\leq \frac{1}{b} \sum_{k=n}^{m-1} q^{k-n_0} \omega(x_{n_0}, T(x_{n_0})) \\ &\leq \frac{1}{b} \left( \frac{q^{n-n_0}}{1-q} \right) \omega(x_{n_0}, T(x_{n_0})). \end{aligned} \quad (2.67)$$

Thus we conclude that  $\{x_n\}$  is a Cauchy sequence. Now, proceeding the proof of Theorem 2.1, we get some  $v_0 \in X$  such that  $f(v_0) = \omega(v_0, T(v_0)) = 0$  and  $v_0 \in T(v_0)$ .  $\square$

Following the proof of Theorem 2.2, we can obtain the following result.

**Theorem 2.4.** *Suppose that all the hypotheses of Theorem 2.3 except (iii) hold. Assume that*

$$\inf\{\omega(x, v) + \omega(x, T(x)) : x \in X\} > 0, \quad (2.68)$$

for every  $v \in X$  with  $v \notin T(v)$ . Then  $\text{Fix}(T) \neq \emptyset$ .

Now, we present a result which is a generalization of Theorem 1.4 due to Klim and Wardowski [3] and Ciric [4, Theorem 7].

**Theorem 2.5.** *Let  $(X, d)$  be a complete metric space with a  $w$ -distance  $\omega$ . Let  $T : X \rightarrow Cl(X)$  be a multivalued map. Assume that the following conditions hold:*

(i) *there exists a function  $\varphi : [0, \infty) \rightarrow [0, 1)$  such that for each  $t \in [0, \infty)$*

$$\limsup_{r \rightarrow t^+} \varphi(r) < 1, \quad (2.69)$$

(ii) for any  $x \in X$ , there exists  $y \in T(x)$  satisfying

$$\begin{aligned}\omega(x, y) &= \omega(x, T(x)), \\ \omega(y, T(y)) &\leq \varphi(\omega(x, y))\omega(x, y),\end{aligned}\tag{2.70}$$

(iii) the map  $f : X \rightarrow \mathbb{R}$  defined by  $f(x) = \omega(x, T(x))$  is lower semicontinuous.

Then there exists  $v_0 \in X$  such that  $f(v_0) = 0$ . Further if  $\omega(v_0, v_0) = 0$ , then  $v_0 \in T(v_0)$ .

*Proof.* Let  $x_0 \in X$ , be any initial point. Then from (ii) we can choose  $x_1 \in T(x_0)$  such that

$$\begin{aligned}\omega(x_0, x_1) &= \omega(x_0, T(x_0)) \\ \omega(x_1, T(x_1)) &\leq \varphi(\omega(x_0, x_1))\omega(x_0, x_1) \quad \varphi(\omega(x_0, x_1)) < 1.\end{aligned}\tag{2.71}$$

Using the analogous method like in the proof of Lemma 2.1 [10], we obtain the existence of Cauchy sequence  $\{x_n\}$  such that  $x_n \in T(x_{n-1})$  and satisfying

$$\begin{aligned}\omega(x_n, x_{n+1}) &= \omega(x_n, T(x_n)), \\ \omega(x_{n+1}, T(x_{n+1})) &\leq \varphi(\omega(x_n, x_{n+1}))\omega(x_n, x_{n+1}), \quad \varphi(\omega(x_n, x_{n+1})) < 1.\end{aligned}\tag{2.72}$$

Consequently, there exists  $v_0 \in X$  such that  $\lim_{n \rightarrow \infty} x_n = v_0$ . Since  $f$  is lower semicontinuous, we have

$$0 \leq f(v_0) \leq \liminf_{n \rightarrow \infty} f(x_n) = 0,\tag{2.73}$$

thus,  $f(v_0) = \omega(v_0, T(v_0)) = 0$ . Further by closedness of  $T(v_0)$  and since  $\omega(v_0, v_0) = 0$ , it follows from Lemma 1.5 that  $v_0 \in T(v_0)$ .  $\square$

### 3. Examples

The following example shows that Theorem 2.1 is a genuine generalization of Ciric [4, Theorem 5].

*Example 3.1.* Let  $X = [0, 1]$  with the usual metric  $d$ . Define a function  $\omega : X \times X \rightarrow [0, \infty)$ , by

$$\omega(x, y) = y \quad \forall x, y \in X.\tag{3.1}$$

Clearly,  $\omega$  is a  $w$ -distance on  $X$  and  $\omega \neq d$ . Let  $T : X \rightarrow Cl(X)$  be such that

$$T(x) = \begin{cases} \left\{ \frac{1}{2}x^2 \right\}, & \text{for } x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right], \\ \left\{ \frac{17}{96}, \frac{1}{4} \right\}, & \text{for } x = \frac{15}{32}. \end{cases}\tag{3.2}$$

Define now  $\varphi : [0, \infty) \rightarrow [0, 1)$  as follows

$$\varphi(t) = \begin{cases} \frac{8}{5}t, & \text{for } t \in \left[0, \frac{1}{2}\right), \\ \frac{4}{5}, & \text{for } t \in \left[\frac{1}{2}, \infty\right). \end{cases} \quad (3.3)$$

Note that

$$f(x) = \omega(x, T(x)) = \begin{cases} \frac{1}{2}x^2, & \text{for } x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right], \\ \frac{17}{96}, & \text{for } x = \frac{15}{32} \end{cases} \quad (3.4)$$

and  $f$  is lower semicontinuous. Moreover for each  $x \in [0, 15/32) \cup (15/32, 1]$  we have  $T(x) = \{(1/2)x^2\}$ . Take  $y = (1/2)x^2$ , then we have

$$\omega(x, y) = \omega(x, T(x)) = \omega\left(x, \frac{1}{2}x^2\right) = \frac{1}{2}x^2. \quad (3.5)$$

Further, note that

$$\begin{aligned} \omega(x, y) &= \frac{1}{2}x^2 \leq [2 - \varphi(\omega(x, y))] \frac{1}{2}x^2 = [2 - \varphi(\omega(x, y))] \omega(x, T(x)), \\ \omega(y, T(y)) &= \omega\left(\frac{1}{2}x^2, \frac{1}{8}x^4\right) = \left(\frac{1}{4}x^2\right) \omega(x, y) < \frac{8}{5} \left(\frac{1}{2}x^2\right) \omega(x, y) = \varphi(\omega(x, y)) \omega(x, y). \end{aligned} \quad (3.6)$$

Hence, for all  $x \in [0, 1], x \neq 15/32$ ,  $T$  satisfies all the conditions of Theorem 2.1. Now, if  $x = 15/32$ , then we have  $T(x) = \{17/96, 1/4\}$ , and

$$\omega(x, T(x)) = \omega\left(\frac{15}{32}, \left\{\frac{17}{96}, \frac{1}{4}\right\}\right) = \inf\left\{\frac{17}{96}, \frac{1}{4}\right\} = \frac{17}{96}. \quad (3.7)$$

Note that for  $x = 15/32$  there is  $y = 17/96 \in T(x)$  such that

$$\begin{aligned} \omega(x, y) &= \frac{17}{96} < \left[2 - \frac{8}{5} \left(\frac{17}{96}\right)\right] \frac{17}{96} = [2 - \varphi(\omega(x, y))] \omega(x, y), \\ \omega(y, T(y)) &= \omega\left(\frac{17}{96}, \frac{1}{2} \left(\frac{17}{96}\right)^2\right) = \frac{1}{2} \left(\frac{17}{96}\right)^2 \\ &< \frac{8}{5} \left(\frac{17}{96}\right) \left(\frac{17}{96}\right) = \varphi(\omega(x, y)) \omega(x, y). \end{aligned} \quad (3.8)$$

Thus,  $T$  also satisfies all the conditions of Theorem 2.1 for  $x = 15/32$ . Hence it follows from Theorem 2.1 that  $\text{Fix}(T) \neq \emptyset$ . Note that  $\text{Fix}(T) = \{0\}$ . Clearly,  $T$  does not satisfy the hypotheses of Ciric [4, Theorem 5] because  $\omega$  is not the metric  $d$ .

Finally, we present an example which shows that Theorem 2.5 is a genuine generalization of Theorem 1.4 due to Klim-Wardowski [3].

*Example 3.2.* Let  $X = [0, \infty)$  with the usual metric  $d$ . Define a function  $\omega : X \times X \rightarrow [0, \infty)$ , by

$$\omega(x, y) = x + y \quad \forall x, y \in X. \quad (3.9)$$

Then  $\omega$  is a  $w$ -distance on  $X$ . Note that  $\omega \neq d$ . Now, for any real number  $a > 1$ , define  $T : X \rightarrow Cl(X)$  by

$$T(x) = \left\{ \frac{x}{a} \right\} \cup [(1 + 2x), \infty), \quad \forall x \in [0, \infty), \quad (3.10)$$

and define a constant function  $\varphi : [0, \infty) \rightarrow [0, 1)$  by

$$\varphi(t) = \frac{1}{a}, \quad \forall t \in [0, \infty). \quad (3.11)$$

Note that  $\varphi(t) < 1$  for all  $t \in [0, \infty)$ . And for each  $x \in X$  we have

$$f(x) = \omega(x, T(x)) = x + \frac{x}{a} = \left( \frac{a+1}{a} \right) x. \quad (3.12)$$

Thus,  $f$  is continuous. Now for each  $x \in [0, \infty)$  there exists  $y = (x/a) \in T(x)$  satisfying

$$\begin{aligned} \omega(x, y) &= \omega\left(x, \frac{x}{a}\right) = \omega(x, T(x)) \\ \omega(y, T(y)) &= \frac{x}{a} + \frac{x}{a^2} = \frac{1}{a} \left( \frac{a+1}{a} \right) x = \varphi(\omega(x, y)) \omega(x, y). \end{aligned} \quad (3.13)$$

Therefore, all assumptions of Theorem 2.5 are satisfied and  $\text{Fix}(T) = \{0\}$ . Note that  $T(x)$  is not compact for all  $x \in X$  and the  $w$ -distance  $\omega$  is not a metric  $d$ , so  $T$  do not satisfy the hypotheses of Theorem 1.4.

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