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Research Article

A Note on Asymptotic Contractions

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We provide sufficient conditions for the iterates of an asymptotic contraction on a complete metric space X to converge to its unique fixed point, uniformly on each bounded subset of X.

1. Introduction

Let (X,d) be a complete metric space. The following theorem is the main result of Chen [1]. It improves upon Kirk's original theorem [2]. In this connection, see also [3, 4].

Theorem 1.1. *Let $T : X \to X$ be such that*

$$d(T^n x, T^n y) \leq \phi_n(d(x,y)) \tag{1.1}$$

for all $x, y \in X$ and all natural numbers n, where $\phi_n : [0,\infty) \to [0,\infty)$ and $\lim_{n\to\infty} \phi_n = \phi$, uniformly on any bounded interval $[0,b]$. Suppose that ϕ is upper semicontinuous and that $\phi(t) < t$ for all $t > 0$. Furthermore, suppose that there exists a positive integer n_* such that ϕ_{n_*} is upper semicontinuous and $\phi_{n_*}(0) = 0$. If there exists $x_0 \in X$ which has a bounded orbit $O(x_0) = \{x_0, Tx_0, T^2x_0, \dots\}$, then T has a unique fixed point $x_* \in X$ and $\lim_{n\to\infty} T^n x = x_*$ for all $x \in X$.

Note that Theorem 1.1 does not provide us with uniform convergence of the iterates of T on bounded subsets of X, although this does hold for many classes of mappings of contractive type (e.g., [5, 6]). This property is important because it yields stability of the convergence of iterates even in the presence of computational errors [7]. In the present paper we show that this conclusion can be derived in the setting of Theorem 1.1. To this end, we first prove a somewhat more general result (Theorem 1.2) which, when combined with Theorem 1.1, yields our strengthening of Chen's result (Theorem 1.3).

Theorem 1.2. *Let $x_* \in X$ be a fixed point of $T : X \to X$. Assume that*

$$d(T^n x, x_*) \le \phi_n(d(x, x_*)) \quad \forall x \in X \text{ and all natural numbers } n, \tag{1.2}$$

where $\phi_n : [0,\infty) \to [0,\infty)$ and $\lim_{n\to\infty} \phi_n = \phi$, uniformly on any bounded interval $[0,b]$. Suppose that ϕ is upper semicontinuous and that $\phi(t) < t$ for all $t > 0$. Then $T^n x \to x_*$ as $n \to \infty$, uniformly on each bounded subset of X.

Theorem 1.3. *Let $T : X \to X$ be such that*

$$d(T^n x, T^n y) \le \phi_n(d(x,y)) \tag{1.3}$$

for all $x, y \in X$ and all natural numbers n, where $\phi_n : [0,\infty) \to [0,\infty)$ and $\lim_{n\to\infty} \phi_n = \phi$, uniformly on any bounded interval $[0,b]$. Suppose that ϕ is upper semicontinuous and that $\phi(t) < t$ for all $t > 0$. Furthermore, suppose that there exists a positive integer n_* such that ϕ_{n_*} is upper semicontinuous and $\phi_{n_*}(0) = 0$. If there exists $x_0 \in X$ which has a bounded orbit $O(x_0) = \{x_0, Tx_0, T^2x_0, \dots\}$, then T has a unique fixed point $x_* \in X$ and $\lim_{n\to\infty} T^n x = x_*$, uniformly on each bounded subset of X.

2. Proof of Theorem 1.2

We may assume without loss of generality that $\phi(0) = 0$ and $\phi_n(0) = 0$ for all integers $n \ge 1$.

For each $x \in X$ and each $r > 0$, set

$$B(x,r) = \{y \in X : d(x,y) \le r\}. \tag{2.1}$$

We first prove three lemmas.

Lemma 2.1. *Let $K > 0$. Then there exists a natural number $\overline{q}$ such that for all integers $s \ge \overline{q}$,*

$$T^s(B(x_*,K)) \subset B(x_*,K+1). \tag{2.2}$$

Proof. There exists a natural number $\overline{q}$ such that for all integers $s \ge \overline{q}$,

$$|\phi_s(t) - \phi(t)| < 1 \quad \forall t \in [0,K]. \tag{2.3}$$

Let $s \geq \overline{q}$ be an integer. Then for all $x \in B(x_*, K)$,

$$d(T^s x, x_*) \leq \phi_s(d(x, x_*)) < \phi(d(x, x_*)) + 1 < d(x, x_*) + 1 < K + 1. \tag{2.4}$$

Lemma 2.1 is proved. □

Lemma 2.2. *Let $0 < \epsilon_1 < \epsilon_0$. Then there exists a natural number q such that for each integer $j \geq q$,*

$$T^j(B(x_*, \epsilon_1)) \subset B(x_*, \epsilon_0). \tag{2.5}$$

Proof. There exists an integer $q \geq 1$ such that for each integer $j \geq q$,

$$|\phi_j(t) - \phi(t)| < (\epsilon_0 - \epsilon_1)/2 \quad \forall t \in [0, \epsilon_0]. \tag{2.6}$$

Assume that

$$j \in \{q, q+1, \dots\}, \qquad x \in B(x_*, \epsilon_1). \tag{2.7}$$

By (1.2) and (2.6),

$$\begin{aligned} d(T^j x, x_*) &\leq \phi_j(d(x, x_*)) < \phi(d(x, x_*)) + \frac{(\epsilon_0 - \epsilon_1)}{2} \\ &\leq \epsilon_1 + \frac{(\epsilon_0 - \epsilon_1)}{2} = \frac{(\epsilon_0 + \epsilon_1)}{2}. \end{aligned} \tag{2.8}$$

Lemma 2.2 is proved. □

Lemma 2.3. *Let $K, \epsilon > 0$. Then there exists a natural number q such that for each $x \in B(x_*, K)$,*

$$\min\{d(T^j x, x_*) : j = 1, \dots, q\} \leq \epsilon. \tag{2.9}$$

Proof. By Lemma 2.1, there is a natural number $\overline{q}$ such that

$$T^n(B(x_*, K)) \subset B(x_*, K+1) \text{ for all natural numbers } n \geq \overline{q}. \tag{2.10}$$

We may assume without loss of generality that $\epsilon < K/8$. Since the function $t - \phi(t)$, $t \in (0, \infty)$, is lower semicontinuous and positive, there is

$$\delta \in \left(0, \frac{\epsilon}{8}\right) \tag{2.11}$$

such that

$$t - \phi(t) \geq 2\delta \quad \forall t \in \left[\frac{\epsilon}{2}, K+1\right]. \tag{2.12}$$

There is a natural number $s \geq \overline{q}$ such that

$$|\phi(t) - \phi_s(t)| \leq \delta \quad \forall t \in [0, K+1]. \tag{2.13}$$

By (2.12) and (2.13), we have, for all $t \in [\epsilon/2, K+1]$,

$$\phi_s(t) \leq \phi(t) + \delta \leq t - 2\delta + \delta = t - \delta. \tag{2.14}$$

In view of (2.13) and (2.11), we have, for all $t \in [0, \epsilon/2]$,

$$\phi_s(t) \leq \phi(t) + \delta \leq t + \delta \leq \frac{\epsilon}{2} + \delta < \frac{3}{4}\epsilon. \tag{2.15}$$

Choose a natural number p such that

$$p > 4 + \delta^{-1}(K+1). \tag{2.16}$$

Let

$$x \in B(x_*, K). \tag{2.17}$$

We will show that

$$\min\{d(T^j x, x_*) : j = 1, 2, \ldots, ps\} \leq \epsilon. \tag{2.18}$$

Let us assume the contrary. Then

$$d(T^j x, x_*) > \epsilon \quad \forall j = s, \ldots, ps. \tag{2.19}$$

By (2.17) and (2.10),

$$T^j x \in B(x_*, K+1), \quad j = s, \ldots, ps. \tag{2.20}$$

Let a natural number i satisfy $i \leq p - 1$. By (2.19) and (2.20),

$$d(T^{is}x, x_*) > \epsilon, \qquad d(T^{is}x, x_*) \leq K+1. \tag{2.21}$$

It follows from (1.2), (2.21), and (2.14) that

$$d(T^s(T^{is}x), x_*) \leq \phi_s(d(T^{is}x, x_*)) \leq d(T^{is}x, x_*) - \delta. \tag{2.22}$$

Thus for each natural number $i \le p-1$,

$$d(T^{(i+1)s}x,x_*) \le d(T^{is}x,x_*) - \delta. \tag{2.23}$$

This inequality implies that

$$d(T^{ps}x,x_*) \le d(T^{(p-1)s}x,x_*) - \delta \le \cdots \le d(T^{s}x,x_*) - (p-1)\delta. \tag{2.24}$$

When combined with (2.20) and (2.16), this implies, in turn, that

$$d(T^{ps}x,x_*) \le K + 1 - (p-1)\delta < 0. \tag{2.25}$$

The contradiction we have reached proves (2.18) and completes the proof of Lemma 2.3. □

Completion of the proof of Theorem 1.2. Let $K,\epsilon > 0$. Choose $\epsilon_1 \in (0,\epsilon)$. By Lemma 2.2, there exists a natural number q_1 such that

$$T^{j}(B(x_*,\epsilon_1)) \subset B(x_*,\epsilon) \text{ for all integers } j \ge q_1. \tag{2.26}$$

By Lemma 2.3, there exists a natural number q_2 such that

$$\min\{d(T^{j}x,x_*) : j = 1,\dots,q_2\} \le \epsilon_1 \quad \forall x \in B(x_*,K). \tag{2.27}$$

Assume that

$$x \in B(x_*,K). \tag{2.28}$$

By (2.27), there is a natural number $j_1 \le q_2$ such that

$$d(T^{j_1}x,x_*) \le \epsilon_1. \tag{2.29}$$

In view of (2.29) and (2.26),

$$T^{j}(T^{j_1}x) \in B(x_*,\epsilon) \text{ for all integers } j \ge q_1. \tag{2.30}$$

Inclusion (2.30) and the inequality $j_1 \le q_2$ now imply that

$$T^{i}x \in B(x_*,\epsilon) \text{ for all integers } i \ge q_1 + q_2. \tag{2.31}$$

Theorem 1.2 is proved.

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