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Research Article

Common Fixed Point and Approximation Results for Noncommuting Maps on Locally Convex Spaces

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Common fixed point results for some new classes of nonlinear noncommuting maps on a locally convex space are proved. As applications, related invariant approximation results are obtained. Our work includes improvements and extension of several recent developments of the existing literature on common fixed points. We also provide illustrative examples to demonstrate the generality of our results over the known ones.

1. Introduction and Preliminaries

In the sequel, (E,τ) will be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is said to be an associated family of seminorms for τ if the family $\{\gamma U : \gamma > 0\}$, where $U = \bigcap_{i=1}^{n} U_{\alpha_i}$ and $U_{\alpha_i} = \{x : p_{\alpha_i}(x) < 1\}$, forms a base of neighborhoods of zero for τ. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is called an augmented associated family for τ if $\{p_\alpha : \alpha \in I\}$ is an associated family with property that the seminorm $\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in I\}$ for any $\alpha, \beta \in I$. The associated and augmented associated families of seminorms will be denoted by $A(\tau)$ and $A^*(\tau)$, respectively. It is well known that given a locally convex space (E,τ), there always exists a family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E such that $\{p_\alpha : \alpha \in I\} = A^*(\tau)$ (see [1, page 203]).

The following construction will be crucial. Suppose that M is a τ-bounded subset of E. For this set M we can select a number $\lambda_\alpha > 0$ for each $\alpha \in I$ such that $M \subset \lambda_\alpha U_\alpha$, where $U_\alpha = \{x : p_\alpha(x) \le 1\}$. Clearly, $B = \bigcap_\alpha \lambda_\alpha U_\alpha$ is τ-bounded, τ-closed, absolutely convex and contains M. The linear span E_B of B in E is $\bigcup_{n=1}^{\infty} nB$. The Minkowski functional of B is a norm $\|\cdot\|_B$ on E_B. Thus $(E_B, \|\cdot\|_B)$ is a normed space with B as its closed unit ball and $\sup_\alpha p_\alpha(x/\lambda_\alpha) = \|x\|_B$ for each $x \in E_B$ (for details see [1–3]).

Let M be a subset of a locally convex space (E,τ). Let $I,J : M \to M$ be mappings. A mapping $T : M \to M$ is called (I,J)-Lipschitz if there exists $k \geq 0$ such that $p_\alpha(Tx - Ty) \leq kp_\alpha(Ix - Jy)$ for any $x,y \in M$ and for all $p_\alpha \in A^*(\tau)$. If $k < 1$ (resp., $k = 1$), then T is called an (I,J)-contraction (resp., (I,J)-nonexpansive). A point $x \in M$ is a common fixed (coincidence) point of I and T if $x = Ix = Tx(Ix = Tx)$. The set of coincidence points of I and T is denoted by $C(I,T)$, and the set of fixed points of T is denoted by $F(T)$. The pair $\{I,T\}$ is called:

(1) commuting if $TIx = ITx$ for all $x \in M$;

(2) R-weakly commuting if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists $R > 0$ such that $p_\alpha(ITx - TIx) \leq Rp_\alpha(Ix - Tx)$. If $R = 1$, then the maps are called weakly commuting [4];

(3) compatible [5] if for all $p_\alpha \in A^*(\tau)$, $\lim_n p_\alpha(TIx_n - ITx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some t in M;

(4) weakly compatible if they commute at their coincidence points, that is, $ITx = TIx$ whenever $Ix = Tx$.

Suppose that M is q-starshaped with $q \in F(I)$ and is both T- and I-invariant. Then T and I are called:

(5) R-subcommuting on M if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists a real number $R > 0$ such that $p_\alpha(ITx - TIx) \leq (R/k)p_\alpha(((1-k)q + kTx) - Ix)$ for each $k \in (0,1)$. If $R = 1$, then the maps are called 1-subcommuting [6];

(6) R-subweakly commuting on M (see [7]) if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists a real number $R > 0$ such that $p_\alpha(ITx - TIx) \leq Rd_{p_\alpha}(Ix, [q,Tx])$, where $[q,x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ and $d_{p_\alpha}(u,M) = \inf\{p_\alpha(x-u) : x \in M\}$;

(7) C_q-commuting [8, 9] if $ITx = TIx$ for all $x \in C_q(I,T)$, where $C_q(I,T) = \cup\{C(I,T_k) : 0 \leq k \leq 1\}$ and $T_k x = (1-k)q + kTx$.

If $u \in E, M \subseteq E$, then we define the set, $P_M(u)$, of best M-approximations to u as $P_M(u) = \{y \in M : p_\alpha(y-u) = d_{p_\alpha}(u,M)$, for all $p_\alpha \in A^*(\tau)\}$. A mapping $T : M \to E$ is called demiclosed at 0 if $\{x_\alpha\}$ converges weakly to x and $\{Tx_\alpha\}$ converges to 0, then we have $Tx = 0$. A locally convex space E satisfies Opial's condition if for every net $\{x_\beta\}$ in E weakly convergent to $x \in$ X, the inequality

$$\liminf_{\beta\to\infty} p_\alpha(x_\beta - x) < \liminf_{\beta\to\infty} p_\alpha(x_\beta - y) \tag{1.1}$$

holds for all $y \neq x$ and $p_\alpha \in A^*(\tau)\}$.

In 1963, Meinardus [10] employed the Schauder fixed point theorem to prove a result regarding invariant approximation. Singh [11], Sahab et al. [12], and Jungck and Sessa [13] proved similar results in best approximation theory. Recently, Hussain and Khan [6] have proved more general invariant approximation results for 1-subcommuting maps which extend the work of Jungck and Sessa [13] and Al-Thagafi [14] to locally convex spaces. More recently, with the introduction of noncommuting maps to this area, Pant [15], Pathak et al. [16], Hussain and Jungck [7], and Jungck and Hussain [9] further extended and improved the above-mentioned results; details on the subject may be found in [17, 18]. For applications of fixed point results of nonlinear mappings in simultaneous best approximation theory and variational inequalities, we refer the reader to [19–21]. Fixed point theory of nonexpansive and noncommuting mappings is very rich in Banach spaces and metric spaces [13–17]. However, some partial results have been obtained for these mappings in the setup of locally convex spaces (see [22] and its references). It is remarked that the generalization of a known result in Banach space setting to the case of locally convex spaces is neither trivial nor easy (see, e.g., [2, 22]).

The following general common fixed point result is a consequence of Theorem 3.1 of Jungck [5], which will be needed in the sequel.

Theorem 1.1. *Let (X, d) be a complete metric space, and let T, f, g be selfmaps of X. Suppose that f and g are continuous, the pairs $\{T, f\}$ and $\{T, g\}$ are compatible such that $T(X) \subset f(X) \cap g(X)$. If there exists $r \in (0,1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq r \max\left\{ d(fx, gy), d(Tx, fx), d(Ty, gy), \frac{1}{2}\left[d(fx, Ty) + d(Tx, gy)\right] \right\}, \tag{1.2}$$

then there is a unique point z in X such that $Tz = fz = gz = z$.

The aim of this paper is to extend the above well-known result of Jungck to locally convex spaces and establish general common fixed point theorems for generalized (f, g)-nonexpansive subcompatible maps in the setting of a locally convex space. We apply our theorems to derive some results on the existence of common fixed points from the set of best approximations. We also establish common fixed point and approximation results for the newly defined class of Banach operator pairs. Our results extend and unify the work of Al-Thagafi [14], Chen and Li [23], Hussain [24], Hussain and Berinde [25], Hussain and Jungck [7], Hussain and Khan [6], Hussain and Rhoades [8], Jungck and Sessa [13], Khan and Akbar [19, 20], Pathak and Hussain [21], Sahab et al. [12], Sahney et al. [26], Singh [11, 27], Tarafdar [3], and Taylor [28].

2. Subcompatible Maps in Locally Convex Spaces

Recently, Khan et al. [29] introduced the class of subcompatible mappings as follows:

Definition 2.1. Let M be a q-starshaped subset of a normed space E. For the selfmaps I and T of M with $q \in F(I)$, we define $S_q(I, T) := \cup\{S(I, T_k) : 0 \leq k \leq 1\}$, where $T_k x = (1-k)q + kTx$ and $S(I, T_k) = \{\{x_n\} \subset M : \lim_n I x_n = \lim_n T_k x_n = t \in M\}$. Now I and T are subcompatible if $\lim_n \|ITx_n - TIx_n\| = 0$ for all sequences $\{x_n\} \in S_q(I, T)$.

We can extend this definition to a locally convex space by replacing the norm with a family of seminorms.

Clearly, subcompatible maps are compatible but the converse does not hold, in general, as the following example shows.

Example 2.2 (see [29]). Let $X = R$ with usual norm and $M = [1, \infty)$. Let $I(x) = 2x - 1$ and $T(x) = x^2$, for all $x \in M$. Let $q = 1$. Then M is q-starshaped with $Iq = q$. Note that I and T are compatible. For any sequence $\{x_n\}$ in M with $\lim_n x_n = 2$, we have, $\lim_n I x_n = \lim_n T_{2/3} x_n = 3 \in M$. However, $\lim_n \|ITx_n - TIx_n\| \neq 0$. Thus I and T are not subcompatible maps.

Note that R-subweakly commuting and R-subcommuting maps are subcompatible. The following simple example reveals that the converse is not true, in general.

Example 2.3 (see [29]). Let $X = R$ with usual norm and $M = [0, \infty)$. Let $I(x) = x/2$ if $0 \le x < 1$ and $Ix = x$ if $x \ge 1$, and $T(x) = 1/2$ if $0 \le x < 1$ and $Tx = x^2$ if $x \ge 1$. Then M is 1-starshaped with $I1 = 1$ and $S_q(I, T) = \{\{x_n\} : 1 \le x_n < \infty\}$. Note that I and T are subcompatible but not R-weakly commuting for all $R > 0$. Thus I and T are neither R-subweakly commuting nor R-subcommuting maps.

We observe in the following example that the weak commutativity of a pair of selfmaps on a metric space depends on the choice of the metric; this is also true for compatibility, R-weak commutativity, and other variants of commutativity of maps.

Example 2.4 (see [30]). Let $X = R$ with usual metric and $M = [0, \infty)$. Let $I(x) = 1 + x$ and $T(x) = 2+x^2$. Then $|ITx-TIx| = 2x$ and $|Ix-Tx| = |x^2-x+1|$. Thus the pair (I, T) is not weakly commuting on M with respect to usual metric. But if X is endowed with the discrete metric d, then $d(ITx, TIx) = 1 = d(Ix, Tx)$ for $x > 1$. Thus the pair (I, T) is weakly commuting on M with respect to discrete metric.

Next we establish a positive result in this direction in the context of linear topologies utilizing Minkowski functional; it extends [6, Lemma 2.1].

Lemma 2.5. *Let I and T be compatible selfmaps of a τ-bounded subset M of a Hausdorff locally convex space (E, τ). Then I and T are compatible on M with respect to $\|\cdot\|_B$.*

Proof. By hypothesis, $\lim_{n\to\infty} p_\alpha(ITx_n - TIx_n) = 0$ for each $p_\alpha \in A^*(\tau)$ whenever $\lim_{n\to\infty} p_\alpha(Tx_n - t) = 0 = \lim_{n\to\infty} p_\alpha(Ix_n - t)$ for some $t \in M$. Taking supremum on both sides, we get

$$\sup_\alpha \lim_{n\to\infty} p_\alpha\left(\frac{ITx_n - TIx_n}{\lambda_\alpha}\right) = \sup_\alpha\left(\frac{0}{\lambda_\alpha}\right), \tag{2.1}$$

whenever

$$\sup_\alpha \lim_{n\to\infty} p_\alpha\left(\frac{Tx_n - t}{\lambda_\alpha}\right) = \sup_\alpha\left(\frac{0}{\lambda_\alpha}\right) = \sup_\alpha \lim_{n\to\infty} p_\alpha\left(\frac{Ix_n - t}{\lambda_\alpha}\right). \tag{2.2}$$

This implies that

$$\lim_{n\to\infty} \sup_\alpha\, p_\alpha\left(\frac{ITx_n - TIx_n}{\lambda_\alpha}\right) = 0, \tag{2.3}$$

whenever

$$\lim_{n\to\infty} \sup_\alpha\, p_\alpha\left(\frac{Tx_n - t}{\lambda_\alpha}\right) = 0 = \lim_{n\to\infty} \sup_\alpha p_\alpha\left(\frac{Ix_n - t}{\lambda_\alpha}\right). \tag{2.4}$$

Hence $\lim_{n\to\infty}\|ITx_n - TIx_n\|_B = 0$, whenever $\lim_{n\to\infty}\|Tx_n - t\|_B = 0 = \lim_{n\to\infty}\|Ix_n - t\|_B$ as desired. □

There are plenty of spaces which are not normable (see [31, page 113]). So it is natural and essential to consider fixed point and approximation results in the context of a locally convex space. An application of Lemma 2.5 provides the following general common fixed point result.

Theorem 2.6. *Let M be a nonempty τ-bounded, τ-complete subset of a Hausdorff locally convex space (E,τ) and let T, f, and g be selfmaps of M. Suppose that f and g are nonexpansive, the pairs $\{T,f\}$ and $\{T,g\}$ are compatible such that $T(M) \subset f(M) \cap g(M)$. If there exists $r \in (0,1)$ such that for all $x,y \in M$, and for all $p_\alpha \in A^*(\tau)$*

$$p_\alpha(Tx-Ty) \le r \max\left\{p_\alpha(fx-gy), p_\alpha(Tx-fx), p_\alpha(Ty-gy), \frac{1}{2}\left[p_\alpha(fx-Ty)+p_\alpha(Tx-gy)\right]\right\}, \tag{2.5}$$

then there is a unique point z in M such that $Tz = fz = gz = z$.

Proof. Since the norm topology on E_B has a base of neighbourhoods of 0 consisting of τ-closed sets and M is τ-sequentially complete, therefore M is $\|\cdot\|_B$- sequentially complete in $(E_B, \|\cdot\|_B)$; see [3, the proof of Theorem 1.2]. By Lemma 2.5, the pairs $\{T,f\}$ and $\{T,g\}$ are $\|\cdot\|_B$–compatible maps of M. From (2.5) we obtain for any $x,y \in M$,

$$\begin{aligned}\sup_\alpha p_\alpha\left(\frac{Tx-Ty}{\lambda_\alpha}\right) \le r \max\Bigg\{&\sup_\alpha p_\alpha\left(\frac{fx-gy}{\lambda_\alpha}\right), \sup_\alpha p_\alpha\left(\frac{Tx-fx}{\lambda_\alpha}\right), \sup_\alpha p_\alpha\left(\frac{Ty-gy}{\lambda_\alpha}\right), \\ &\frac{1}{2}\left[\sup_\alpha p_\alpha\left(\frac{fx-Ty}{\lambda_\alpha}\right) + \sup_\alpha p_\alpha\left(\frac{Tx-gy}{\lambda_\alpha}\right)\right]\Bigg\}.\end{aligned} \tag{2.6}$$

Thus

$$\begin{aligned}\|Tx-Ty\|_B \le r \max\Big\{&\|fx-gy\|_B, \|Tx-fx\|_B, \|Ty-gy\|_B, \\ &\frac{1}{2}\left[\|fx-Ty\|_B + \|Tx-gy\|_B\right]\Big\}.\end{aligned} \tag{2.7}$$

As f and g are nonexpansive on τ-bounded set M, f, and g are also nonexpansive with respect to $\|\cdot\|_B$ and hence continuous (cf. [6]). A comparison of our hypothesis with that of Theorem 1.1 tells that we can apply Theorem 1.1 to M as a subset of $(E_B, \|\cdot\|_B)$ to conclude that there exists a unique z in M such that $Tz = fz = gz = z$. □

We now prove the main result of this section.

Theorem 2.7. *Let M be a nonempty τ-bounded, τ-sequentially complete, q-starshaped subset of a Hausdorff locally convex space (E,τ) and let T, f, and g be selfmaps of M. Suppose that f and g are affine and nonexpansive with $q \in F(f) \cap F(g)$, and $T(M) \subset f(M) \cap g(M)$. If the pairs $\{T,f\}$ and*

$\{T, g\}$ are subcompatible and, for all $x, y \in M$ and for all $p_\alpha \in A^*(\tau)$,

$$p_\alpha(Tx - Ty) \le \max\Big\{p_\alpha(fx - gy), d_{p_\alpha}(fx, [Tx, q]), d_{p_\alpha}(gy, [Ty, q]), \frac{1}{2}[d_{p_\alpha}(fx, [Ty, q]) + d_{p_\alpha}(gy, [Tx, q])]\Big\}, \tag{2.8}$$

then $F(T) \cap F(f) \cap F(g) \neq \emptyset$ provided that one of the following conditions holds:

(i) *$cl(T(M))$ is τ-sequentially compact, and T is continuous (cl stands for closure);*

(ii) *M is τ-sequentially compact, and T is continuous;*

(iii) *M is weakly compact in (E, τ), and $f - T$ is demiclosed at 0.*

Proof. Define $T_n : M \to M$ by

$$T_n x = (1 - k_n)q + k_n T x \tag{2.9}$$

for all $x \in M$ and a fixed sequence of real numbers k_n $(0 < k_n < 1)$ converging to 1. Then, each T_n is a selfmap of M and for each $n \ge 1$, $T_n(M) \subset f(M) \cap g(M)$ since f and g are affine and $T(M) \subset f(M) \cap g(M)$. As f is affine and the pair $\{T, f\}$ is subcompatible, so for any $\{x_m\} \subset M$ with $\lim_m f x_m = \lim_m T_n x_m = t \in M$, we have

$$\begin{aligned}\lim_m p_\alpha(T_n f x_m - f T_n x_m) &= k_n \lim_m p_\alpha(T f x_m - f T x_m)\\ &= 0.\end{aligned} \tag{2.10}$$

Thus the pair $\{T_n, f\}$ is compatible on M for each n. Similarly, the pair $\{T_n, g\}$ is compatible for each $n \ge 1$.

Also by (2.8),

$$\begin{aligned}p_\alpha(T_n x - T_n y) &= k_n p_\alpha(Tx - Ty)\\ &\le k_n \max\Big\{p_\alpha(fx - gy), d_{p_\alpha}(fx, [Tx, q]), d_{p_\alpha}(gy, [Ty, q]),\\ &\qquad \frac{1}{2}[d_{p_\alpha}(fx, [Ty, q]) + d_{p_\alpha}(gy, [Tx, q])]\Big\}\\ &\le k_n \max\Big\{p_\alpha(fx - gy), p_\alpha(fx - T_n x), p_\alpha(gy - T_n y),\\ &\qquad \frac{1}{2}[p_\alpha(fx - T_n y) + p_\alpha(gy - T_n x)]\Big\},\end{aligned} \tag{2.11}$$

for each $x, y \in M$, $p_\alpha \in A^*(\tau)$, and $0 < k_n < 1$. By Theorem 2.6, for each $n \ge 1$, there exists $x_n \in M$ such that $x_n = f x_n = g x_n = T_n x_n$.

(i) The compactness of $cl(T(M))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ and a $z \in cl(T(M))$ such that $Tx_m \to z$ as $m \to \infty$. Since $k_m \to 1$, $x_m = T_m x_m = (1-k_m)q + k_m T x_m$ also converges to z. Since T, f, and g are continuous, we have $z \in F(T) \cap F(f) \cap F(g)$. Thus $F(T) \cap F(f) \cap F(g) \neq \emptyset$.

(ii) Proof follows from (i).

(iii) Since M is weakly compact, there is a subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to some $y \in M$. But, f and g being affine and continuous are weakly continuous, and the weak topology is Hausdorff, so we have $fy = y = gy$. The set M is bounded, so $(f-T)(x_m) = (1-(k_m)^{-1})(x_m - q) \to 0$ as $m \to \infty$. Now the demiclosedness of $f-T$ at 0 guarantees that $(f-T)y = 0$ and hence $F(T) \cap F(f) \cap F(g) \neq \emptyset$. □

Theorem 2.7 extends and improves [14, Theorem 2.2], [7, Theorems 2.2-2.3, and Corollaries 2.4–2.7], [13, Theorem 6], and the main results of Tarafdar [3] and Taylor [28] (see also [6, Remarks 2.4]).

Theorem 2.8. *Let M be a nonempty τ-bounded, τ-sequentially complete, q-starshaped subset of a Hausdorff locally convex space (E,τ) and let T, f, and g be selfmaps of M. Suppose that f and g are affine and nonexpansive with $q \in F(f) \cap F(g)$, and $T(M) \subset f(M) \cap g(M)$. If the pairs $\{T, f\}$ and $\{T, g\}$ are subcompatible and T is (f,g)-nonexpansive, then $F(T) \cap F(f) \cap F(g) \neq \emptyset$, provided that one of the following conditions holds*

(i) *$cl(T(M))$ is τ-sequentially compact;*

(ii) *M is τ-sequentially compact;*

(iii) *M is weakly compact in (E,τ), $f-T$ is demiclosed at* 0.

(iv) *M is weakly compact in an Opial space (E,τ).*

Proof. (i)–(iii) follow from Theorem 2.7.

(iv) As in (iii) we have $fy = y = gy$ and $\|fx_m - Tx_m\| \to 0$ as $m \to \infty$. If $fy \neq Ty$, then by the Opial's condition of E and (f,g)-nonexpansiveness of T we get,

$$
\begin{aligned}
\liminf_{n\to\infty} p_\alpha(fx_m - gy) &= \liminf_{n\to\infty} p_\alpha(fx_m - fy) < \liminf_{n\to\infty} p_\alpha(fx_m - Ty) \\
&\leq \liminf_{n\to\infty} p_\alpha(fx_m - Tx_m) + \liminf_{n\to\infty} p_\alpha(Tx_m - Ty) \\
&= \liminf_{n\to\infty} p_\alpha(Tx_m - Ty) \leq \liminf_{n\to\infty} p_\alpha(fx_m - gy),
\end{aligned}
\tag{2.12}
$$

which is a contradiction. Thus $fy = Ty$ and hence $F(T) \cap F(f) \cap F(g) \neq \emptyset$. □

As 1-subcommuting maps are subcompatible, so by Theorem 2.8, we obtain the following recent result of Hussain and Khan [6] without the surjectivity of f. Note that a continuous and affine map is weakly continuous, so the weak continuity of f is not required as well.

Corollary 2.9 ([6, Theorem 2.2]). *Let M be a nonempty τ-bounded, τ-sequentially complete, q-starshaped subset of a Hausdorff locally convex space (E,τ) and let T, f be selfmaps of M. Suppose that f is affine and nonexpansive with $q \in F(f)$, and $T(M) \subset f(M)$. If the pair $\{T, f\}$ is* 1*-subcommuting* *and T is f-nonexpansive, then $F(T) \cap F(f) \neq \emptyset$, provided that one of the following conditions holds:*

(i) *$cl(T(M)$ is τ-sequentially compact;*

(ii) *M is τ-sequentially compact;*

(iii) *M is weakly compact in (E,τ), $f - T$ is demiclosed at 0.*

(iv) *M is weakly compact in an Opial space (E,τ).*

The following theorem improves and extends the corresponding approximation results in [6–8, 11–14, 25, 27].

Theorem 2.10. *Let M be a nonempty subset of a Hausdorff locally convex space (E,τ) and let $f, g, T : E \to E$ be mappings such that $u \in F(T) \cap F(f) \cap F(g)$ for some $u \in E$ and $T(\partial M \cap M) \subset M$. Suppose that f and g are affine and nonexpansive on $P_M(u)$ with $q \in F(f) \cap F(g)$, $P_M(u)$ is τ-bounded, τ-sequentially complete, q-starshaped and $f(P_M(u)) = P_M(u) = g(P_M(u))$. If the pairs (T, f) and (T, g) are subcompatible and, for all $x \in P_M(u) \cup \{u\}$ and $p_\alpha \in A^*(\tau)$,*

$$p_\alpha(Tx - Ty) \leq \begin{cases} p_\alpha(fx - gu), & \text{if } y = u, \\ \max\Big\{p_\alpha(fx - gy), d_{p_\alpha}(fx, [q, Tx]), d_{p_\alpha}(gy, [q, Ty]), \\ \qquad \dfrac{1}{2}\left[d_{p_\alpha}(fx, [q, Ty]) + d_{p_\alpha}(gy, [q, Tx])\right]\Big\}, & \text{if } y \in P_M(u), \end{cases} \tag{2.13}$$

then $P_M(u) \cap F(f) \cap F(g) \cap F(T) \neq \emptyset$, provided that one of the following conditions holds

(i) *$cl(T(P_M(u)))$ is τ-sequentially compact, and T is continuous;*

(ii) *$P_M(u)$ is τ-sequentially compact, and T is continuous;*

(iii) *$P_M(u)$ is weakly compact, and $(f - T)$ is demiclosed at 0.*

Proof. Let $x \in P_M(u)$. Then for each p_α, $p_\alpha(x - u) = d_{p_\alpha}(u, M)$. Note that for any $k \in (0,1)$, $p_\alpha(ku + (1-k)x - u) = (1-k)p_\alpha(x - u) < d_{p_\alpha}(u, M)$.

It follows that the line segment $\{ku + (1-k)x : 0 < k < 1\}$ and the set M are disjoint. Thus x is not in the interior of M and so $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subset M$, Tx must be in M. Also since $fx \in P_M(u)$, $u \in F(T) \cap F(f) \cap F(g)$, and T, f, g satisfy (2.13), we have for each p_α,

$$p_\alpha(Tx - u) = p_\alpha(Tx - Tu) \leq p_\alpha(fx - gu) = p_\alpha(fx - u) = d_{p_\alpha}(u, M). \tag{2.14}$$

Thus $Tx \in P_M(u)$. Consequently, $T(P_M(u)) \subset P_M(u) = f(P_M(u)) = g(P_M(u))$. Now Theorem 2.7 guarantees that $P_M(u) \cap F(f) \cap F(g) \cap F(T) \neq \emptyset$. □

Remark 2.11. One can now easily prove on the lines of the proof of the above theorem that the approximation results are similar to those of Theorems 2.11-2.12 due to Hussain and Jungck [7] in the settingof a Hausdorff locally convex space.

We define $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$ and denote by $\Im_0$ the class of closed convex subsets of E containing 0. For $M \in \Im_0$, we define $M_u = \{x \in M : p_\alpha(x) \leq 2p_\alpha(u)$ for each $p_\alpha \in A^*(\tau)\}$. It is clear that $P_M(u) \subset M_u \in \Im_0$.

The following result extends [14, Theorem 4.1] and [7, Theorem 2.14].

Theorem 2.12. *Let f, g, T be selfmaps of a Hausdorff locally convex space (E, τ) with $u \in F(T) \cap F(f) \cap F(g)$ and $M \in \Im_0$ such that $T(M_u) \subset f(M) \subset M = g(M)$. Suppose that $p_\alpha(fx - u) = p_\alpha(x - u)$ and $p_\alpha(gx - u) = p_\alpha(x - u)$ for all $x \in M_u$ and for each p_α where $cl f(M)$ is compact. Then*

(i) *$P_M(u)$ is nonempty, closed, and convex,*

(ii) *$T(P_M(u)) \subset f(P_M(u)) \subset P_M(u) = g(P_M(u))$,*

(iii) *$P_M(u) \cap F(f) \cap F(g) \cap F(T) \neq \emptyset$ provided f and g are subcompatible, affine, and nonexpansive on M, and, for some $q \in P_M(u)$ and for all $x, y \in P_M(u)$,*

$$p_\alpha(fx - fy) \leq \max\Big\{p_\alpha(gx - gy), d_{p_\alpha}(gx, [q, fx]), d_{p_\alpha}(gy, [q, fy]), \frac{1}{2}[d_{p_\alpha}(gx, [q, fy]) + d_{p_\alpha}(gy, [q, fx])]\Big\}, \tag{2.15}$$

T is continuous, the pairs $\{T, f\}$ and $\{T, g\}$ are subcompatible on $P_M(u)$ and satisfy for all $q \in F(f) \cap F(g)$,

$$p_\alpha(Tx - Ty) \leq \max\Big\{p_\alpha(fx - gy), d_{p_\alpha}(fx, [q, Tx]), d_{p_\alpha}(gy, [q, Ty]), \frac{1}{2}[d_{p_\alpha}(fx, [q, Ty]) + d_{p_\alpha}(gy, [q, Tx])]\Big\} \tag{2.16}$$

for all $x, y \in P_M(u)$ and for each $p_\alpha \in A^*(\tau)$.

Proof. (i) We follow the arguments used in [7] and [8]. Let $r = d_{p_\alpha}(u, M)$ for each p_α.

Then there is a minimizing sequence $\{y_n\}$ in M such that $\lim_n p_\alpha(u - y_n) = r$. As $cl(f(M))$ is compact so $\{fy_n\}$ has a convergent subsequence $\{fy_m\}$ with $\lim_m fy_m = x_0$ (say) in M. Now by using

$$p_\alpha(fx - u) \leq p_\alpha(x - u) \tag{2.17}$$

we get for each p_α,

$$r \leq p_\alpha(x_0 - u) = \lim_m p_\alpha(fy_m - u) \leq \lim_m p_\alpha(y_m - u) = \lim_n p_\alpha(y_n - u) = r. \tag{2.18}$$

Hence $x_0 \in P_M(u)$. Thus $P_M(u)$ is nonempty closed and convex.

(i) Follows from [7, Theorem 2.14].

(ii) By Theorem 2.7(i), $P_M(u) \cap F(f) \cap F(g) \neq \emptyset$, so it follows that there exists $q \in P_M(u)$ such that $q \in F(f) \cap F(g)$. Hence (iii) follows from Theorem 2.7(i). □

3. Banach Operator Pair in Locally Convex Spaces

Utilizing similar arguments as above, the following result can be proved which extends recent common fixed point results due to Hussain and Rhoades [8, Theorem 2.1] and Jungck and Hussain [9, Theorem 2.1] to the setup of a Hausdorff locally convex space which is not necessarily metrizable.

Theorem 3.1. *Let M be a τ-bounded subset of a Hausdorff locally convex space (E,τ), and let I and let T be weakly compatible self-maps of M. Assume that $\tau - cl(T(M)) \subset I(M)$, $\tau - cl(T(M))$ is τ-sequentially complete, and T and I satisfy, for all $x, y \in M$, $p_\alpha \in A^*(\tau)$ and for some $0 \le k < 1$,*

$$p_\alpha(Tx - Ty) \le k \max\{p_\alpha(Ix - Iy), p_\alpha(Ix - Tx), p_\alpha(Iy - Ty), p_\alpha(Ix - Ty), p_\alpha(Iy - Tx)\}. \tag{3.1}$$

Then $F(I) \cap F(T)$ is a singleton.

As an application of Theorem 3.1, the analogue of all the results due to Hussain and Berinde [25], and Hussain and Rhoades [8] can be established for C_q-commuting maps I and T defined on a τ-bounded subset M of a Hausdorff locally convex space. We leave details to the reader.

Recently, Chen and Li [23] introduced the class of Banach operator pairs, as a new class of noncommuting maps and it has been further studied by Hussain [24], Ciric et al. [32], Khan and Akbar [19, 20], and Pathak and Hussain [21]. The pair (T, f) is called a Banach operator pair, if the set $F(f)$ is T-invariant, namely, $T(F(f)) \subseteq F(f)$. Obviously, commuting pair (T, f) is a Banach operator pair but converse is not true, in general; see [21, 23]. If (T, f) is a Banach operator pair, then (f, T) need not be a Banach operator pair (cf. [23, Example 1]).

Chen and Li [23] proved the following.

Theorem 3.2 ([23, Theorems 3.2-3.3]). *Let M be a q-starshaped subset of a normed space X and let T, I be self-mappings of M. Suppose that $F(I)$ is q-starshaped and I is continuous on M. If $cl(T(M))$ is compact (resp., I is weakly continuous, $cl(T(M))$ is complete, M is weakly compact, and either $I - T$ is demiclosed at 0 or X satisfies Opial's condition), (T, I) is a Banach operator pair, and T is I-nonexpansive on M, then $M \cap F(T) \cap F(I) \ne \emptyset$.*

In this section, we extend and improve the above-mentioned common fixed point results of Chen and Li [23] in the setup of a Hausdorff locally convex space.

Lemma 3.3. *Let M be a nonempty τ-bounded subset of Hausdorff locally convex space (E,τ), and let T, f, and g be self-maps of M. If $F(f) \cap F(g)$ is nonempty, $\tau - cl(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$, $\tau - cl(T(M))$ is τ-sequentially complete, and T, f, and g satisfy for all $x, y \in M$ and for some $0 \le k < 1$,*

$$p_\alpha(Tx - Ty) \le k \max\{p_\alpha(fx - gy), p_\alpha(fx - Tx), p_\alpha(gy - Ty), p_\alpha(fx - Ty), p_\alpha(gy - Tx)\} \tag{3.2}$$

then $M \cap F(T) \cap F(f) \cap F(g)$ is singleton.

Proof. Note that $\tau - cl(T(F(f) \cap F(g)))$ being a subset of $\tau - cl(T(M))$ is τ-sequentially complete. Further, for all $x,\ y \in F(f) \cap F(g)$, we have

$$\begin{aligned} p_\alpha(Tx - Ty) &\le k \max\{p_\alpha(fx - gy), p_\alpha(fx - Tx), p_\alpha(gy - Ty), p_\alpha(fx - Ty), p_\alpha(gy - Tx)\} \\ &= k \max\{p_\alpha(x - y), p_\alpha(x - Tx), p_\alpha(y - Ty), p_\alpha(x - Ty), p_\alpha(y - Tx)\}. \end{aligned} \tag{3.3}$$

Hence T is a generalized contraction on $F(f) \cap F(g)$ and $\tau - cl(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$. By Theorem 3.1 (with I = identity map), T has a unique fixed point z in $F(f) \cap F(g)$ and consequently, $F(T) \cap F(f) \cap F(g)$ is singleton. □

The following result generalizes [19, Theorem 2.3], [24, Theorem 2.11], and [21, Theorem 2.2] and improves [14, Theorem 2.2] and [13, Theorem 6].

Theorem 3.4. *Let M be a nonempty τ-bounded subset of Hausdorff locally convex (resp., complete) space (E, τ) and let T, f, and g be self-maps of M. Suppose that $F(f) \cap F(g)$ is q-starshaped, $\tau - cl(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$(resp., $\tau - wcl(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$), $\tau - cl(T(M))$ is compact (resp., $\tau - wcl(T(M))$ is weakly compact), T is continuous on M (resp., $I - T$ is demiclosed at 0, where I stands for identity map) and*

$$\begin{aligned} p_\alpha(Tx - Ty) \le \max\{&p_\alpha(fx - gy), d_{p_\alpha}(fx, [q, Tx]), d_{p_\alpha}(gy, [q, Ty]), \\ &d_{p_\alpha}(gy, [q, Tx]), d_{p_\alpha}(fx, [q, Ty])\}. \end{aligned} \tag{3.4}$$

For all $x, y \in M$, then $M \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Proof. Define $T_n : F(f) \cap F(g) \to F(f) \cap F(g)$ by $T_n x = (1 - k_n)q + k_n Tx$ for all $x \in F(f) \cap F(g)$ and a fixed sequence of real numbers k_n $(0 < k_n < 1)$ converging to 1. Since $F(f) \cap F(g)$ is q-starshaped and $\tau - cl(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$ (resp., $\tau - wcl(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$), so $\tau - cl(T_n(F(f) \cap F(g))) \subseteq F(f) \cap F(g))$ (resp., $\tau - wcl(T_n(F(f) \cap F(g))) \subseteq F(f) \cap F(g))$ for each $n \ge 1$. Also by (3.4),

$$\begin{aligned} p_\alpha(T_n x - T_n y) &= k_n p_\alpha(Tx - Ty) \\ &\le k_n \max\{p_\alpha(fx - gy), d_{p_\alpha}(fx, [q, Tx]), \\ &\qquad d_{p_\alpha}(gy, [q, Ty]), d_{p_\alpha}(fx, [q, Ty]), d_{p_\alpha}(gy, [q, Tx])\} \\ &\le k_n \max\{p_\alpha(fx - gy), p_\alpha(fx - T_n x), p_\alpha(gy - T_n y), \\ &\qquad p_\alpha(gy - T_n x), p_\alpha(fx - T_n y)\}, \end{aligned} \tag{3.5}$$

for each $x, y \in F(f) \cap F(g)$ and some $0 < k_n < 1$.

If $\tau - cl(T(M))$ is compact, for each $n \in \mathbb{N}$, $\tau - cl(T_n(M))$ is τ-compact and hence τ-sequentially complete. By Lemma 3.3, for each $n \ge 1$, there exists $x_n \in F(f) \cap F(g)$ such that $x_n = fx_n = gx_n = T_n x_n$. The compactness of $\tau - cl(T(M))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to z \in cl(T(M))$ as $m \to \infty$. Since $\{Tx_m\}$ is a sequence in $T(F(f)\cap F(g))$ and $\tau-cl(T(F(f)\cap F(g)))\subseteq F(f)\cap F(g)$, therefore $z\in F(f)\cap F(g)$. Further, $x_m = T_m x_m = (1-k_m)q+k_m T x_m \to z$. By the continuity of T, we obtain $Tz = z$. Thus, $M\cap F(T)\cap F(f)\cap F(g)\neq\emptyset$ proves the first case.

The weak compactness of $\tau - wcl(T(M))$ implies that $\tau - wcl(T_n(M))$ is weakly compact and hence τ-sequentially complete due to completeness of X. From Lemma 3.3, for each $n\geq 1$, there exists $x_n\in F(f)\cap F(g)$ such that $x_n = fx_n = gx_n = T_n x_n$. Moreover, we have $p_\alpha(x_n - Tx_n)\to 0$ as $n\to\infty$. The weak compactness of $\tau - wcl(T(M))$ implies that there is a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ converging weakly to $y\in\tau - wcl(T(M))$ as $m\to\infty$. Since $\{Tx_m\}$ is a sequence in $T(F(f)\cap F(g))$, therefore $y\in\tau - wcl(T(F(f)\cap F(g)))\subseteq F(f)\cap F(g)$. Also we have, $x_m - Tx_m \to 0$ as $m\to\infty$. If $I-T$ is demiclosed at 0, then $y = Ty$. Thus $M\cap F(T)\cap F(f)\cap F(g)\neq\emptyset$. □

Corollary 3.5. *Let M be a nonempty τ-bounded subset of Hausdorff locally convex (resp., complete) space (E,τ) and let T, f, and g be self-maps of M. Suppose that $F(f)\cap F(g)$ is q-starshaped, and τ-closed (resp., τ-weakly closed), $\tau-cl(T(M))$ is compact (resp., $\tau-wcl(T(M))$ is weakly compact), T is continuous on M (resp., $I-T$ is demiclosed at 0), (T,f) and (T,g) are Banach operator pairs and satisfy (3.4) for all $x,y\in M$, then $M\cap F(T)\cap F(f)\cap F(g)\neq\emptyset$.*

Let $C = P_M(u)\cap C_M^{f,g}(u)$, where $C_M^{f,g}(u) = C_M^f(u)\cap C_M^g(u)$ and $C_M^f(u) = \{x\in M : fx\in P_M(u)\}$. It is important to note here that $P_M(u)$ is always bounded.

Corollary 3.6. *Let E be a Hausdorff locally convex (resp., complete) space and $T,\ f$, and g be self-maps of E. If $u\in E$, $D\subseteq C$, $D_0 := D\cap F(f)\cap F(g)$ is q-starshaped, $\tau - cl(T(D_0))\subseteq D_0$ (resp., $\tau - wcl(T(D_0))\subseteq D_0$], $\tau - cl(T(D))$ is compact (resp., $\tau - wcl(T(D))$ is weakly compact), T is continuous on D (resp., $I-T$ is demiclosed at 0), and (3.4) holds for all $x,y\in D$, then $P_M(u)\cap F(T)\cap F(f)\cap F(g)\neq\emptyset$.*

Corollary 3.7. *Let E be a Hausdorff locally convex (resp., complete) space and T, f, and g be self-maps of E. If $u\in E$, $D\subseteq P_M(u)$, $D_0 := D\cap F(f)\cap F(g)$ is q-starshaped, $\tau - cl(T(D_0))\subseteq D_0$ (resp., $\tau - wcl(T(D_0))\subseteq D_0$), $\tau - cl(T(D))$ is compact (resp., $\tau - wcl(T(D))$ is weakly compact), T is continuous on D (resp., $I-T$ is demiclosed at 0), and (3.4) holds for all $x,y\in D$, then $P_M(u)\cap F(T)\cap F(f)\cap F(g)\neq\emptyset$.*

Remark 3.8. Khan and Akbar [19, Corollaries 2.4–2.8] and Chen and Li [23, Theorems 4.1 and 4.2] are particular cases of Corollaries 3.5 and 3.6.

The following result extends [14, Theorem 4.1], [7, Theorem 2.14], [19, Theorem 2.9], and [21, Theorems 2.7–2.11].

Theorem 3.9. *Let $f,\ g,\ T$ be self-maps of a Hausdorff locally convex space E. If $u\in E$ and $M\in\Im_0$ such that $T(M_u)\subseteq M$, $\tau - cl(T(M_u))$ is compact and $\|Tx-u\|\leq\|x-u\|$ for all $x\in M_u$, then $P_M(u)$ is nonempty, closed, and convex with $T(P_M(u))\subseteq P_M(u)$. If, in addition, $D\subseteq P_M(u)$, $D_0 := D\cap F(f)\cap F(g)$ is q-starshaped, $\tau - cl(T(D_0))\subseteq D_0$, T is continuous on D, and (3.4) holds for all $x,y\in D$, then $P_M(u)\cap F(T)\cap F(f)\cap F(g)\neq\emptyset$.*

Proof. We utilize Corollary 3.5 instead of Theorem 2.7 in the proof of Theorem 2.12. □

Remark 3.10. (1) The class of Banach operator pairs is different from that of weakly compatible maps; see for example [21, 23, 32].

(2) In Example 2.2, the pair (T,f) is a Banach operator but T and f are not C_q-commuting maps and hence not a subcompatible pair.

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