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Common coupled coincidence and coupled fixed point results in two generalized metric spaces

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Abstract

In this article, we prove the existence of common coupled coincidence and coupled fixed point of generalized contractive type mappings in the context of two generalized metric spaces. These results generalize several comparable results from the current literature. We also provide illustrative examples in support of our new results.

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1 Introduction and preliminaries

The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity [1-5]. Mustafa and Sims [4] generalized the concept of a metric space and call it a generalized metric space. Based on the notion of generalized metric spaces, Mustafa et al. [5-9] obtained some fixed point theorems for mappings satisfying different contractive conditions. Abbas and Rhoades [10] initiated the study of common fixed point theory in generalized metric spaces (see also [11]). Saadati et al. [12] proved some fixed point results for contractive mappings in partially ordered G-metric spaces. Abbas et al. [13] obtained some periodic point results in generalized metric spaces. Shatanawi [14] obtained some fixed point results for contractive mappings satisfying Φ-maps in G-metric spaces (see also [15]).

Bhashkar and Lakshmikantham [16] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ (a nonempty set) and established some coupled fixed point theorems in partially ordered complete metric spaces. Later, Lakshmikantham and Ćirić [3] proved coupled coincidence and coupled common fixed point results for nonlinear mappings $F : X \times X \to X$ and $g : X \to X$ satisfying certain contractive conditions in partially ordered complete metric spaces. Recently, Abbas et al. [17] obtained some coupled common fixed point results in two generalized metric spaces. Choudhury and Maity [18] also proved the existence of coupled fixed points in generalized metric spaces. Recently, Aydi et al. [19] generalized the results of Choudhury and Maity [18]. For other works on G-metric spaces, we refer the reader to [20,21].

The aim of this article is to prove some common coupled coincidence and coupled fixed points results for mappings defined on a set equipped with two generalized metrics. It is worth mentioning that our results do not rely on continuity of mappings involved therein. Our results extend and unify various comparable results in [17,22,23].

Consistent with Mustafa and Sims [4], the following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \to R^+$ satisfies:

(a) $G(x, y, z) = 0$ if $x = y = z$;
(b) $0 < G(x, y, z)$ for all $x, y \in X$, with $x \neq y$;
(c) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$;
(d) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables); and
(e) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then, G is called a G-metric on X and (X, G) is called a G-metric space.

Definition 1.2. A sequence $\{x_n\}$ in a G-metric space X is:

(i) a *G-Cauchy* sequence if, for any $\varepsilon > 0$, there is an $n_0 \in N$ (the set of natural numbers) such that for all $n, m, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$,
(ii) a *G-convergent* sequence if, for any $\varepsilon > 0$, there is an $x \in X$ and an $n_0 \in N$, such that for all $n, m \geq n_0$, $G(x, x_n, x_m) < \varepsilon$.

A G-metric space on X is said to be G-complete if every G-Cauchy sequence in X is G-convergent in X. It is known that $\{x_n\}$ G-converges to $x \in X$ if and only if $G(x_m, x_n, x) \to 0$ as $n, m \to \infty$ [4].

Proposition 1.3. [4] Let X be a G-metric space. Then, the following are equivalent:

1. $\{x_n\}$ is G-convergent to x.
2. $G(x_n, x_n, x) \to 0$ as $n \to \infty$.
3. $G(x_n, x, x) \to 0$ as $n \to \infty$.
4. $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.

Definition 1.4. [16] An element $(x, y) \in X \times X$ is called:

(C_1) a coupled fixed point of mapping $T : X \times X \to X$ if $x = T(x, y)$ and $y = T(y, x)$;

(C_2) a coupled coincidence point of mappings $T : X \times X \to X$ and $f : X \to X$ if $f(x) = T(x,y)$ and $f(y) = T(y,x)$, and in this case (fx,fy) is called coupled point of coincidence;

(C_3) a common coupled fixed point of mappings $T : X \times X \to X$ and $f : X \to X$ if $x = f(x) = T(x, y)$ and $y = f(y) = T(y, x)$.

Definition 1.5. An element $(x, y) \in X \times X$ is called:

(CC_1) a common coupled coincidence point of the mappings $T, S : X \times X \to X$ and $f : X \to X$ if $T(x, y) = S(x, y) = fx$ and $T(y, x) = S(y, x) = fy$, and in this case (fx, fy) is called a common coupled point of coincidence;

(CC_2) a common coupled fixed point of mappings $T, S : X \times X \to X$ and $f :$

$$X \to X \text{ if } T(x, y) = S(x, y) = f(x) = x \text{ and } T(y, x) = S(y, x) = f(y) = y.$$

Definition 1.6. [22] Mappings $T : X \times X \to X$ and $f : X \to X$ are called

(W_1) w-compatible if $f(T(x, y)) = T(fx,fy)$ whenever $f(x) = T(x,y)$ and $f(y) = T(y, x)$;

(W_2) w^*-compatible if $f(T(x,x)) = T(fx, fx)$ whenever $f(x) = T(x,x)$.

2 Common coupled fixed points

We extend some recent results of Abbas et al. [17,22] and Sabetghadam [23] to the setting of two generalized metric spaces.

Theorem 2.1. Let G_1 and G_2 be two G-metrics on X such that $G_2(x,y,z) \leq G_1(x,y,z)$ for all $x, y, z \in X$, $S,T : X \times X \to X$, and $f : X \to X$ be mappings satisfying

$$\begin{aligned} &G_1\left(S(x,y), T(u,v), T(s,t)\right) \\ &\leq a_1 G_2\left(fx, fu, fs\right) + a_2 G_2\left(S(x,y), fx, fx\right) + a_3 G_2\left(T(x,v), fu, fs\right) \\ &\quad + a_4 G_2\left(fy, fv, ft\right) + a_5 G_2\left(S(x,y), fu, fs\right) + a_6 G_2\left(T(u,v), T(s,t), fx\right) \end{aligned} \tag{2.1}$$

for all $x, y, u, v, s, t \in X$, where $a_i \geq 0$, for $i = 1, 2, \ldots, 6$ and $a_1 + a_4 + a_5 + 2(a_2 + a_3 + a_6) < 1$. If $S(X \times X) \subseteq f(X)$, $T(X \times X) \subseteq f(X)$, $f(X)$ is G_1-complete subset of X, then S, T, and f have a unique common coupled coincidence point. Moreover, if S or T is w^*-compatible with f, then f, S, and T have a unique common coupled fixed point.

Proof. As S, T, and f satisfy condition (2.1), so for all $x, y, u, v \in X$, we have

$$\begin{aligned} &G_1\left(S(x,y), T(u,v), T(s,v)\right) \\ &\leq a_1 G_2\left(fx, fu, fs\right) + a_2 G_2\left(S(x,y), fx, fx\right) + a_3 G_2\left(T(x,v), fu, fu\right) \\ &\quad + a_4 G_2\left(fy, fv, fv\right) + a_5 G_2\left(S(x,y), fu, fu\right) + a_6 G_2\left(T(u,v), T(u,v), fx\right). \end{aligned} \tag{2.2}$$

Let $x_0, y_0 \in X$. We choose $x_1, y_1 \in X$ such that $fx_1 = S(x_0, y_0)$ and $fy_1 = S(y_0, x_0)$, this can be done in view of $S(X \times X) \subseteq f(X)$. Similarly, we can choose $x_2, y_2 \in X$ such that $fx_2 = T(x_1, y_1)$ and $fy_2 = T(y_1, x_1)$ since $T(X \times X) \subseteq f(X)$. Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$fx_{2n+1} = S\left(x_{2n}, y_{2n}\right), \quad fx_{2n+2} = T\left(x_{2n+1}, y_{2n+1}\right) \tag{2.3}$$

and

$$fy_{2n+1} = S\left(y_{2n}, x_{2n}\right), \quad fy_{2n+2} = T\left(y_{2n+1}, x_{2n+1}\right). \tag{2.4}$$

From (2.2), we have

$$\begin{aligned} &G_1\left(fx_{2n+1}, fx_{2n+2}, fx_{2n+2}\right) \\ = \ &G_1\left(S\left(x_{2n}, y_{2n}\right), T\left(x_{2n+1}, y_{2n+1}\right), T\left(x_{2n+1}, y_{2n+1}\right)\right) \\ \leq \ &a_1 G_2\left(fx_{2n}, fx_{2n+1}, fx_{2n+1}\right) + a_2 G_2\left(S\left(x_{2n}, y_{2n}\right), fx_{2n}, fx_{2n}\right) \\ &+ a_3 G_2\left(T\left(x_{2n+1}, y_{2n+1}\right), fx_{2n+1}, fx_{2n+1}\right) + a_4 G_2\left(fy_{2n}, fy_{2n+1}, fy_{2n+1}\right) \\ &+ a_5 G_2\left(S\left(x_{2n}, y_{2n}\right), fx_{2n+1}, fx_{2n+1}\right) + a_6 G_2\left(T\left(x_{2n+1}, y_{2n+1}\right), T\left(x_{2n+1}, y_{2n+1}\right), fx_{2n}\right) \\ = \ &a_1 G_2\left(fx_{2n}, fx_{2n+1}, fx_{2n+1}\right) + a_2 G_2\left(fx_{2n+1}, fx_{2n}, fx_{2n}\right) \\ &+ a_3 G_2\left(fx_{2n+2}, fx_{2n+1}, fx_{2n+1}\right) + a_4 G_2\left(fy_{2n}, fy_{2n+1}, fy_{2n+1}\right) \\ &+ a_5 G_2\left(fx_{2n+1}, fx_{2n+1}, fx_{2n+1}\right) + a_6 G_2\left(fx_{2n+2}, fx_{2n+2}, fx_{2n}\right) \\ \leq \ &\left(a_1 + 2a_2 + a_6\right) G_2\left(fx_{2n}, fx_{2n+1}, fx_{2n+1}\right) + \left(2a_3 + a_6\right) G_2\left(fx_{2n+1}, fx_{2n+2}, fx_{2n+2}\right) \\ &+ a_4 G_2\left(fy_{2n}, fy_{2n+1}, fy_{2n+1}\right), \end{aligned}$$

which implies that

$$\begin{aligned} &G_1(fx_{2n+1}, fx_{2n+2}, fx_{2n+2}) \\ &\leq \frac{1}{1 - 2a_3 - a_6}\left[(a_1 + 2a_2 + a_6)G_2(fx_{2n+1}, fx_{2n+1}, fx_{2n+1}) + a_4 G_2(fy_{2n}, fy_{2n+1}, fy_{2n+1})\right]. \end{aligned} \tag{2.5}$$

Similarly, we obtain

$$G_1(fy_{2n+1}, fy_{2n+2}, fy_{2n+2}) \\ \leq \frac{1}{1-2a_3-a_6}[(a_1+2a_2+a_6)G_2(fy_{2n}, fy_{2n+1}, fy_{2n+1}) + a_4 G_2(fx_{2n}, fx_{2n+1}, fx_{2n+1})]. \quad (2.6)$$

Now, from (2.5) and (2.6), we obtain

$$G_1(fx_{2n+1}, fx_{2n+2}, fx_{2n+2}) + G_1(fy_{2n+1}, fy_{2n+2}, fy_{2n+2}) \\ \leq \lambda[G_2(fx_{2n}, fx_{2n+1}, fx_{2n+1}) + G_2(fy_{2n}, fy_{2n+1}, fy_{2n+1})],$$

where $\lambda = \dfrac{a_1+a_4+2a_2+a_6}{1-2a_3-a_6}$. Obviously, $0 \leq \lambda < 1$.

In a similar way, we obtain

$$G_1(fx_{2n}, fx_{2n+1}, fx_{2n+1}) + G_1(fy_{2n}, fy_{2n+1}, fy_{2n+1}) \\ \leq \lambda[G_2(fx_{2n-1}, fx_{2n}, fx_{2n}) + G_2(fy_{2n-1}, fy_{2n}, fy_{2n})].$$

Thus, for all $n \geq 0$,

$$G_1(fx_n, fx_{n+1}, fx_{n+1}) + G_1(fy_n, fy_{n+1}, fy_{n+1}) \\ \leq \lambda[G_2(fx_{n-1}, fx_n, fx_n) + G_2(fy_{n-1}, fy_n, fy_n)].$$

Repetition of above process n times gives

$$\begin{aligned} & G_1(fx_n, fx_{n+1}, fx_{n+1}) + G_1(fy_n, fy_{n+1}, fy_{n+1}) \\ \leq\ & \lambda[G_2(fx_{n-1}, fx_n, fx_n) + G_2(fy_{n-1}, fy_n)] \\ \leq\ & \lambda^2[G_2(fx_{n-2}, fx_{n-1}, fx_{n-1}) + G_2(fy_{n-2}, fy_{n-1}, fy_{n-1})] \\ \leq\ & \cdots \leq \lambda^n[G_2(fx_0, fx_1, fx_1) + G_2(fy_0, fy_1, fy_1)]. \end{aligned}$$

For any $m > n \geq 1$, repeated use of property (e) of G-metric gives

$$\begin{aligned} & G_1(fx_n, fx_m, fx_m) + G_1(fy_n, fy_m, fy_m) \\ \leq\ & G_2(fx_n, fx_{n+1}, fx_{n+1}) + G_2(fx_{n+1}, x_{x+2}, x_{n+2}) + G_2(fy_n, fy_{n+1}, fy_{n+1}) \\ & + G_2(fx_{y+1}, x_{y+2}, x_{y+2}) + \cdots + G_2(fx_{m-1}, fx_m, fx_m) + G_2(fy_{m-1}, fy_m, fy_m) \\ \leq\ & (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1})[G_2(fx_0, fx_1, fx_1) + G_2(fy_0, fy_1, fy_1)] \\ \leq\ & \frac{\lambda^n}{1-\lambda}[G_2(fx_0, fx_1, fx_1) + G_2(fy_0, fy_1, fy_1)], \end{aligned}$$

and so $G_1(fx_n fx_m, fx_m) + G_1(fy_n, fy_m, fy_m) \to 0$ as $n, m \to \infty$. Hence, $\{fx_n\}$ and $\{fy_n\}$ are G_1-Cauchy sequences in $f(X)$. By G_1-completeness of $f(X)$, there exists $fx, fy \in f(X)$ such that $\{fx_n\}$ and $\{fy_n\}$ converge to fx and fy, respectively.

Now, we prove that $S(x,y) = fx$ and $T(y,x) = fy$. Using (2.2), we have

$$\begin{aligned} & G_1(fx, T(x,y), T(x,y)) \\ \leq\ & G_1(fx_{2n+1}, T(x,y), T(x,y)) + G_1(fx, fx_{2n+1}, fx_{2n+1}) \\ =\ & G_1(S(s_{2n}, y_{2n}), T(x,y), T(x,y)) + G_1(fx_{2n+1}, fx_{2n+1}, fx) \\ \leq\ & a_1 G_2(fx_{2n}, fx, fx) + a_2 G_2(S(x_{2n}, y_{2n}), fx_{2n}, fx_{2n}) + a_3 G_2(T(x,y), fx, fx) \\ & + a_4 G_2(fy_{2n}, fy, fy) + a_5 G_2(S(x_{2n}, y_{2n}), fx, fx) \\ & + a_6 G_2(T(x,y), T(x,y), fx_{2n}) + G_1(fx_{2n+1}, fx_{2n+1}, fx) \\ \leq\ & a_1 G_2(fx_{2n}, fx, fx) + a_2 G_1(fx_{2n+1}, fx_{2n}, fx_{2n}) + 2a_3 G_3(T(x,y), T(x,y), fx) \\ & + a_4 G_2(fy_{2n}, fy, fy) + a_5 G_2(fx_{2n+1}, fx, fx) \\ & + a_6 G_2(T(x,y), T(x,y), fx_{2n}) + G_1(fx_{2n+1}, fx_{2n+1}, fx), \end{aligned}$$

which further implies that

$$\begin{aligned} & G_1(fx, T(x,y), T(x,y)) \\ \le & \frac{1}{1-2a_3}[a_1 G_2(fx_{2n}, fx, fx) + a_2 G_2(fx_{2n}, fx_{2n}) + a_4 G_2(fy_{2n}, fy, fy) \\ & + a_5 G_2(fx_{2n+1}, fx, fx) + a_6 G_2(T(x,y), T(x,y), fx_{2n}) + G_1(fx_{2n+1}, fx_{2n+1}, fx)]. \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$G_1(fx, T(x,y), T(x,y)) \le \frac{a_6}{1-2a_3} G_1(T(x,y), T(x,y), fx).$$

As $\frac{a_6}{1-2a_3} < 1$, so we have $G_1(fx, T(x, y), T(x, y)) = 0$, and $T(x, y) = fx$.

Again from (2.2), we have

$$\begin{aligned} & G_1(S(x,y), fx, fx) \\ = & G_1(S(x,y), T(x,y), T(x,y)) \\ \le & a_1 G_2(fx, fx, fx) + a_2 G_2(S(x,y), fx, fx) + a_3 G_2(T(x,y), fx, fx) \\ & + a_4 G_2(fy, fy, fy) + a_5 G_2(S(x,y), fx, fx) \\ & + a_6 G_2(T(x,y), T(x,y), fx) \\ = & (a_2 + a_5) G_2(S(x,y), fx, fx) \\ \le & (a_2 + a_5) G_1(S(x,y), fx, fx). \end{aligned}$$

That is $G_1(S(x,y), fx, fx) = 0$, and $S(x,y) = fx$. Thus, $T(x,y) = S(x,y) = fx$. Similarly, it can be shown that $T(y, x) = S(y, x) = fy$. Thus, (fx, fy) is a coupled point of coincidence of mappings f, S, and T.

To show that $fx = fy$, we proceed as follows: Note that

$$\begin{aligned} & G_1(fx_{2n+1}, fy_{2n+2}, fy_{2n+2}) \\ = & G_1(S(x_{2n}, y_{2n}), T(y_{2n+1}, x_{2n+1}), T(y_{2n+1}, x_{2n+1})) \\ \le & a_1 G_2(fx_{2n}, fy_{2n+1}, fy_{2n+1}) + a_2 G_2(S(x_{2n}, y_{2n}), fx_{2n}, fx_{2n}) \\ & + a_3 G_2(T(y_{2n+1}, x_{2n+1}), fy_{2n+1}, fy_{2n+1}) + a_4 G_2(fy_{2n}, fx_{2n+1}, fx_{2n+1}) \\ & + a_5 G_2(S(x_{2n}, y_{2n}), fy_{2n+1}, fy_{2n+1}) + a_6 G_2(T(y_{2n+1}, x_{2n+1}), T(y_{2n+1}, x_{2n+1}), fx_{2n}) \\ = & a_1 G_2(fx_{2n}, fy_{2n+1}, fy_{2n+1}) + a_2 G_2(fx_{2n+1}, fx_{2n}, fx_{2n}) \\ & + a_3 G_2(fy_{2n+2}, fy_{2n+1}, fy_{2n+1}) + a_4 G_2(fy_{2n}, fx_{2n+1}, fx_{2n+1}) \\ & + a_5 G_2(fx_{2n+1}, fy_{2n+1}, fy_{2n+1}) + a_6 G_2(fy_{2n+2}, fy_{2n+2}, fx_{2n}). \end{aligned}$$

Taking limit as $n \to \infty$, we obtain

$$G_1(fx, fy, fy) \le (a_1 + a_5 + a_6) G_2(fx, fy, fy) + a_4 G_2(fx, fx, fy).$$

This implies that

$$G_1(fx, fy, fy) \le \frac{a_4}{1-(a_1 + a_5 + a_6)} G_1(fx, fx, fy). \tag{2.7}$$

In the similar way, we can show that

$$G_1(fy, fx, fx) \le \frac{a_4}{1-(a_1 + a_5 + a_6)} G_1(fy, fy, fx). \tag{2.8}$$

Since $\frac{a_4}{1-(a_1+a_5+a_6)} < 1$, from (2.7) and (2.8), we must have $G_1(fx, fy, fy) = 0$. So that $fx = fy$. Thus, (fx, fx) is a coupled point of coincidence of mappings f, S and T. Now, if there is another $x^* \in X$ such that (fx^*, fx^*) is a coupled point of coincidence of mappings f, S, and T, then

$$\begin{aligned}
& G_1(fx, fx^*, fx^*) \\
= \;& G_1(S(x,x), T(x^*,x^*), T(x^*,x^*)) \\
\leq \;& a_1G_2(fx, fx^*, fx^*) + a_2G_2(S(x,x), fx, fx) \\
& + a_3G_2(T(x^*,x^*), fx^*, fx^*) + a_4G_2(fx, fx^*, fx^*) \\
& + a_5G_2(S(x,x), fx^*, fx^*) + a_6G_2(T(x^*,x^*), T(x^*,x^*), fx) \\
= \;& a_1G_2(fx, fx^*, fx^*) + a_2G_2(fx, fx, fx) \\
& + a_3G_2(fx^*, fx^*, fx^*) + a_4G_2(fx, fx^*, fx^*) \\
& + a_5G_2(fx, fx^*, fx^*) + a_6G_2(fx^*, fx^*, fx) \\
\leq \;& (a_1 + a_4 + a_5 + a_6)G_2(fx, fx^*, fx^*)
\end{aligned}$$

implies that $G_1(fx, fx^*, fx^*) = 0$ and so $fx^* = fx$. Hence, (fx, fx) is a unique coupled point of coincidence of mappings f, S, and T.

Now, we show that f, S, and T have common coupled fixed point.

For this, let $f(x) = u$. Then, we have $u = fx = T(x, x)$. By w^*-compatibility of f and T, we have

$$f(u) = f(fx) = f(T(x,x)) = T(fx, fx) = T(u,u).$$

Then, (fu, fu) is a coupled point of coincidence of f, S, and T. By the uniqueness of coupled point of coincidence, we have $fu = fx$. Therefore, (u, u) is the common coupled fixed point of f, S, and T.

To prove the uniqueness, let $v \in X$ with $u \neq v$ such that (v, v) is the common coupled fixed point of f, S, and T. Then, using (2.2),

$$\begin{aligned}
& G_1(u, v, v) \\
= \;& G_1(s(u,u), T(v,v), T(v,v)) \\
\leq \;& a_1G_2(fu, fv, fv) + a_2G_2(S(u,u), fu, fu) + a_3G_2(T(v,v), fv, fv) \\
& + a_4G_2(fu, fv, fv) + a_5G_2(S(u,u), fv, fv) + a_6G_2(T(v,v), T(v,v), fu) \\
= \;& (a_1 + a_4 + a_5 + a_6)\, G_2(fu, fv, fv) = (a_1 + a_4 + a_5 + a_6)\, G_2(u, v, v) \\
\leq \;& (a_1 + a_4 + a_5 + a_6)\, G_1(u, v, v).
\end{aligned}$$

Since $a_1 + a_4 + a_5 + a_6 < 1$, so that $G_1(u, v, v) = 0$ and $u = u^*$. Thus, f, S, and T have a unique common coupled fixed point.

In Theorem 2.1, take $S = T$, to obtain Theorem 2.1 of Abbas et al. [22] as the following corollary.

Corollary 2.2. Let G_1 and G_2 be two G-metrics on X such that $G_2(x, y, z) \leq G_1(x, y, z)$, for all $x, y, z \in X$, $T : X \times X \to X$, and $f : X \to X$ be mappings satisfying

$$\begin{aligned}
& G_1(T(x,y), T(u,v), T(s,t)) \\
\leq \;& a_1G_2(fx, fu, fs) + a_2G_2(T(x,y), fx, fx) + a_3G_2(T(u,v), fu, fs) \\
& + a_4G_2(fy, fv, ft) + a_5G_2(T(x,y), fu, fs) + a_6G_2(T(u,v), T(s,t), fx)
\end{aligned} \tag{2.9}$$

for all $x, y, u, v, s, t \in X$, where $a_i \geq 0$, for $i = 1, 2, \ldots, 6$ and $a_1 + a_4 + a_5 + 2(a_2 + a_3 + a_6) < 1$. If $T(X \times X) \subseteq f(X)$, $f(X)$ is G_1-complete subset of X, then T and f have a unique common coupled coincidence point. Moreover, if T is w^*-compatible with f, then T and f have a unique common coupled fixed point.

In Theorem 2.1, take $s = u$ and $t = v$, to obtain the following corollary which extends and generalizes the corresponding results of [17,22,23].

Corollary 2.3 Let G_1 and G_2 be two G-metrics on X such that $G_2(x, y, z) \leq G_1(x, y, z)$, for all $x, y, z \in X$, $S, T : X \times X \to X$, and $f : X \to X$ be mappings satisfying

$$\begin{aligned} & G_1\left(S(x,y), T(u,v), T(u,v)\right) \\ \leq\ & a_1 G_2(fx, fu, fu) + a_2 G_2\left(S(x,y), fx, fx\right) + a_3 G_2\left(T(u,v), fu, fu\right) \\ & + a_4 G_2(fy, fv, fv) + a_5 G_2\left(S(x,y), fu, fu\right) + a_6 G_2\left(T(u,v), T(s,t), fx\right) \end{aligned} \tag{2.10}$$

for all $x, y, u, v \in X$, where $a_i \geq 0$, for $i = 1, 2, \ldots, 6$ and $a_1 + a_4 + a_5 + 2(a_2 + a_3 + a_6) < 1$. If $S(X \times X) \subseteq f(X)$, $T(X \times X) \subseteq f(X)$, $f(X)$ is G_1-complete subset of X, then S, T, and f have a unique common coupled coincidence point. Moreover, if S or T is w^*-compatible with f, then f, S, and T have a unique common coupled fixed point.

Example 2.4. Let $X = 0,1$, G-metrics G_1 and G_2 on X be given as (in [22]):

$$G_1(a, b, c) = |a - b| + |b - c| + |c - a|$$
$$G_2(a, b, c) = \frac{1}{2}|a - b| + |b - c| + |c - a|.$$

Define $S, T : X \times X \to X$ and $f : X \to X$ as

$$S(x,y) = \frac{x^2}{8},$$
$$T(x,y) = 0 \text{ and}$$
$$f(x) = x^2 \text{ for all } x, y \in X.$$

For $x, y, u, v \in X$, we have

$$\begin{aligned} G_1\left(S(x,y), T(u,v), T(u,v)\right) &= G_1\left(\frac{x^2}{8}, 0, 0\right) \\ &= \frac{x^2}{4} \\ &= \frac{1}{4}\left(\frac{1}{2}(2x^2)\right) \\ &= \frac{1}{4} G_2\left(0, 0, x^2\right) \\ &= \frac{1}{4} G_2\left(T(u,v), T(u,v), fx\right). \end{aligned}$$

Thus, (2.10) is satisfied with $a_1 = a_2 = a_3 = a_4 = a_5 = 0$ and $a_6 = \frac{1}{4}$, where $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 < 1$. It is obvious to note that S is w^*-compatible with f. Hence, all the conditions of Corollary 2.4 are satisfied. Moreover, (0, 0) is the unique common coupled fixed point of S, T, and f.

If we take $\alpha = a_1$, $\beta = a_4$, $\gamma = a_5$, and $a_2 = a_3 = a_6 = 0$ in Theorem 2.1, then the following corollary is obtained which extends and generalizes the comparable results of [17,22,23].

Corollary 2.5. Let G_1 and G_2 be two G-metrics on X such that $G_2(x, y, z) \le G_1(x, y, z)$, for all $x, y, z \in X$, and $S, T : X \times X \to X, f : X \to X$ be mappings satisfying

$$\begin{aligned} & G_1\left(S\left(x,y\right), T\left(u,v\right), T\left(s,t\right)\right) \\ \le\ & \alpha G_2\left(fx, fu, fs\right) + \beta G_2\left(fy, fv, ft\right) + \gamma G_2\left(S\left(x,y\right), fu, fs\right) \end{aligned} \tag{2.11}$$

for all $x, y, u, v, s, t \in X$, where $\alpha, \beta, \gamma \ge 0$, and $\alpha + \beta + \gamma < 1$. If $S(X \times X) \subseteq f(X)$, $T(X \times X) \subseteq f(X)$, $f(X)$ is G_1-complete subset of X, then S, T, and f have a unique common coupled coincidence point. Moreover, if S or T is w^*-compatible with f, then f, S, and T have a unique common coupled fixed point.

Corollary 2.6. Let G_1 and G_2 be two G-metrics on X such that $G_2(x, y, z) \le G_1(x, y, z)$, for all $x, y, z \in X$, $T : X \times X \to X$, and $f : X \to X$ be mappings satisfying

$$\begin{aligned} & G_1\left(T\left(x,y\right), T\left(u,v\right), T\left(s,t\right)\right) \\ \le\ & \alpha G_2\left(fx, fu, fs\right) + \beta G_2\left(fy, fv, ft\right) + \gamma G_2\left(S\left(x,y\right), fu, fs\right) \end{aligned}$$

for all $x, y, u, v, s, t \in X$, where $\alpha, \beta, \gamma \ge 0$, and $\alpha + \beta + \gamma < 1$. If $T(X \times X) \subseteq f(X)$, $f(X)$ is G_1-complete subset of X, then T and f have a unique common coupled coincidence point. Moreover, if T is w^*-compatible with f, then f and T have a unique common coupled fixed point.

Example 2.7. Let $X = [0,1]$, and two G-metrics G_1, G_2 on X be given as (in [22]):

$$\begin{aligned} & G_1\left(a,b,c\right) = |a-b| + |b-c| + |c-a| \quad \text{and} \\ & G_2\left(a,b,c\right) = \frac{1}{2}|a-b| + |b-c| + |c-a|. \end{aligned}$$

Define $T : X \times X \to X$ and $f : X \to X$ as

$$\begin{aligned} T(x,y) &= \frac{x+y}{16} \quad \text{and} \\ f(x) &= \frac{x}{2} \quad \text{for all } x, y \in X. \end{aligned}$$

Now, for $x, y \in X$,

$$\begin{aligned} & G_1\left(T\left(x,y\right), T\left(u,v\right), T\left(s,t\right)\right) \\ =\ & \frac{1}{16}\left[\left|x+y-(u+v)\right| + \left|u+v-(s+t)\right| + \left|s+t-(x+y)\right|\right] \\ \le\ & \frac{1}{16}\left[|x-u| + |y-v| + |u-s| + |v-t| + |s-x| + |t-y|\right] \\ \le\ & \frac{1}{16}\left[|x-u| + |y-v| + |u-s| + |v-t| + |s-x| + |t-y| \right. \\ & \left. + \left|\frac{x+y}{9} - u\right| + |u-s| + \left|s - \frac{x+y}{8}\right|\right] \\ =\ & \frac{1}{16}\left[|x-u| + |u-s| + |s-x| + |y-v| + |v-t| + |t-y| \right. \\ & \left. + \left|\frac{x+y}{8} - u\right| + |u-s| + \left|s - \frac{x+y}{8}\right|\right] \\ =\ & \frac{1}{4}\left[\frac{1}{2}\left(\frac{1}{2}|x-u| + \frac{1}{2}|u-s| + \frac{1}{2}|s-x|\right)\right] \\ & + \frac{1}{4}\left[\frac{1}{2}\left(\frac{1}{2}|y-v| + \frac{1}{2}|v-t| + \frac{1}{2}|t-y|\right)\right] \\ & + \frac{1}{4}\left[\frac{1}{2}\left(\frac{1}{2}\left|\frac{x+y}{8} - u\right| + \frac{1}{2}|u-s| + \frac{1}{2}\left|s - \frac{x+y}{8}\right|\right)\right] \\ =\ & \alpha G_2\left(\frac{x}{2}, \frac{u}{2}, \frac{s}{2}\right) + \beta G_2\left(\frac{y}{2}, \frac{v}{2}, \frac{t}{2}\right) + \gamma G_2\left(\frac{x+y}{16}, \frac{u}{2}, \frac{s}{2}\right) \\ =\ & \alpha G_2\left(fx, fu, fs\right) + \beta G_2\left(fy, fv, ft\right) + \gamma G_2\left(T\left(x,y\right), fu, fs\right). \end{aligned}$$

Thus, (2.11) is satisfied with $\alpha = \beta = \gamma = \frac{1}{4}$ where $\alpha + \beta + \gamma < 1$. It is obvious to note that T is w^*-compatible with f. Hence, all the conditions of *Corollary 2.5* are satisfied. Moreover, (0,0) is the unique common coupled fixed point of T and f.

Corollary 2.8. Let G_1 and G_2 be two G-metrics on X with $G_2(x, y, z) \le G_1(x, y, z)$, for all $x, y, z \in X$ and $S,T : X \times X \to X$, $f : X \to X$ be two mappings such that

$$\begin{aligned} & G_1\left(S\left(x,y\right),T\left(u,v\right),T\left(u,v\right)\right) \\ \le\ & \alpha G_2\left(fx,fu,fs\right)+\beta G_2\left(fy,fv,fu\right)+\gamma G_2\left(S\left(x,y\right),fu,fu\right) \end{aligned} \tag{2.12}$$

for all $x, y, u, v \in X$, where $\alpha, \beta, \gamma \ge 0$ and $\alpha + \beta + \gamma < 1$. If $S(X \times X) \subseteq f(X)$, $T(X \times X) \subseteq f(X)$, $f(X)$ is G_1-complete subset of X, then S, T, and f have a unique common coupled coincidence point. Moreover, if S or T is w^*-compatible with f, then f, S, and T have a unique common coupled fixed point.

Theorem 2.9. Let G_1 and G_2 be two G-metrics on X such that $G_2(x, y, z) \le G_1(x, y, z)$, for all $x, y, z \in X$, and S, $T : X \times X \to X$, $f : X \to X$ be mappings satisfying

$$\begin{aligned} & G_1\left(S\left(x,y\right),T\left(u,v\right),T\left(s,t\right)\right) \\ \le\ & k\max\left\{G_2\left(fx,fu,fs\right)+G_2\left(fy,fv,ft\right)+G_2\left(S\left(x,y\right),fu,fs\right)\right\} \end{aligned} \tag{2.13}$$

for all $x, y, u, v, s, t \in X$, where $0 \le k < \frac{1}{2}$. If $S(X \times X) \subseteq f(X)$, $T(X \times X) \subseteq f(X)$, $f(X)$ is G_1-complete subset of X, then S, T, and f have a unique common coupled coincidence point. Moreover, if S or T is w^*-compatible with f, then f, S, and T have a unique common coupled fixed point.

Proof. Let $x_0, y_0 \in X$. We choose $x_1, y_1 \in X$ such that $fx_1 = S(x_0, y_0)$ and $fy_1 = S(y_0, x_0)$, this can be done in view of $S(X \times X) \subseteq f(X)$. Similarly, we can choose $x_2, y_2 \in X$ such that $fx_2 = T(x_1, y_1)$ and $fy_2 = T(y_1,x_1)$ since $T(X \times X) \subseteq f(X)$. Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$fx_{2n+1}=S\left(x_{2n},y_{2n}\right),\ fx_{2n+2}=T\left(x_{2n+1},y_{2n+1}\right)$$

and

$$fy_{2n+1}=S\left(y_{2n},x_{2n}\right),\ fy_{2n+2}=T\left(y_{2n+1},x_{2n+1}\right).$$

Now,

$$\begin{aligned} & G_1\left(fx_{2n+1},fx_{2n+2},fx_{2n+2}\right) \\ =\ & G_1\left(S\left(x_{2n},y_{2n}\right),T\left(x_{2n+1},y_{2n+1}\right),T\left(x_{2n+1},y_{2n+1}\right)\right) \\ \le\ & k\max\left\{G_2\left(fx_{2n},fx_{2n+1},fx_{2n+1}\right),G_2\left(fy_{2n},fy_{2n+1},fy_{2n+1}\right),\right. \\ & \left.G_2\left(S\left(x_{2n},y_{2n}\right),fx_{2n+1},fx_{2n+1}\right)\right\} \\ =\ & k\max\left\{G_2\left(fx_{2n},fx_{2n+1},fx_{2n+1}\right),G_2\left(fy_{2n},fy_{2n+1},fy_{2n+1}\right),\right. \\ & \left.G_2\left(fx_{2n+1},fx_{2n+1},fx_{2n+1}\right)\right\}, \end{aligned}$$

which implies that

$$\begin{aligned} & G_1\left(fx_{2n+1},fx_{2n+2},fx_{2n+2}\right) \\ \le\ & k\max\left\{G_2\left(fx_{2n},fx_{2n+1},fx_{2n+1}\right),G_2\left(fy_{2n},fy_{2n+1},fy_{2n+1}\right)\right\}. \end{aligned} \tag{2.14}$$

Similarly, we can show that

$$\begin{aligned} & G_1\left(fy_{2n+1}, fy_{2n+2}, fy_{2n+2}\right) \\ \leq\ & k\max\left\{G_2\left(fy_{2n}, fy_{2n+1}, fy_{2n+1}\right), G_2\left(fx_{2n}, fx_{2n+1}, fx_{2n+1}\right)\right\}. \end{aligned} \tag{2.15}$$

Now, from (2.14) and (2.15), we obtain

$$\begin{aligned} & G_1\left(fx_{2n+1}, fx_{2n+2}, fx_{2n+2}\right) + G_1\left(fy_{2n+1}, fy_{2n+2}, fy_{2n+2}\right) \\ \leq\ & k\left[\max\left\{G_2\left(fx_{2n}, fx_{2n+1}, fx_{2n+1}\right), G_2\left(fy_{2n}, fy_{2n+1}, fy_{2n+1}\right)\right\}\right. \\ & \left.+\max\left\{G_2\left(fy_{2n}, fy_{2n+1}, fy_{2n+1}\right), G_2\left(fx_{2n}, fx_{2n+1}, fx_{2n+1}\right)\right\}\right] \\ \leq\ & 2k\left[G_2\left(fx_{2n}, fx_{2n+1}, fx_{2n+1}\right) + G_2\left(fy_{2n}, fy_{2n+1}, fy_{2n+1}\right)\right]. \end{aligned}$$

In a similar way, we can obtain

$$\begin{aligned} & G_1\left(fx_{2n}, fx_{2n+1}, fx_{2n+1}\right) + G_1\left(fy_{2n}, fy_{2n+1}, fy_{2n+1}\right) \\ \leq\ & 2k\left[G_2\left(fx_{2n-1}, fx_{2n}, fx_{2n}\right) + G_2\left(fy_{2n-1}, fy_{2n}, fy_{2n}\right)\right]. \end{aligned}$$

Thus, for all $n \geq 0$,

$$\begin{aligned} & G_1\left(fx_n, fx_{n+1}, fx_{n+1}\right) + G_1\left(fy_n, fy_{n+1}, fy_{n+1}\right) \\ \leq\ & 2k\left[G_2\left(fx_{n-1}, fx_n, fx_n\right) + G_2\left(fy_{n-1}, fy_n, fy_n\right)\right]. \end{aligned}$$

Since $0 \leq 2\kappa < 1$. Therefore, repetition of above process n times gives

$$\begin{aligned} & G_1\left(fx_n, fx_{n+1}, fx_{n+1}\right) + G_1\left(fy_n, fy_{n+1}, fy_{n+1}\right) \\ \leq\ & 2k\left[G_2\left(fx_{n-1}, fx_n, fx_n\right) + G_2\left(fy_{n-1}, fy_n, fy_n\right)\right] \\ \leq\ & (2k)^2\left[G_2\left(fx_{n-2}, fx_{n-1}, fx_{n-1}\right) + G_2\left(fy_{n-2}, fy_{n-1}, fy_{n-1}\right)\right] \\ \leq\ & \ldots \leq (2k)^n\left[G_2\left(fx_0, fx_1, fx_1\right) + G_2\left(fy_0, fy_1, fy_1\right)\right]. \end{aligned}$$

For any $m > n \geq 1$, repeated use of property (e) of G-metric gives

$$\begin{aligned} & G_1\left(fx_n fx_m, fx_m\right) + G_1\left(fy_n, fy_m, fy_m\right) \\ \leq\ & G_2\left(fx_n, fx_{n+1}, fx_{n+1}\right) + G_2\left(fx_{n+1}, x_{x+2}, x_{n+2}\right) + G_2\left(fy_{n+1}, fy_{n+1}\right) \\ & +G_2\left(fx_{y+1}, x_{y+2}, x_{y+2}\right) + \ldots + G_2\left(fx_{m-1}, fx_m, fx_m\right) + G_2\left(fy_{m-1}, fy_m, fy_m\right) \\ \leq\ & \left((2k)^n + (2k)^{n+1} + \ldots + (2k)^{m-1}\right)\left[G_2\left(fx_0, fx_1, fx_1\right) + G_2\left(fy_0, fy_1, fy_1\right)\right] \\ \leq\ & \frac{(2k)^n}{1-2k}\left[G_2\left(fx_0, fx_1, fx_1\right) + G_2\left(fy_0, fy_1, fy_1\right)\right] \end{aligned}$$

and so $G_1(fx_n, fx_m, fx_m) + G_1(fy_n, fy_m, fy_m) \to 0$ as $n, m \to \infty$. Hence, $\{fx_n\}$ and $\{fy_n\}$ are G_1-Cauchy sequences in $f(X)$. By G_1-completeness of $f(X)$, there exists $fx, fy \in f(X)$ such that $\{fx_n\}$ and $\{fy_n\}$ converges to fx and fy, respectively.

Now, we prove that $S(x,y) = fx$ and $T(y,x) = fy$. Using (2.13), we have

$$\begin{aligned} & G_1\left(fx, T(x,y), T(x,y)\right) \\ \leq\ & G_1\left(fx_{2n+1}, T(x,y), T(x,y)\right) + G_1\left(fx, fx_{2n+1}, fx_{2n+1}\right) \\ =\ & G_1\left(S\left(x_{2n}, y_{2n}\right), T(x,y), T(x,y)\right) + G_1\left(fx_{2n+1}, fx_{2n+1}, fx\right) \\ \leq\ & k\max\left\{G_2\left(fx_{2n}, fx, fx\right), G_2\left(fy_{2n}, fy, fy\right), G_2\left(S\left(x_{2n}, y_{2n}\right), fx, fx\right)\right\} \\ & + G_1\left(fx_{2n+1}, fx_{2n+1}, fx\right) \\ =\ & k\max\left\{G_2\left(fx_{2n}, fx, fx\right), G_2\left(fy_{2n}, fy, fy\right), G_2\left(fx_{2n+1}, fx_n, fx\right)\right\} \\ & +G_1\left(fx_{2n+1}, fx_{2n+1}, fx\right). \end{aligned}$$

Taking limit as $n \to \infty$, implies that $G_1(fx, T(x, y), T(x, y)) = 0$, and $T(x, y) = fx$. Also, further from (2.13), we have

$$\begin{aligned} & G_1\left(S(x,y), fx, fx\right) \\ = & \; G_1\left(S(x,y), T(x,y), T(x,y)\right) \\ \leq & \; k \max\left\{G_2\left(fx,fx,fx\right), G_2\left(fy,fy,fy\right), G_2\left(S(x,y),fx,fx\right)\right\} \\ = & \; kG_2\left(S(x,y),fx,fx\right) \\ \leq & \; kG_1\left(S(x,y),fx,fx\right), \end{aligned}$$

that is $G_1(S(x, y), fx, fx) = 0$, and $S(x, y) = fx$. Thus, $T(x, y) = S(x, y) = fx$. Similarly, it can be shown that $T(y, x) = S(y, x) = fy$. Thus, (fx, fy) is coupled point of coincidence of mappings f, S, and T.

Now, we shall show that $fx = fy$. So that

$$\begin{aligned} & G_1\left(fx_{2n+1}, fy_{2n+2}, fy_{2n+2}\right) \\ = & \; G_1\left(S\left(x_{2n}, y_{2n}\right), T\left(y_{2n+1}, x_{2n+1}\right), T\left(y_{2n+1}, x_{2n+1}\right)\right) \\ \leq & \; k \max\left\{G_2\left(fx_{2n}, fx_{2n+1}, fx_{2n+1}\right), G_2\left(fy_{2n}, fy_{2n+1}, fy_{2n+1}\right),\right. \\ & \left. G_2\left(S\left(x_{2n}, y_{2n}\right), fy_{2n+1}, fy_{2n+1}\right)\right\} \\ \leq & \; k \max\left\{G_2\left(fy_{2n}, fy_{2n+1}, fy_{2n+1}\right), G_2\left(fx_{2n}, fx_{2n+1}, fx_{2n+1}\right),\right. \\ & \left. G_2\left(fx_{2n+1}, fy_{2n+1}, fy_{2n+1}\right)\right\}. \end{aligned}$$

On taking the limit as $n \to \infty$, we obtain that

$$\begin{aligned} G_1\left(fx,fy,fy\right) &\leq k \max\left\{G_2\left(fx,fy,fy\right), G_2\left(fx,fx,fy\right)\right\} \\ &= kG_2\left(fx,fx,fy\right) \leq kG_1\left(fx,fx,fy\right). \end{aligned} \tag{2.16}$$

In the similar way, we can show that

$$G_1\left(fy,fx,fx\right) \leq kG_1\left(fy,fy,fx\right). \tag{2.17}$$

From (2.16) and (2.17), we must have $G_1(fx, fy, fy) = 0$ which implies that $fx = fy$. Thus, (fx, fx) is a coupled point of coincidence of mappings f, S, and T. Now, if there is another $x^* \in X$ such that (fx^*,fx^*) is a coupled point of coincidence of mappings f, S, and T, then

$$\begin{aligned} & G_1(fx, fx^*, fx^*) \\ = & \; G_1(S(x,x), T(x^*,x^*), T(x^*,x^*)) \\ \leq & \; k \max\left\{G_2\left(fx,fx^*,fx^*\right), G_2\left(fx,fx^*,fx^*\right), G_2\left(S(x,x),fx^*,fx^*\right)\right\} \\ = & \; kG_2\left(fx,fx^*,fx^*\right) \end{aligned}$$

implies that $G_1(fx, fx^*, fx^*) = 0$ and so $fx^* = fx$. Hence, (fx, fx) is a unique coupled point of coincidence of mappings f, S, and T.

Now, we show that f, S, and T have common coupled fixed point.

For this, let $f(x) = u$. Then, we have $u = fx = T(x, x)$. By w^*-compatibility of f and T, we have

$$f(u) = f(fx) = f\left(T(x,x)\right) = T(fx,fx) = T(u,u). \tag{2.18}$$

That is, (fu, fu) is a coupled point of coincidence of f, S, and T. By the uniqueness of coupled point of coincidence, we have $fu = fx$. Therefore, (u, u) is the common coupled fixed point of f, S, and T.

To prove the uniqueness, we proceed as follows: let $v \in X$ with $u \neq v$ such that (v, v) is the common coupled fixed point of f, S and T. Using (2.13), we have

$$\begin{aligned} & G_1(u,v,v) \\ = \; & G_1\left(S(u,u),T(v,v),T(u,v)\right) \\ \leq \; & k\max\left\{G_2(fu,fv,fv),G_2(fu,fv,fv),G_2\left(S(u,u),fv,fv\right)\right\} \\ = \; & kG_2(fu,fv,fv)=kG_2(u,v,v) \\ \leq \; & kG_1(u,v,v), \end{aligned}$$

so that $G_1(u, v, v) = 0$ and $u = u^*$. Thus, f, S, and T have a unique common coupled fixed point.

In Theorem 2.9, take $S = T$, to obtain the following Theorem 2.6 of [22].

Corollary 2.10. Let G_1 and G_2 be two G-metrics on X such that $G_2(x, y, z) \leq G_1(x, y, z)$, for all $x, y, z \in X$, $T : X \times X \to X$, and $f : X \to X$ be mappings satisfying

$$\begin{aligned} & G_1\left(T(x,y),T(u,v),T(s,t)\right) \\ \leq \; & k\max\left\{G_2(fx,fu,fs),G_2(fy,fv,ft),G_2(T(x,y),fu,fs)\right\} \end{aligned} \tag{2.19}$$

for all $x, y, u, v, s, t \in X$, where $0 \leq k < \frac{1}{2}$. If $T(X \times X) \subseteq f(X)$, $f(X)$ is G_1-complete subset of X, then T and f have a unique common coupled coincidence point. Moreover, if T is w^*-compatible with f, then T and f have a unique common coupled fixed point.

In Theorem 2.9, take $s = u$ and $t = v$, to obtain the following corollary which extends and generalizes the corresponding results of [17,22,23].

Corollary 2.11 Let G_1 and G_2 be two G-metrics on X such that $G_2(x, y, z) \leq G_1(x, y, z)$, for all $x, y, z \in X$, $S, T : X \times X \to X$, and $f : X \to X$ be mappings satisfying

$$\begin{aligned} & G_1\left(S\left(x,y\right),T\left(u,v\right),T\left(s,v\right)\right) \\ \leq \; & k\max\left\{G_2\left(fx,fu,fu\right)+G_2\left(fy,fv,fv\right)+G_2\left(S\left(x,y\right),fu,fv\right)\right\} \end{aligned} \tag{2.20}$$

for all $x, y, u, v \in X$, where $0 \leq k < \frac{1}{2}$. If $S(X \times X) \subseteq f(X)$, $T(X \times X) \subseteq f(X)$, $f(X)$ is G_1-complete subset of X, then S, T, and f have a unique common coupled coincidence point. Moreover, if S or T is w^*-compatible with f, then f, S, and T have a unique common coupled fixed point.

Corollary 2.12. Let G_1 and G_2 be two G-metrics on X such that $G_2(x, y, z) \leq G_1(x, y, z)$, for all $x, y, z \in X$, $S, T : X \times X \to X$, and $f : X \to X$ be mappings satisfying

$$G_1\left(S(x,y),T(u,v),T(s,t) \leq hG_2(fx,fu,fs)\right) \tag{2.21}$$

for all $x, y, u, v, s, t \in X$, where $0 \leq h < 1$. If $S(X \times X) \subseteq f(X)$, $T(X \times X) \subseteq f(X)$, $f(X)$ is G_1-complete subset of X, then S, T, and f have a unique common coupled coincidence point. Moreover, if S or T is w^*-compatible with f, then f, S, and T have a unique common coupled fixed point.

Remark 2.13. By the equivalence of some metrics and cone metric fixed point results in [24]:

(a) Theorem 2.1 can be viewed as an extension and generalization of (i) Theorem 2.2, Corollary 2.3, Theorem 2.6, Corollary 2.7 and Corollary 2.8 in [23], (ii) Theorem 2.1, Corollary 2.2, Corollary 2.5 and Corollary 2.5 in [22], (iii) Theorem 2.4 and Corollary 2.5 in [17].

(b) Theorem 2.9 is a generalization and improvement of (i) Theorem 2.2 and Corollary 2.3 in [23], Theorem 2.6, Corollary 2.7 and Corollary 2.8 in [22].

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All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Competing interests
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