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Fixed point theorems for contraction mappings in modular metric spaces

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Abstract

In this article, we study and prove the new existence theorems of fixed points for contraction mappings in modular metric spaces.

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1 Introduction

Let (X, d) be a metric space. A mapping $T : X \to X$ is a contraction if

$$d(T(x), T(y)) \leq kd(x, y), \tag{1.1}$$

for all $x, y \in X$, where $0 \leq k < 1$. The Banach Contraction Mapping Principle appeared in explicit form in Banach's thesis in 1922 [1]. Since its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis. Banach contraction principle has been extended in many different directions, see [2-10]. The notion of modular spaces, as a generalize of metric spaces, was introduced by Nakano [11] and was intensively developed by Koshi, Shimogaki, Yamamuro [11-13] and others. Further and the most complete development of these theories are due to Luxemburg, Musielak, Orlicz, Mazur, Turpin [14-18] and their collaborators. A lot of mathematicians are interested fixed points of Modular spaces, for example [4,19-26].

In 2008, Chistyakov [27] introduced the notion of modular metric spaces generated by F-modular and develop the theory of this spaces, on the same idea he was defined the notion of a modular on an arbitrary set and develop the theory of metric spaces generated by modular such that called the modular metric spaces in 2010 [28].

In this article, we study and prove the existence of fixed point theorems for contraction mappings in modular metric spaces.

2 Preliminaries

We will start with a brief recollection of basic concepts and facts in modular spaces and modular metric spaces (see [14,15,27-29] for more details).

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Definition 2.1. *Let X be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A functional $\rho : X \to [0, \infty]$ is called a* modular *if for arbitrary x and y, elements of X satisfies the following three conditions :*

(A.1) $\rho(x) = 0$ *if and only if* $x = 0$;

(A.2) $\rho(\alpha x) = \rho(x)$ *for all scalar* α *with* $|\alpha| = 1$;

(A.3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, *whenever* $\alpha, \beta \geq 0$ *and* $\alpha + \beta = 1$.

If we replace (*A*.3) *by*

(A.4) $\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y)$, *for* $\alpha, \beta \geq 0$, $\alpha^s + \beta^s = 1$ *with an* $s \in (0, 1]$, *then the modular* ρ *is called s-convex modular, and if* $s = 1$, ρ *is called a convex modular.*

If ρ is modular in X, then the set defined by

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0^+\} \tag{2.1}$$

is called a modular space. X_ρ is a vector subspace of X it can be equipped with an *F-norm* defined by setting

$$\|x\|_\rho = \inf\{\lambda > 0 : \rho\left(\tfrac{x}{\lambda}\right) \leq \lambda\}, \quad x \in X_\rho. \tag{2.2}$$

In addition, if ρ is convex, then the modular space X_ρ coincides with

$$X_\rho^* = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \rho(\lambda x) < \infty\} \tag{2.3}$$

and the functional $\|x\|_\rho^* = \inf\{\lambda > 0 : \rho\left(\tfrac{x}{\lambda}\right) \leq 1\}$is an ordinary norm on X_ρ^* which is equivalence to $\|x\|_\rho$(see [16]).

Let X be a nonempty set, $\lambda \in (0, \infty)$ and due to the disparity of the arguments, function $w : (0, \infty) \times X \times X \to [0, \infty]$ will be written as $w_\lambda(x, y) = w(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$.

Definition 2.2. [[28], Definition 2.1] *Let X be a nonempty set. A function* $w : (0, \infty) \times X \times X \to [0, \infty]$ *is said to be a metric modular on X if satisfying, for all* $x, y, z \in X$ *the following condition holds:*

(*i*) $w_\lambda(x, y) = 0$ *for all* $\lambda > 0$ *if and only if* $x = y$;

(*ii*) $w_\lambda(x, y) = w_\lambda(y, x)$ *for all* $\lambda > 0$;

(*iii*) $w_{\lambda + \mu}(x, y) \leq w_\lambda(x, z) + w_\mu(z, y)$ *for all* $\lambda, \mu > 0$.

If instead of (*i*), *we have only the condition*

(*i'*) $w_\lambda(x, x) = 0$ *for all* $\lambda > 0$, *then w is said to be a (metric) pseudomodular on X.*

The main property of a (pseudo) modular w on a set X is a following: given $x, y \in X$, the function $0 < \lambda \mapsto w_\lambda(x, y) \in [0, \infty]$ is a nonincreasing on $(0, \infty)$.

In fact, if $0 < \mu < \lambda$, then (iii), (i') and (ii) imply

$$w_\lambda(x, y) \leq w_{\lambda-\mu}(x, x) + w_\mu(x, y) = w_\mu(x, y). \tag{2.4}$$

It follows that at each point $\lambda > 0$ the right limit $w_{\lambda+0}(x, y) := \lim_{\varepsilon \to +0} w_{\lambda+\varepsilon}(x, y)$ and the left limit $w_{\lambda-0}(x, y) := \lim_{\varepsilon \to +0} w_{\lambda-\varepsilon}(x, y)$ exists in $[0, \infty]$ and the following two inequalities hold :

$$w_{\lambda+0}(x, y) \leq w_\lambda(x, y) \leq w_{\lambda-0}(x, y). \tag{2.5}$$

Definition 2.3. [[28], Definition 3.3] *A function* $w : (0, \infty) \times X \times X \to [0, \infty]$ *is said to be a convex (metric) modular on X if it is satisfies the conditions* (*i*) *and* (*ii*) *from Definition 2.2 as well as this condition holds;*

$$(iv)\ w_{\lambda+\mu}(x, y) = \frac{\lambda}{\lambda+\mu} w_\lambda(x, z) + \frac{\mu}{\lambda+\mu} w_\mu(z, y)\ \text{for all } \lambda, \mu > 0 \text{ and } x, y, z \in X.$$

If instead of (i), we have only the condition (i') from Definition 2.2, then w is called a convex(metric) pseudomodular on X.

From [27,28], we know that, if $x_0 \in X$, the set $X_w = \{x \in X : \lim_{\lambda\to\infty} w_\lambda(x, x_0) = 0\}$ is a metric space, called *a modular space*, whose metric is given by $d_w^\circ(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \le \lambda\}$ for all $x, y \in X_w$. Moreover, if w is convex, the modular set X_w is equal to $X_w^* = \{x \in X : \exists \lambda = \lambda(x) > 0$ such that $w_\lambda(x, x_0) < \infty\}$ and metrizable by $d_w^*(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \le 1\}$for all $x, y \in X_w^*$. We know that (see [[28], Theorem 3.11]) if X is a real linear space, $\rho : X \to [0, \infty]$ and

$$w_\lambda(x, y) = \rho\left(\frac{x - y}{\lambda}\right) \text{ for all } \lambda > 0 \text{ and } x, y \in X, \tag{2.6}$$

then ρ is *modular (convex modular)* on X in the sense of (A.1)-(A.4) if and only if w is *metric modular (convex metric modular, respectively)* on X. On the other hand, if w satisfy the following two conditions (i) $w_\lambda(\mu x, 0) = w_{\lambda/\mu}(x, 0)$ for all $\lambda, \mu > 0$ and $x \in X$, (ii) $w_\lambda(x + z, y + z) = w_\lambda(x, y)$ for all $\lambda > 0$ and $x, y, z \in X$, if we set $\rho(x) = w_1(x, 0)$ with (2.6) holds, where $x \in X$, then

(i) $X_\rho = X_w$ is a linear subspace of X and the functional $\|x\|_\rho = d_w^\circ(x, 0)$, $x \in X_\rho$, is an *F-norm* on X_ρ;
(ii) if w is convex, $X_\rho^* \equiv X_w^*(0) = X_\rho$ is a linear subspace of X and the functional $\|x\|_\rho = d_w^*(x, 0)$, $x \in X_\rho^*$, is an norm on X_ρ^*.

Similar assertions hold if replace the word *modular* by *pseudomodular*. If w is metric modular in X, we called the set X_w is *modular metric space*.

By the idea of property in metric spaces and modular spaces, we defined the following:

Definition 2.4. *Let* X_w *be a modular metric space.*

(1) The sequence $(x_n)_{n\in\mathbb{N}}$ *in* X_w *is said to be convergent to* $x \in X_w$ *if* $w_\lambda(x_n, x) \to 0$, *as* $n \to \infty$ *for all* $\lambda > 0$.

(2) The sequence $(x_n)_{n\in\mathbb{N}}$ *in* X_w *is said to be Cauchy if* $w_\lambda(x_m, x_n) \to 0$, *as* $m, n \to \infty$ *for all* $\lambda > 0$.

(3) A subset C of X_w *is said to be closed if the limit of a convergent sequence of C always belong to C.*

(4) A subset C of X_w *is said to be complete if any Cauchy sequence in C is a convergent sequence and its limit is in C.*

(5) A subset C of X_w *is said to be bounded if for all* $\lambda > 0$ $\delta_w(C) = \sup\{w_\lambda(x, y); x, y \in C\} < \infty$.

3 Main results

In this section, we prove the existence of fixed points theorems for contraction mapping in modular metric spaces.

Definition 3.1. *Let w be a metric modular on X and X_w be a modular metric space induced by w and $T : X_w \to X_w$ be an arbitrary mapping. A mapping T is called a contraction if for each $x, y \in X_w$ and for all $\lambda > 0$ there exists $0 \le k < 1$ such that*

$$w_\lambda(Tx, Ty) \le k w_\lambda(x, y). \tag{3.1}$$

Theorem 3.2. *Let w be a metric modular on X and X_w be a modular metric space induced by w. If X_w is a complete modular metric space and $T : X_w \to X_w$ is a contraction mapping, then T has a unique fixed point in X_w. Moreover, for any $x \in X_w$, iterative sequence $\{T^n x\}$ converges to the fixed point.*

Proof. Let x_0 ba an arbitrary point in X_w and we write $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$, and in general, $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. Then,

$$\begin{aligned} w_\lambda(x_{n+1}, x_n) &= w_\lambda(Tx_n, Tx_{n-1}) \\ &\le k w_\lambda(x_n, x_{n-1}) \\ &= k w_\lambda(Tx_{n-1}, Tx_{n-2}) \\ &\le k^2 w_\lambda(x_{n-1}, x_{n-2}) \\ &\ \ \vdots \\ &\le k^n w_\lambda(x_1, x_0) \end{aligned}$$

for all $\lambda > 0$ and for each $n \in \mathbb{N}$. Therefore, $\lim_{n\to\infty} w_\lambda(x_{n+1}, x_n) = 0$ for all $\lambda > 0$. So for each $\lambda > 0$, we have for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $w_\lambda(x_n, x_{n+1}) < \epsilon$ for all $n \in \mathbb{N}$ with $n \ge n_0$. Without loss of generality, suppose $m, n \in \mathbb{N}$ and $m > n$. Observe that, for $\frac{\lambda}{m-n} > 0$, there exists $n_{\lambda/(m-n)} \in \mathbb{N}$ such that

$$w_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) < \frac{\varepsilon}{m-n}$$

for all $n \ge n_{\lambda/(m-n)}$. Now, we have

$$\begin{aligned} w_\lambda(x_n, x_m) &\le w_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) + w_{\frac{\lambda}{m-n}}(x_{n+1}, x_{n+2}) + \cdots + w_{\frac{\lambda}{m-n}}(x_{m-1}, x_m) \\ &< \frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + \cdots + \frac{\varepsilon}{m-n} \\ &= \varepsilon \end{aligned}$$

for all $m, n \ge n_{\lambda/(m-n)}$. This implies $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of X_w, there exists a point $x \in X_w$ such that $x_n \to \times$ as $n \to \infty$.

By the notion of metric modular w and the contraction of T, we get

$$\begin{aligned} w_\lambda(Tx, x) &\le w_{\frac{\lambda}{2}}(Tx, Tx_n) + w_{\frac{\lambda}{2}}(Tx_n, x) \\ &\le k w_{\frac{\lambda}{2}}(x, x_n) + w_{\frac{\lambda}{2}}(x_{n+1}, x) \end{aligned} \tag{3.2}$$

for all $\lambda > 0$ and for each $n \in \mathbb{N}$. Taking $n \to \infty$ in (3.2) implies that $w_\lambda(Tx, x) = 0$ for all $\lambda > 0$ and thus $Tx = x$. Hence, x is a fixed point of T. Next, we prove that x is a unique fixed point. Suppose that z is another fixed point of T. We see that

$$\begin{aligned} w_\lambda(x,z) &= w_\lambda(Tx,Tz) \\ &\le kw_\lambda(x,z) \end{aligned}$$

for all $\lambda > 0$. Since $0 \le k < 1$, we get $w_\lambda(x, z) = 0$ for all $\lambda > 0$ this implies that $x = z$. Therefore, x is a unique fixed point of T and the proof is complete. □

Theorem 3.3. *Let w be a metric modular on X and X_w be a modular metric space induced by w. If X_w is a complete modular metric space and $T : X_w \to X_w$ is a contraction mapping. Suppose $x^* \in X_w$ is a fixed point of T, $\{\varepsilon_n\}$ is a sequence of positive numbers for which $\lim_{n\to\infty} \varepsilon_n = 0$, and $\{y_n\} \subseteq X_w$ satisfies*

$$w_\lambda(y_{n+1},\ Ty_n) \le \varepsilon_n$$

for all $\lambda > 0$. Then, $\lim_{n\to\infty} y_n = x^*$.

Proof. For each $m \in \mathbb{N}$, we observe that

$$\begin{aligned} w_\lambda(T^{m+1}x, y_{m+1}) &= w_{\frac{\lambda\cdot m}{m}}(T^{m+1}x, y_{m+1}) \\ &\le w_{\frac{\lambda\cdot(m-1)}{m}}(T^{m+1}x, Ty_m) + w_{\frac{\lambda}{m}}(Ty_m, y_{m+1}) \\ &\le kw_{\frac{\lambda\cdot(m-1)}{m}}(T^{m}x, y_m) + \varepsilon_m \\ &\le kw_{\frac{\lambda\cdot(m-2)}{m}}(T^{m}x, Ty_{m-1}) + kw_{\frac{\lambda}{m}}(Ty_{m-1}x, y_m) + \varepsilon_m \\ &\le k^2w_{\frac{\lambda\cdot(m-2)}{m}}(T^{m-1}x, y_{m-1}) + k\varepsilon_{m-1} + \varepsilon_m \\ &\ \ \vdots \\ &\le \sum_{i=0}^{m} k^{m-i}\varepsilon_i \end{aligned} \tag{3.3}$$

for all $\lambda > 0$. Thus, we get

$$\begin{aligned} w_\lambda(y_{m+1}, x^*) &\le w_{\frac{\lambda}{2}}(y_{m+1}, T^{m+1}x) + w_{\frac{\lambda}{2}}(T^{m+1}x, x^*) \\ &\le \sum_{i=0}^{m} k^{m-i}\varepsilon_i + w_{\frac{\lambda}{2}}(T^{m+1}x, x^*). \end{aligned} \tag{3.4}$$

Next, we claimed that $\lim_{m\to\infty} w_\lambda(y_{m+1}, x^*) = 0$ for all $\lambda > 0$.

Now let $\varepsilon > 0$. Since $\lim_{n\to\infty} \varepsilon_n = 0$, there exists $N \in \mathbb{N}$ such that for $m \ge N$, $\varepsilon_m \le \varepsilon$. Thus,

$$\begin{aligned} \sum_{i=0}^{m} k^{m-i}\varepsilon_i &= \sum_{i=0}^{N} k^{m-i}\varepsilon_i + \sum_{i=N+1}^{m} k^{m-i}\varepsilon_i \\ &\le k^{m-N}\sum_{i=0}^{N} k^{N-i}\varepsilon_i + \varepsilon\sum_{i=N+1}^{m} k^{m-i}. \end{aligned} \tag{3.5}$$

Taking limit as $m \to \infty$ in (3.5), we have

$$\lim_{m\to\infty} \sum_{i=0}^{m} k^{m-i}\varepsilon_i = 0. \tag{3.6}$$

Since x_0 is a fixed point of T and using result of Theorem 3.2, we get the sequence $\{T^n x\}$ converge to x^*. This implies that

$$\lim_{m\to\infty} w_{\frac{\lambda}{2}}(T^{m+1}x, x^*) = 0 \tag{3.7}$$

for all $\lambda > 0$. From (3.4), (3.6) and (3.7), we have

$$\lim_{m\to\infty} w_\lambda(y_{m+1}, x^*) = 0 \tag{3.8}$$

for all $\lambda > 0$ which implies that $\lim_{n\to\infty} y_n = x^*$. □

Theorem 3.4. *Let w be a metric modular on X and X_w be a modular metric space induced by w. If X_w is a complete modular metric space and $T : X_w \to X_w$ is a mapping, which T^N is a contraction mapping for some positive integer N. Then, T has a unique fixed point in X_w.*

Proof. By Theorem 3.2 , T^N has a unique fixed point $u \in X_w$. From $T^N(T_u) = T^{N+1}u = T(T^N u) = Tu$, so Tu is a fixed point of T^N. By the uniqueness of fixed point of T^N, we have $Tu = u$. Thus, u is a fixed point of T. Since fixed point of T is also fixed point of T^N, we can conclude that T has a unique fixed point in X_w. □

Theorem 3.5. *Let w be metric modular on X, X_w be a complete modular metric space induced by w and for $x^* \in X_w$ we define*

$$B_w(x^*, \gamma) := \{x \in X_w | w_\lambda(x, x^*) \le \gamma \quad \textit{for all} \quad \lambda > 0\}.$$

If $T : B_w(x^*, \gamma) \to X_w$ is a contraction mapping with

$$w_{\frac{\lambda}{2}}(Tx^*, x^*) \le (1-k)\gamma \tag{3.9}$$

for all $\lambda > 0$, where $0 \le k < 1$. Then, T has a unique fixed point in $B_w(x^*, \gamma)$.

Proof. By Theorem 3.2 , we only prove that $B_w(x^*, \gamma)$ is complete and $Tx \in B_w(x^*, \gamma)$, for all $x \in B_w(x^*, \gamma)$. Suppose that $\{x_n\}$ is a Cauchy sequence in $B_w(x^*, \gamma)$, also $\{x_n\}$ is a Cauchy sequence in X_w. Since X_w is complete, there exists $x \in X_w$ such that

$$\lim_{n\to\infty} w_{\frac{\lambda}{2}}(x_n, x) = 0 \tag{3.10}$$

for all $\lambda > 0$. Since for each $n \in \mathbb{N}$, $x_n \in B_w(x^*, \gamma)$, using the property of metric modular, we get

$$\begin{aligned} w_\lambda(x^*, x) &\le w_{\frac{\lambda}{2}}(x^*, x_n) + w_{\frac{\lambda}{2}}(x_n, x) \\ &\le \gamma + w_{\frac{\lambda}{2}}(x_n, x^*) \end{aligned} \tag{3.11}$$

for all $\lambda > 0$. It follows the inequalities (3.10) and (3.11), we have $w_\lambda(x^*, x) \le \gamma$ which implies that $x \in B_w(x^*, \gamma)$. Therefore, $\{x_n\}$ is convergent sequence in $B_w(x^*, \gamma)$ and also $B_w(x^*, \gamma)$ is complete.

Next, we prove that $Tx \in B_w(x^*, \gamma)$ for all $x \in B_w(x^*, \gamma)$. Let $x \in B_w(x^*, \gamma)$. From the inequalities (3.9), using the contraction of T and the notion of metric modular, we have

$$\begin{aligned} w_\lambda(x^*, Tx) &\le w_{\frac{\lambda}{2}}(x^*, Tx^*) + w_{\frac{\lambda}{2}}(Tx^*, Tx) \\ &\le (1-k)\gamma + kw_{\frac{\lambda}{2}}(x^*, x) \\ &\le (1-k)\gamma + k\gamma \\ &= \gamma. \end{aligned}$$

Therefore, $Tx \in B_w(x^*, \gamma)$ and the proof is complete.

Theorem 3.6. *Let w be a metric modular on X, X_w be a complete modular metric space induced by w and $T : X_w \to X_w$. If*

$$w_\lambda(Tx, Ty) \le k(w_{2\lambda}(Tx, x) + w_{2\lambda}(Ty, y)) \tag{3.12}$$

for all $x, y \in X_w$ and for all $\lambda > 0$, where $k \in [0, \frac{1}{2})$, then T has a unique fixed point in X_w. Moreover, for any $x \in X_w$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. Let x_0 be an arbitrary point in X_w and we write $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$, and in general, $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. If $Tx_{n_0-1} = Tx_{n_0}$for some $n_0 \in \mathbb{N}$, then $Tx_{n_0} = x_{n_0}$. Thus, x_{n_0} is a fixed point of T. Suppose that $Tx_{n-1} \ne Tx_n$ for all $n \in \mathbb{N}$. For $k \in [0, \frac{1}{2})$, we have

$$\begin{aligned} w_\lambda(x_{n+1}, x_n) &= w_\lambda(Tx_n, Tx_{n-1}) \\ &\le k(w_{2\lambda}(Tx_n, x_n) + w_{2\lambda}(Tx_{n-1}, x_{n-1})) \\ &\le k(w_\lambda(x_{n+1}, x_n) + w_\lambda(x_n, x_{n-1})) \end{aligned} \tag{3.13}$$

for all $\lambda > 0$ and for all $n \in \mathbb{N}$. Hence,

$$w_\lambda(x_{n+1}, x_n) \le \frac{k}{1-k} w_\lambda(x_n, x_{n-1}) \tag{3.14}$$

for all $\lambda > 0$ and for all $n \in \mathbb{N}$. Put $\beta := \frac{k}{1-k}$, since $k \in [0, \frac{1}{2})$, we get $\beta \in [0, 1)$ and hence

$$\begin{aligned} w_\lambda(x_{n+1}, x_n) &\le \beta w_\lambda(x_n, x_{n-1}) \\ &\le \beta^2 w_\lambda(x_{n-1}, x_{n-2}) \\ &\;\;\vdots \\ &\le \beta^n w_\lambda(x_1, x_0) \end{aligned} \tag{3.15}$$

for all $\lambda > 0$ and for all $n \in \mathbb{N}$. Similar to the proof of Theorem 3.2, we can conclude that $\{x_n\}$ is a Cauchy sequence and by the completeness of X_w there exists a point $x \in X_w$ such that $x_n \to x$ as $n \to \infty$. By the property of metric modular and the inequality (3.12), we have

$$\begin{aligned} w_\lambda(Tx, x) &\le w_{\frac{\lambda}{2}}(Tx, Tx_n) + w_{\frac{\lambda}{2}}(Tx_n, x) \\ &\le k(w_\lambda(Tx, x) + w_\lambda(Tx_n, x_n)) + w_{\frac{\lambda}{2}}(Tx_n, x) \\ &\le k(w_\lambda(Tx, x) + w_{\frac{\lambda}{2}}(Tx_n, x) + w_{\frac{\lambda}{2}}(x, x_n)) + w_{\frac{\lambda}{2}}(Tx_n, x) \\ &= k(w_\lambda(Tx, x) + w_{\frac{\lambda}{2}}(x_{n+1}, x) + w_{\frac{\lambda}{2}}(x, x_n)) + w_{\frac{\lambda}{2}}(x_{n+1}, x) \end{aligned} \tag{3.16}$$

for all $\lambda > 0$ and for all $n \in \mathbb{N}$. Taking $n \to \infty$ in the inequality (3.16), we obtained that

$$w_\lambda(Tx, x) \le kw_\lambda(Tx, x). \tag{3.17}$$

Since $k \in [0, \frac{1}{2})$, we have $Tx = x$. Thus, x is a fixed point of T. Next, we prove that x is a unique fixed point. Suppose that z be another fixed point of T. We note that

$$\begin{aligned} w_\lambda(x,z) &= w_\lambda(Tx,Tz) \\ &\leq k(w_{\frac{\lambda}{2}}(Tx,x) + w_{\frac{\lambda}{2}}(Tz,z)) \\ &= 0 \end{aligned}$$

for all $\lambda > 0$, which implies that $x = z$. Therefore, x is a unique fixed point of T. □

Now, we shall give a validate example of Theorem 3.2 .

Example 3.7. *Let* $X = \{(a, 0) \in \mathbb{R}^2 | 0 \leq a \leq 1\} \cup \{(0, b) \in \mathbb{R}^2 | 0 \leq b \leq 1\}$.

Defined the mapping $w : (0, \infty) \times X \times X \to [0, \infty]$ *by*

$$w_\lambda((a_1, 0), (a_2, 0)) = \frac{4|a_1 - a_2|}{3\lambda},$$

$$w_\lambda((0, b_1), (0, b_2)) = \frac{|b_1 - b_2|}{\lambda},$$

and

$$w_\lambda((a, 0), (0, b)) = \frac{4a}{3\lambda} + \frac{b}{\lambda} = w_\lambda((0, b), (a, 0)).$$

We note that if we take $\lambda \to \infty$, *then we see that* $X = X_w$ *and also* X_w *is a complete modular metric space. We let a mapping* $T : X_w \to X_w$ *is define by*

$$T((a, 0)) = (0, a)$$

and

$$T((0, b)) = \left(\frac{b}{2}, 0\right).$$

Simple computations show that

$$w_\lambda(T((a_1, b_1)), T((a_2, b_2))) \leq \frac{3}{4} w_\lambda((a_1, b_1), (a_2, b_2))$$

for all $(a_1, b_1), (a_2, b_2) \in X_w$. Thus, T is a contraction mapping with constant $k = \frac{3}{4}$. *Therefore, T has a unique fixed point that is* $(0, 0) \in X_w$.

On the Euclidean metric d on X_w, *we see that*

$$d(T((0, 0)), T((1, 0))) = d((0, 0), (0, 1)) = 1 > k = kd((0, 0), (1, 0))$$

for all $k \in [0, 1)$. *Thus, T is not a contraction mapping and then the Banach contraction mapping cannot be applied to this example.*

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Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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