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Fixed points of monotone nonexpansive mappings on a hyperbolic metric space with a graph

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Abstract
In this work, we define the concept of *G*-monotone nonexpansive multivalued mappings defined on a metric space with a graph *G*. Then we obtain sufficient conditions for the existence of fixed points for such mappings in hyperbolic metric spaces. This is the first kind of such results in this direction.

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1 Introduction

Fixed point theorems for monotone single-valued mappings in a metric space endowed with a partial ordering have been widely investigated. These theorems are hybrids of the two most fundamental and useful theorems in fixed point theory: the Banach contraction principle ([1], Theorem 2.1) and the Tarski fixed point theorem [2, 3]. Generalizing the Banach contraction principle for multivalued mapping to metric spaces, Nadler [4] obtained the following result.

Theorem 1.1 [4] *Let (X,d) be a complete metric space. Denote by $\mathcal{CB}(X)$ the set of all nonempty closed bounded subsets of X. Let $F : X \to \mathcal{CB}(X)$ be a multivalued mapping. If there exists $k \in [0,1)$ such that*

$$H\big(F(x),F(y)\big) \le kd(x,y)$$

for all $x,y \in X$, where H is the Hausdorff metric on $\mathcal{CB}(X)$, then F has a fixed point in X.

A number of extensions and generalizations of the Nadler theorem were obtained by different authors; see for instance [5, 6] and references cited therein. The Tarski theorem was extended to multivalued mappings by different authors; see [5, 7–9]. Investigation of the existence of fixed points for single-valued mappings in partially ordered metric spaces was initially considered by Ran and Reurings in [10] who proved the following result.

Theorem 1.2 [10] *Let $(X,\preceq)$ be a partially ordered set such that every pair $x,y \in X$ has an upper and lower bound. Let d be a metric on X such that (X,d) is a complete metric* *space. Let $f : X \to X$ be a continuous monotone (either order preserving or order reversing) mapping. Suppose that the following conditions hold:*

(1) *There exists $k \in [0,1)$ with*

$$d\big(f(x),f(y)\big) \le kd(x,y) \quad \text{for all } x,y \in X \text{ such that } x \succeq y.$$

(2) *There exists an $x_0 \in X$ with $x_0 \preceq f(x_0)$ or $x_0 \succeq f(x_0)$.*

Then f is a Picard operator (PO), that is, f has a unique fixed point $x^* \in X$ and for each $x \in X$, $\lim_{n\to\infty} f^n(x) = x^*$.

After this, different authors considered the problem of existence of a fixed point for contraction mappings in partially ordered metric spaces; see [8, 11–13] and references cited therein. Nieto *et al.* in [13] extended the ideas of [10] to prove the existence of solutions to some differential equations. Recently, two results have appeared, giving sufficient conditions for f to be a PO, if (X,d) is endowed with a graph. The first of which was given by Jachymski [14] and the second one was given by Jachymski and Lukawska [15], generalizing the results of [11, 13, 16, 17] to a single-valued mapping in metric spaces with a graph instead of a partial ordering.

The aim of this paper is two folds: first to give a correct definition of monotone multivalued mappings, second to extend the conclusion of Theorem 1.2 to the case of monotone multivalued mappings in metric spaces endowed with a graph.

2 Preliminaries

It seems that the terminology of graph theory instead of partial ordering gives a clearer picture and yield interesting generalization of the Banach contraction principle. Let us begin this section with terminology for metric spaces which will be used throughout.

Let G be a directed graph (digraph) with set of vertices $V(G)$ and a set of edges $E(G)$ contains all the loops, *i.e.*, $(x,x) \in E(G)$ for any $x \in V(G)$. We also assume that G has no parallel edges (arcs) and so we can identify G with the pair $(V(G),E(G))$. Our graph theory notations and terminology are standard and can be found in all graph theory books, like [18, 19] and [20]. Moreover, we may treat G as a weighted graph (see [20], p.309]) by assigning to each edge the distance between its vertices. By G^{-1} we denote the conversion of a graph G, *i.e.*, the graph obtained from G by reversing the direction of edges. Thus we have

$$E\big(G^{-1}\big) = \big\{(y,x) \mid (x,y) \in E(G)\big\}.$$

A digraph G is called an oriented graph if whenever $(u,v) \in E(G)$, then $(v,u) \notin E(G)$. The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\widetilde{G}) = E(G) \cup E\big(G^{-1}\big).$$

We call (V',E') a subgraph of G if $V' \subseteq V(G)$, $E' \subseteq E(G)$ and for any edge $(x,y) \in E'$, $x,y \in V'$.

If x and y are vertices in a graph G, then a (directed) path in G from x to y of length N is a sequence $(x_i)_{i=1}^{i=N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$, and $(x_{n-1},x_n) \in E(G)$ for

$i = 1, \ldots, N$. A graph G is connected if there is a directed path between any two vertices. G is weakly connected if $\widetilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation $\mathcal{R}$ defined on $V(G)$ by the rule:

$y \mathcal{R} z$ if there is a (directed) path in G from y to z.

Clearly G_x is connected.

Next we introduce the concept of hyperbolic metric spaces. Indeed let (X, d) be a metric space. Suppose that there exists a family $\mathcal{F}$ of metric segments such that any two points x, y in X are endpoints of a unique metric segment $[x, y] \in \mathcal{F}$ ($[x, y]$ is an isometric image of the real line interval $[0, d(x, y)]$). We shall denote by $\beta x \oplus (1-\beta)y$ the unique point z of $[x, y]$ which satisfies

$$d(x,z) = (1-\beta)d(x,y) \quad \text{and} \quad d(z,y) = \beta d(x,y),$$

where $\beta \in [0,1]$. Such metric spaces with a family $\mathcal{F}$ of metric segments are usually called *convex metric spaces* [21]. Moreover, if we have

$$d\bigl(\alpha p \oplus (1-\alpha)x, \alpha q \oplus (1-\alpha)y\bigr) \le \alpha d(p,q) + (1-\alpha)d(x,y)$$

for all p, q, x, y in X, and $\alpha \in [0,1]$, then X is said to be a *hyperbolic metric space* (see [22]).

Obviously, normed linear spaces are hyperbolic spaces. As nonlinear examples, one can consider Hadamard manifolds [23], the Hilbert open unit ball equipped with the hyperbolic metric [24], and CAT(0) spaces [25–27]. We will say that a subset C of a hyperbolic metric space X is convex if $[x, y] \subset C$ whenever x, y are in C.

Definition 2.1 Let (X, d) be a hyperbolic metric space. A graph G on X is said to be convex if and only if for any $x, y, z, w \in X$ and $\alpha \in [0,1]$, we have

$$(x,z) \in E(G) \quad \text{and} \quad (y,w) \in E(G) \quad \Longrightarrow \quad \bigl(\alpha x \oplus (1-\alpha)y, \alpha z \oplus (1-\alpha)w\bigr) \in E(G).$$

Next we introduce the concept of monotone multivalued mappings. In [9], the authors offered the following definition.

Definition 2.2 ([9], Definition 2.6) Let $F : X \to 2^X$ be a multivalued mapping with nonempty closed and bounded values. The mapping F is said to be a G-contraction if there exists $k \in [0,1)$ such that

$$H\bigl(F(x), F(y)\bigr) \le k d(x,y) \quad \text{for all } (x,y) \in E(G)$$

and if $u \in F(x)$ and $v \in F(y)$ are such that

$$d(u,v) \le k d(x,y) + \alpha \quad \text{for each } \alpha > 0,$$

then $(u, v) \in E(G)$.

In particular, this definition implies that if $u \in F(x)$ and $v \in F(y)$ are such that

$$d(u,v) \le kd(x,y),$$

then $(u,v) \in E(G)$, which is very restrictive. In fact in the proof of Theorem 3.1 in [9], the authors try to construct an orbit (x_n) such that $(x_n, x_{n+1}) \in E(G)$, for any $n \ge 1$, but this fails to happen according to Definition 2.2. Our definition of G-contraction multivalued mappings is more appropriate. It finds its roots in [28]. In the sequel, we assume that (X,d) is a metric space, and G is a directed graph (digraph) with a set of vertices $V(G) = X$ and the set of edges $E(G)$ contains all the loops, *i.e.*, $(x,x) \in E(G)$, for any $x \in X$.

Definition 2.3 Let (X,d) be a metric space and C a nonempty subset of X.

(i) We say that a mapping $T : C \to C$ is G-edge preserving if

$$\forall x,y \in C, \quad (x,y) \in E(G) \quad \Rightarrow \quad \big(T(x),T(y)\big) \in E(G).$$

(ii) We say that a mapping $T : C \to C$ is G-contraction if T is G-edge preserving and there exists $k \in [0,1)$ such that

$$\forall x,y \in C, \quad (x,y) \in E(G) \quad \Rightarrow \quad d\big(T(x),T(y)\big) \le kd(x,y).$$

(iii) We say that a mapping $T : C \to C$ is G-nonexpansive if T is G-edge preserving and

$$\forall x,y \in C, \quad (x,y) \in E(G) \quad \Rightarrow \quad d\big(T(x),T(y)\big) \le d(x,y).$$

(iv) A multivalued mapping $T : C \to 2^C$ is said to be monotone increasing (resp. decreasing) G-contraction if there exists $\alpha \in [0,1)$ such that for any $x, y \in C$ with $(x,y) \in E(G)$ and any $u \in T(x)$ there exists $v \in T(y)$ such that

$$(u,v) \in E(G) \qquad \big(\text{resp. } (v,u) \in E(G)\big) \quad \text{and} \quad d(u,v) \le \alpha d(x,y).$$

Similarly we will say that the multivalued mapping $T : C \to 2^C$ is monotone increasing (resp. decreasing) G-nonexpansive if for any $x, y \in C$ with $(x,y) \in E(G)$ and any $u \in T(x)$ there exists $v \in T(y)$ such that

$$(u,v) \in E(G) \qquad \big(\text{resp. } (v,u) \in E(G)\big) \quad \text{and} \quad d(u,v) \le d(x,y).$$

$x \in C$ is called a fixed point of a single-valued mapping T if and only if $T(x) = x$. For a multivalued mapping T, x is a fixed point if and only if $x \in T(x)$. The set of all fixed points of a mapping T is denoted by $\mathrm{Fix}(T)$.

3 Main results

We begin with the following well-known theorem, which gives the existence of a fixed point for monotone single-valued and multivalued contraction mappings in metric spaces endowed with a graph.

Theorem 3.1 [14] *Let (X,d) be a complete metric space, and let the triple (X,d,G) have the following property*:

($*$) *For any $(x_n)_{n\geq 1}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$, for $n \geq 1$, then there is a subsequence $(x_{k_n})_{n\geq 1}$ with $(x_{k_n}, x) \in E(G)$, for $n \geq 1$.*

Let $f : X \to X$ be a G-contraction, $X_f := \{x \in X : (x, f(x)) \in E(G)\}$. Then the following statements hold:

1. $\operatorname{card}\operatorname{Fix} f = \operatorname{card}\{[x]_{\widetilde{G}} : x \in X_f\}$.
2. $\operatorname{Fix} f \neq \emptyset$ *if and only if* $X_f \neq \emptyset$.
3. *f has a unique fixed point if and only if there exists an $x_0 \in X_f$ such that $X_f \subseteq [x_0]_{\widetilde{G}}$.*
4. *For any $x \in X_f$, $f|_{[x]_{\widetilde{G}}}$ is a PO, that is, f has a unique fixed point $x^* \in [x]_{\widetilde{G}}$ and for each $x \in [x]_{\widetilde{G}}$, $\lim_{n\to\infty} f^n(x) = x^*$.*
5. *If $X_f \neq \emptyset$ and G is weakly connected, then f is a PO, that is, f has a unique fixed point $x^* \in X$ and for each $x \in X$, $\lim_{n\to\infty} f^n(x) = x^*$.*

The multivalued version of Theorem 3.1 may be stated as follows.

Theorem 3.2 [29] *Let (X,d) be a complete metric space and suppose that the triple (X,d,G) has property ($*$). We denote by $\mathcal{CB}(X)$ the collection of all nonempty closed and bounded subsets of X. Let $T : X \to \mathcal{CB}(X)$ be a monotone increasing G-contraction mapping and $X_T := \{x \in X; (x,u) \in E(G)$ for some $u \in T(x)\}$. If $X_T \neq \emptyset$, then the following statements hold*:

1. *For any $x \in X_T$, $T|_{[x]_{\widetilde{G}}}$ has a fixed point.*
2. *If $x \in X$ with $(x, \bar{x}) \in E(G)$ where $\bar{x}$ is a fixed point of T, then $\{T^n(x)\}$ converges to $\bar{x}$.*
3. *If G is weakly connected, then T has a fixed point in G.*
4. *If $X' := \bigcup\{[x]_{\widetilde{G}} : x \in X_T\}$, then $T|_{X'}$ has a fixed point in X.*
5. *If $T(X) \subseteq E(G)$ then T has a fixed point.*
6. $\operatorname{Fix} T \neq \emptyset$ *if and only if* $X_T \neq \emptyset$.

Remark 3.1 The missing information in Theorem 3.2 is the uniqueness of the fixed point. In fact we do have a partial positive answer to this question. Indeed if $\bar{u}$ and $\bar{w}$ are two fixed points of T such that $(\bar{u}, \bar{w}) \in E(G)$, then we must have $\bar{u} = \bar{w}$. In general T may have more than one fixed point.

Remark 3.2 If we assume G is such that $E(G) := X \times X$ then clearly G is connected and Theorem 3.2 gives the Nadler theorem [4].

The following is a direct consequence of Theorem 3.2.

Corollary 3.1 *Let (X,d) be a complete metric space. Let G be a graph on X such that the triple (X,d,G) has the Property ($*$). If G is weakly connected then every G-contraction $T : X \to \mathcal{CB}(X)$ such that $(x_0, x_1) \in E(G)$, for some $x_0 \in X$ and $x_1 \in T(x_0)$, has a fixed point.*

Next we discuss some existence results for nonexpansive single-valued and multivalued G-monotone mappings. To the best of our knowledge, these results were never investigated for such mappings.

Theorem 3.3 *Let (X,d) be a complete hyperbolic metric space and suppose that the triple (X,d,G) has property $(*)$. Assume G is convex. Let C be a nonempty, closed, convex, and bounded subset of X. Let $T : C \to C$ be a G-nonexpansive mapping. Assume $C_T := \{x \in C : (x, T(x)) \in E(G)\} \neq \emptyset$. Then*

$$\inf\{d(x, T(x)); x \in C\} = 0.$$

In particular, there exists an approximate fixed point sequence (x_n) in C of T, i.e.,

$$\lim_{n\to\infty} d(x_n, T(x_n)) = 0.$$

Proof Fix $a \in C$. Let $\lambda \in (0,1)$ and define $T_\lambda : C \to C$ by

$$T_\lambda(x) = \lambda a \oplus (1-\lambda)T(x).$$

If $(x,y) \in E(G)$, then we have $(T(x), T(y)) \in E(G)$, since T is G-edge preserving. Moreover, since G is convex and $(a,a) \in E(G)$, we obtain

$$\big(T_\lambda(x), T_\lambda(y)\big) = \big(\lambda a \oplus (1-\lambda)T(x), \lambda a \oplus (1-\lambda)T(y)\big) \in E(G),$$

i.e., T_λ is G-edge preserving, and

$$d\big(\lambda a \oplus (1-\lambda)T(x), \lambda a \oplus (1-\lambda)T(y)\big) \leq (1-\lambda)d\big(T(x), T(y)\big) \leq (1-\lambda)d(x,y),$$

i.e., $d(T_\lambda(x), T_\lambda(y)) \leq (1-\lambda)d(x,y)$. In other words, T_λ is a G-contraction. It is easy to see that $C_T \subset C_{T_\lambda}$. Hence C_{T_λ} is not empty. Theorem 3.1 implies the existence of a fixed point ω_λ of T_λ in C. So we have

$$\omega_\lambda = \lambda a \oplus (1-\lambda)T(\omega_\lambda),$$

which implies

$$d\big(\omega_\lambda, T(\omega_\lambda)\big) \leq \lambda d\big(a, T(\omega_\lambda)\big) \leq \lambda\delta(C),$$

where $\delta(C) = \sup\{d(x,y); x,y \in C\}$ is the diameter of C. Set $x_n = \omega_{1/n}$, for $n \geq 1$. Then we have $d(x_n, T(x_n)) \leq \delta(C)/n$, for $n \geq 1$. In particular, we have

$$\inf\{d(x, T(x)); x \in X\} \leq \lim_{n\to\infty} d\big(x_n, T(x_n)\big) = 0.$$

The proof of Theorem 3.3 is therefore complete. □

In order to obtain a fixed point existence result for G-nonexpansive mappings, we need some extra assumptions.

Definition 3.1 We will say that G is transitive if, for any two vertices x and y that are connected by a directed finite path, we have $(x,y) \in E(G)$.

Note that if the triple (X,d,G) has property $(*)$ and G is transitive, then we have the following property:

$(**)$ For any $(x_n)_{n\geq 1}$ in X, if $x_n \to x$ and $(x_n,x_{n+1}) \in E(G)$, for $n \geq 1$, then $(x_n,x) \in E(G)$, for $n \geq 1$.

Definition 3.2 We will say that a nonempty subset C of X is G-compact if and only if for any $(x_n)_{n\geq 1}$ in C, if $(x_n,x_{n+1}) \in E(G)$, for $n \geq 1$, then there exists a subsequence (x_{k_n}) of (x_n) which is convergent to a point in C.

Note that G-compactness does not necessarily imply compactness. Indeed, consider the metric set X, subset of $\mathbb{R}^3$, built on a cone routed at the origin. All rays are bounded and compact. But X is unbounded. Define the graph G on X by $(x,y) \in E(G)$ if and only if x and y are on the same ray. Then any sequence $(x_n) \in X$ such that $(x_n,x_{n+1}) \in E(G)$, for $n \geq 1$, will belong to a ray. Hence (x_n) has a convergent subsequence. This shows that X is G-compact but fails to be compact.

Theorem 3.4 *Let (X,d) be a complete hyperbolic metric space and suppose that the triple (X,d,G) has property $(*)$. Assume G is convex and transitive. Let C be a nonempty, G-compact and convex subset of X. Let $T : C \to C$ be a G-nonexpansive mapping. Assume $C_T := \{x \in C : (x, T(x)) \in E(G)\} \neq \emptyset$. Then T has a fixed point.*

Proof Since C_T is not empty, choose $x_0 \in C_T$. Let (λ_n) be a sequence of numbers in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 0$. As in the proof of Theorem 3.3, define the mapping $T_1 : C \to C$ by

$$T_1(x) = \lambda_1 x_0 \oplus (1-\lambda_1)T(x).$$

Since $(x_0, T(x_0)) \in E(G)$, we get $(x_0, T_1(x_0)) \in E(G)$. Since T_1 is G-edge preserving, we obtain $(T_1^n(x_0), T_1^{n+1}(x_0)) \in E(G)$ and

$$d\bigl(T_1^n(x_0), T_1^{n+1}(x_0)\bigr) \leq \lambda_1^n d\bigl(x_0, T_1(x_0)\bigr) \quad \text{for } n \geq 1.$$

Hence $(T_1^n(x_0))$ is a Cauchy sequence. Since C is G-compact, we conclude that $(T_1^n(x_0))$ is convergent. Set $\lim_{n\to\infty} T_1^n(x_0) = x_1$. The property $(**)$ implies that $(x_0,x_1) \in E(G)$. By induction, we construct a sequence (x_n) such that x_{n+1} is a fixed point of $T_{n+1} : C \to C$ defined by

$$T_{n+1}(x) = \lambda_{n+1} x_n \oplus (1-\lambda_{n+1})T(x),$$

obtained as the limit of $(T_{n+1}^k(x_n))_{k\geq 1}$. In particular, we have $(x_n,x_{n+1}) \in E(G)$, for any $n \geq 1$. Since C is G-compact, there exists a subsequence (x_{k_n}) which converges to $\omega \in C$. Since G is transitive, the property $(**)$ implies that $(x_{k_n},\omega) \in E(G)$. Using the G-nonexpansiveness of T, we conclude that

$$d\bigl(T(x_{k_n}), T(\omega)\bigr) \leq d(x_{k_n},\omega) \quad \text{for } n \geq 1.$$

Hence $(T(x_{k_n}))$ converges to $T(\omega)$. Since x_{n+1} is a fixed point of T_{n+1}, we get $x_{n+1} = \lambda_{n+1}x_n \oplus (1-\lambda_{n+1})T(x_{n+1})$, which implies

$$d\big(x_{n+1}, T(x_{n+1})\big) \leq \lambda_{n+1} d\big(x_n, T(x_{n+1})\big) \leq \lambda_{n+1}\delta(C) \quad \text{for } n \geq 1,$$

which implies $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. Hence $(T(x_{k_n}))$ converges to ω as well. Therefore we must have $T(\omega) = \omega$, *i.e.*, T has a fixed point. □

Next we investigate the above results for multivalued mappings. The first result for these mappings is the analog to Theorem 3.3.

Theorem 3.5 *Let (X,d) be a complete hyperbolic metric space and suppose that the triple (X,d,G) has property $(*)$. Assume G is convex. Let C be a nonempty, closed, convex, and bounded subset of X. Set $\mathcal{C}(C)$ to be the set of all nonempty closed subsets of C. Let $T : C \to \mathcal{C}(C)$ be a monotone increasing G-nonexpansive mapping. If $C_T := \{x \in C; (x,y) \in E(G)$ for some $y \in T(x)\}$ is not empty, then T has an approximate fixed point sequence $(x_n) \in C$, that is, for any $n \geq 1$, there exists $y_n \in T(x_n)$ such that*

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

In particular, we have $\lim_{n\to\infty} \operatorname{dist}(x_n, T(x_n)) = 0$, *where*

$$\operatorname{dist}\big(x_n, T(x_n)\big) = \inf\big\{d(x_n, y); y \in T(x_n)\big\}.$$

Proof Fix $\lambda \in (0,1)$ and $x_0 \in C$. Define the multivalued map T_λ on C by

$$T_\lambda(x) = \lambda x_0 \oplus (1-\lambda)T(x) = \big\{\lambda x_0 \oplus (1-\lambda)y; y \in T(x)\big\}.$$

Note that $T_\lambda(x)$ is nonempty and closed subset of C. Let us show that T_λ is a monotone increasing G-contraction. Let $x, y \in C$ such that $(x,y) \in E(G)$. Since T is a monotone increasing G-nonexpansive mapping, for any $x^* \in T(x)$ there exists $y^* \in T(y)$ such that $(x^*, y^*) \in E(G)$ and $d(x^*, y^*) \leq d(x,y)$. Since

$$d\big(\lambda x_0 \oplus (1-\lambda)x^*, \lambda x_0 \oplus (1-\lambda)y^*\big) \leq (1-\lambda)d\big(x^*, y^*\big) \leq (1-\lambda)d(x,y),$$

which proves our claim. Since G is convex, we get $(\lambda x_0 \oplus (1-\lambda)x^*, \lambda x_0 \oplus (1-\lambda)y^*) \in E(G)$. This clearly shows that T_λ is a monotone increasing G-contraction as claimed. Note that we have $C_T \subset C_{T_\lambda}$, which implies that C_{T_λ} is nonempty. Using Theorem 3.2 we conclude that T_λ has a fixed point $x_\lambda \in C$. Thus there exists $y_\lambda \in T(x_\lambda)$ such that

$$x_\lambda = \lambda x_0 \oplus (1-\lambda)y_\lambda.$$

In particular we have

$$d(x_\lambda, y_\lambda) \leq \lambda d(x_0, y_\lambda) \leq \lambda\delta(C),$$

which implies $\mathrm{dist}(x_\lambda, T(x_\lambda)) \leq \lambda\delta(C)$. If we choose $\lambda = \frac{1}{n}$, for $n \geq 1$, there exist $x_n \in C$ and $y_n \in T(x_n)$ such that $d(x_n, y_n) \leq \delta(C)/n$, which implies

$$\mathrm{dist}\big(x_n, T(x_n)\big) \leq \frac{1}{n}\delta(C).$$

The proof of Theorem 3.5 is therefore complete. □

The multivalued version of Theorem 3.4 may be stated as follows.

Theorem 3.6 *Let (X,d) be a complete hyperbolic metric space and suppose that the triple (X,d,G) has property $(**)$. Assume G is convex and transitive. Let C be a nonempty, G-compact, and convex subset of X. Then any $T : C \to \mathcal{C}(C)$ monotone increasing G-nonexpansive mapping has a fixed point provided $C_T := \{x \in C; (x,y) \in E(G)$ for some $y \in T(x)\}$ is not empty.*

Proof Since C_T is not empty, choose $x_0 \in C_T$. Let (λ_n) be a sequence of numbers in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 0$. As we did in the proof of Theorem 3.5, define the mapping $T_1 : C \to C$ by

$$T_1(x) = \lambda_1 x_0 \oplus (1-\lambda_1)T(x).$$

Since $C_T \subset C_{T_1}$, there exists $y_0 \in T_1(x_0)$ such that $(x_0, y_0) \in E(G)$. Using the properties of T_1, there exists $y_2 \in T_1(y_1)$ such that $(y_1, y_2) \in E(G)$ and

$$d(y_1, y_2) \leq (1-\lambda_1)d(x_0, y_1).$$

By induction we build a sequence (y_n), with $y_0 = x_0$, such that $y_{n+1} \in T_1(y_n)$, $(y_n, y_{n+1}) \in E(G)$, and

$$d(y_n, y_{n+1}) \leq (1-\lambda_1)d(y_{n-1}, y_n) \leq (1-\lambda_1)^n d(x_0, y_1) \leq (1-\lambda_1)^n \delta(C)$$

for $n \geq 1$. So (y_n) is Cauchy. Set $\lim_{n\to+\infty} y_n = x_1 \in C$. The property $(**)$ implies that $(y_n, x_1) \in E(G)$, for any n. In particular, we have $(x_0, x_1) \in E(G)$. Using the properties of T_1, for any n there exists $z_n \in T(x_1)$ such that

$$d(y_{n+1}, z_n) \leq (1-\lambda_1)d(y_n, x_1).$$

Clearly this implies that (z_n) converges to x_1 as well. Since $T(x_1)$ is closed, we conclude that $x_1 \in T(x_1)$, *i.e.*, x_1 is a fixed point of T_1. By induction, we construct a sequence (x_n) in C such that x_{n+1} is a fixed point of $T_{n+1} : C \to \mathcal{C}(C)$ defined by

$$T_{n+1}(x) = \lambda_{n+1} x_n \oplus (1-\lambda_{n+1})T(x),$$

and $(x_n, x_{n+1}) \in E(G)$. Since C is G-compact, there exists a subsequence (x_{k_n}) which converges to $\omega \in C$. Since G is transitive, the property $(**)$ implies that $(x_n, \omega) \in E(G)$. Since x_n is a fixed point of T_n, there exists $z_n \in T(x_n)$ such that

$$x_n = \lambda_n x_{n-1} \oplus (1-\lambda_n)z_n$$

for any $n \geq 1$. Note that $d(x_n, z_n) \leq \lambda_n d(x_{n_1}, z_n) \leq \lambda_n \delta(C)$, for any $n \geq 1$. In particular we have $\lim_{n\to\infty} d(x_n, z_n) = 0$. Since C is G-compact, there exists a subsequence (x_{k_n}) which converges to some point $\omega \in C$. Clearly (z_{k_n}) also converges to ω. Using the G-nonexpansiveness of T, since $(x_{k_n}, \omega) \in E(G)$, there exists $\omega_n \in T(\omega)$ such that $d(z_{k_n}, \omega_n) \leq d(x_{k_n}, \omega)$, for any n. Therefore we see that (ω_n) converges to ω. Since $T(\omega)$ is closed, we conclude that $\omega \in T(\omega)$, *i.e.*, ω is a fixed point of T. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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